2/12 - Schoen-Simon Compactness

Recall what we're trying to prove.

stable, minimal (codim 1) hypersurface w/ small singular set

Theorem: (Sheeting Theorem)

Let $n \geq 2$. Then, $\exists \varepsilon(n) \in (0,1)$ s.t. if M is stationary, stable, with $\mathcal{H}^{n-2}(\text{sing}(M)) = 0$ and

- $\sup_{M \cap C_2} |x^{n+1}| \leq \frac{1}{2}$
- $E_m^2 := \int_{M \cap C_1} g^2 < \varepsilon^2$

this happens when varifold conv. to a plane

then

$$M \cap C_{\frac{1}{2}} = \bigcup_{i=1}^{q} \text{graph}(u_i), \quad u_i : \overline{B^n_{\frac{1}{2}}}(0) \to \mathbb{R} \text{ smooth minimal graphs w/ } u_i < u_{i+1}$$

Remark: Recall from stratification that "flat" singular points such. The above theorem says that when M is L^2-close to being flat (in the tilt sense), then these bad singularities don't happen.

To accomplish this, we work toward the following result:

Theorem: ($L^2 \to L^\infty$)

Let M be as above. Then,

$$\int_{M \cap C_1} g^2 < \varepsilon^2 \implies \sup_{M \cap C_{\frac{1}{2}}} g \leq \frac{1}{2n}$$

Proof: Recall the weak Caccioppoli inequality from last time: $\forall k \in [0, \frac{1}{2n}]$, $\varphi \in C_c^{0,1}(M)$, the hypothesis gives

we'll drop the superscript for notation

$$\frac{1}{2n} \int_{\{g > k\}} |\nabla^M g|^2 \varphi^2 \left(1 - \frac{k}{g}\right) \leq \int_{\{g > k\}} (g-k)^2 |\nabla^M \varphi|^2$$

We will apply "De Giorgi iteration" to do this.

For $\ell \in \mathbb{N}$, set
$$R_\ell := \tfrac{1}{2} + 2^{-\ell} \downarrow \tfrac{1}{2}$$
$$k_\ell := \tfrac{d}{2n}\left(1 - 2^{-(\ell-1)}\right) \uparrow \tfrac{d}{2n}, \quad d \in (0,1] \text{ fixed param}$$

Using k_ℓ in Caccioppoli,

$$\frac{1}{2n} \int_{\{g > k_{\ell+1}\}} |\nabla g|^2 \varphi^2 \left(1 - \frac{k_\ell}{g}\right) \underset{k_{\ell+1} > k_\ell}{\leq} \frac{1}{2n} \int_{\{g > k_\ell\}} |\nabla g|^2 \varphi^2 \left(1 - \frac{k_\ell}{g}\right) \underset{\text{Caccioppoli}}{\leq} \int_{\{g > k_\ell\}} (g - k_\ell)^2 |\nabla \varphi|^2$$

We know $1-\frac{k_\ell}{g} = \frac{g-k_\ell}{g} \underset{\substack{\uparrow \\ \text{level set}}}{\geq} \frac{k_{\ell+1}-k_\ell}{g} = \frac{d}{2^{\ell+1} n g} \geq \frac{d}{2^{\ell+1} n}$, and so

$$\int_{\{g>k_{\ell+1}\}} |\nabla g|^2 \varphi^2 \le \frac{4n^2 2^\ell}{d} \int_{\{g>k_\ell\}} (g-k_\ell)^2 |\nabla \varphi|^2$$

Note that

$$|\nabla((g-k_{\ell+1})^+\varphi)|^2 \le 2(g-k_{\ell+1})^{+2}|\nabla\varphi|^2 + 2\,\overbrace{\mathbb{1}_{\{g>k_{\ell+1}\}}|\nabla g|^2\varphi^2}^{\text{Integrand of LHS of above!}}$$

Integrating,
$$\int |\nabla[(g-k_{\ell+1})^+\varphi]|^2 \le \frac{c(n)2^\ell}{d}\int_{\{g>k_\ell\}} (g-k_\ell)^2|\nabla\varphi|^2$$

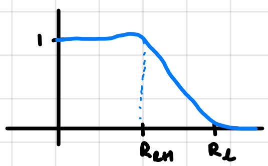
For $n\ge 3$, from Michael-Simon Sobolev ineq,

For $n=2$, there's something slightly different but mostly $\cong$ to do

$$\left(\int |(g-k_{\ell+1})^+\varphi|^{\underbrace{\frac{2n}{n-2}}_{>2}}\right)^{\frac{n-2}{n}} \le C_{MS}(n)\int |\nabla[(g-k_{\ell+1})^+\varphi]|^2$$

$$\Rightarrow \left(\int |(g-k_{\ell+1})^+\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le \frac{c(n)2^\ell}{d}\int_{\{g>k_\ell\}} (g-k_\ell)^2|\nabla\varphi|^2$$

Now take φ to be a cutoff



$\Rightarrow |\nabla\varphi| \le \frac{2}{R_\ell - R_{\ell+1}} \le 8\cdot 2^\ell$

(Note that this is the radial direction: make it symmetric radially, and (fiddly to ensure cpt. supp.) vertically onto the cylinder)

$$\Longrightarrow \left(\int_{\substack{M\cap C_{R_{\ell+1}} \\ \{g>k_{\ell+1}\}}} |(g-k_{\ell+1})^+|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le \frac{c(n)\cdot 8^\ell}{d}\int_{\substack{M\cap C_{R_\ell} \\ \{g>k_\ell\}}} (g-k_\ell)^{+2}$$

By Hölder,

$$\int_{M\cap C_{R_{\ell+1}}} (g-k_{\ell+1})^{+2} \le \left(\int_{M\cap C_{R_{\ell+1}}} (g-k_{\ell+1})^{+\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \cdot \mathcal{H}^n(M\cap C_{R_{\ell+1}}\cap\{g>k_{\ell+1}\})^{\frac{2}{n}}$$

On $\{g>k_{\ell+1}\}$, we know $(g-k_\ell)^+ \ge k_{\ell+1}-k_\ell \ge \frac{d}{n2^{\ell+1}}$. By Markov's ineq ($C_{R_{\ell+1}} \subseteq C_{R_\ell}$),

$$\mathcal{H}^n(M\cap C_{R_{\ell+1}}\cap\{g>k_{\ell+1}\}) \le \frac{n^2 4^{\ell+1}}{d^2}\int_{M\cap C_{R_\ell}} (g-k_\ell)^{+2}$$

$$\int_{\substack{M\cap C_{R_{\ell+1}} \\ \{g>k_{\ell+1}\}}} (g-k_{\ell+1})^{+2} \le \frac{C(n)\cdot 32^\ell}{d^{1+\frac{4}{n}}}\left(\int_{\substack{M\cap C_{R_\ell} \\ \{g>k_\ell\}}} (g-k_\ell)^{+2}\right)^{1+\frac{2}{n}}$$

Setting $G_\ell := \int_{\substack{M\cap C_{R_\ell} \\ \{g>k_\ell\}}} (g-k_\ell)^{+2}$, we have $G_{\ell+1} \le \frac{c(n)}{d^{1+\frac{4}{n}}}\cdot 32^\ell G_\ell^{1+\frac{2}{n}}$

<u>Claim:</u> If $G_1 < \varepsilon(n,d)$, then $G_\ell \to 0$. $\left(\varepsilon(n,d) = c(n)+d^{2+\frac{4}{n}}\right)$

From this, it follows by taking $d=1$ $(R_1=1,\ k_1=0 \Rightarrow G_1 = \int_{M\cap C_1} g^2)$ that if the L^2 tilt excess is small,

$$G_\ell \to 0 \Rightarrow \int_{M\cap C_{\frac{1}{2}}} (g-\tfrac{1}{2n})^{+2} = 0 \Rightarrow g \le \tfrac{1}{2n} \text{ on } M\cap C_{\frac{1}{2}}.$$

$\square$

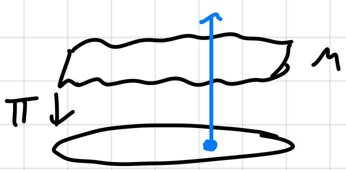
Remark: If we wanted an explicit bound on $\sup_{M\cap C_{\frac{1}{2}}} g^2$ in terms of the tilt excess, we'd track how $\varepsilon(n,d)$ depends on d. This gives something like $L^\infty \lesssim L^2$-type ineq, but with a power.

Proof of Sheeting Thm: We know $g \le \frac{1}{2n}$ on $M \cap C_{\frac{1}{2}}$ by above.

M embedded $\Rightarrow$ $\forall x \in M$, $\exists$ neighborhood $D_x \ni x$ s.t. $M \cap D_x$ is embedded disk

tangent space is fairly flat

We may continuously choose a unit normal on $M \cap D_x$ s.t. $(\nu \cdot e_{n+1}) \ge \sqrt{1 - \frac{1}{(2n)^2}}$
Consider the natural projection $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.
We want the rays $\mathbb{R} \times \{q\}$ (in blue) to intersect M transversely with +1 intersections (i.e. M doesn't do $\supset$).



So, each connected component of M is a graph (no multi-values).
Since $u_i : B^n_{\frac{1}{2}}(0) \setminus \Sigma \to \mathbb{R}$ is a minimal graph and $\mathcal{H}^{n-2}(\Sigma) = 0$, a singularity removal theorem (see Leon Simon in the 70s) gives that u_i extends across Σ.

$\square$

The Sheeting Theorem is the main thing needed to show a great compactness property for sufficiently regular hypersurfaces.

Theorem: (Schoen-Simon Compactness and Regularity)

Suppose $(M_k)_{k \in \mathbb{N}}$ is a sequence of stable minimal hypersurfaces in $B_1^{n+1}(0)$ with $\mathcal{H}^{n-2}(\text{sing}(M_k)) = 0$ and $\limsup_{k \to \infty} \mathcal{H}^n(M_k \cap B_1^{n+1}(0)) < \infty$.

simple case as before

Then, $\exists$ subsequence $(M_{k'})_{k'}$ and a varifold V s.t.

① $M_{k'} \to V$ in $B^{n+1}_{\frac{1}{2}}(0)$ (in the varifold sense)

② $\text{spt}\,\|V\| \cap B^{n+1}_{\frac{1}{2}}(0) = \overline{M} \cap B^{n+1}_{\frac{1}{2}}(0)$, where M is a stable minimal hypersurface with $\dim_{\mathcal{H}}(\text{sing}(M)) \le n-7$.

In particular, taking constant sequences, all stable minimal hypersurfaces with $\mathcal{H}^{n-2}(\text{sing}) = 0$ in fact has $\dim_{\mathcal{H}}(\text{sing}) \le n-7$.

Proof: By compactness of stationary integral varifolds, $\exists$ subseq. $M_{k'}$ and stationary integral varifold V s.t. $M_{k'} \to V$ as varifolds.

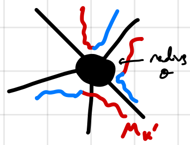
varifold w/ multiplicity 1 associated w/ $M_{k'}$

We stratify! Suppose $x \in \operatorname{sing}(V)$ is a flat singular point ($x \in \tilde{S}_n$). Applying the sheeting thm to $M_{k'}$, this can't happen.

Next, suppose $x \in \tilde{S}_{n-1}$. Zooming in around x (tangent cones),

"smallness of singular set ⇒ very curved hypersurfaces ⇒ ✱ via stability"

V looks like



$\times \mathbb{R}^{n-1}$

Away from a small ball, Sheeting thm. applies and we're flat. So, look in ball.

Since $\mathcal{H}^{n-2}(\operatorname{sing}(M_{k})) = 0$, almost every plane must not do this. So, for a.e. $y \in \mathbb{R}^{n-1}$, $(\mathbb{R}^2 \times \{y\}) \cap \operatorname{sing}(M_{k'}) = \emptyset$. Since things don't curve too much (stability max), $\exists\, \theta = \theta(C) > 0$ s.t.

$$\sup_{\substack{x_1, x_2 \text{ connected} \\ \text{by curve in} \\ \text{the slice}}} |\nu(x_1) - \nu(x_2)| \geq \theta$$

(we may find differing unit normals, otherwise would be flat and Sheeting thm. applies)

However, by general geometry we have $|\nu(x_1) - \nu(x_2)| \leq \int_\Gamma |\nabla^\Gamma \nu| \leq \int_{\substack{M \cap B_\theta(\mathbb{R}^{n-1}) \\ \cap (\mathbb{R}^2 \times \{y\})}} |A|$ (curve joining x_1, x_2)

Integrating over y,

$$\theta \leq c(n) \int_{M_k \cap B_\theta(\mathbb{R}^{n-1})} |A| \overset{\text{Holder}}{\leq} c(n) \underbrace{\left(\int |A|^2\right)^{\frac{1}{2}}}_{\leq c(n) \text{ by stability}} \underbrace{\mathcal{H}^n(M_k \cap B_\theta(\mathbb{R}^{n-1}))^{\frac{1}{2}}}_{\sim \theta^{\frac{1}{2}} c(n)} \Rightarrow \theta \lesssim \theta^{\frac{1}{2}} \quad ✱.$$

The $x \in \tilde{S}_{n-2}$ case is handled by treating the tangent cone $C = \underset{\uparrow}{C_0} \times \mathbb{R}^{n-2}$ (2-dim in $\mathbb{R}^3$). Looking at the link $\Sigma = C_0 \cap S^2$, which can't have any singularity by previous parts $\Rightarrow C_0$ flat $\Rightarrow C$ planar, ✱.

Suppose $x \in \tilde{S}_{n-3} \Rightarrow \ldots \Rightarrow \exists\, C \in \operatorname{VarTan}_x(V)$ of the form $C = C_0 \times \mathbb{R}^{n-3}$. By the above, $\operatorname{sing}(V) \subseteq S_{n-3} \Rightarrow \mathcal{H}^{n-2}(\operatorname{sing}(V)) = 0 \Rightarrow$ we can pass stability to the dim-reduced part of the cone. because sheeting theorem gives graphical convergence

We apply:

<u>Thm:</u> (Simons' Classification)

If $C \subseteq \mathbb{R}^{n+1}$ (n-dim cone) is a minimal, stable cone and $\operatorname{sing}(C) \subseteq \{0\}$,
<u>then</u> $n \in \{3,4,5,6\} \Rightarrow C$ is flat and equals a plane.

So, $\exists\, y \in \operatorname{sing}(C_0) \setminus \{0\} \Rightarrow \exists\, \tilde{y} \in \operatorname{sing}(C) \setminus \overset{\text{spine}}{S(C)}$.
Take a tangent cone to the tangent cone $\tilde{C} \in \operatorname{VarTan}_{\tilde{y}}(C)$

<u>Lemma:</u> $S(\tilde{C}) \supsetneq S(C)$

From the lemma, we know that $\dim(S(\tilde{C})) \geq \dim(S(C)) + 1$ since they're subspaces. Applying the $\tilde{S}_{n-2}$ result on $\tilde{C}$, we get ✱. Clearly, the argument can be iterated until $\tilde{S}_{n-6}$, and so we are done once we prove the lemma.

<u>Proof of lemma:</u> Take $x \in S(C) \Rightarrow \tilde{C} - x = \lim_{\lambda_j \downarrow 0} \frac{C - \tilde{y}}{\lambda_j} - x = \lim_{\lambda_j \downarrow 0} \frac{C - \tilde{y} - \lambda_j x}{\lambda_j} \overset{\substack{C - \lambda_j x = C \\ \text{because } x \in S(C)}}{=} \tilde{C}$
$\Rightarrow x \in S(\tilde{C})$

Also, $\tilde{y} \in S(\tilde{C})$ since $\tilde{C} - \tilde{y} = \lim_{\lambda_j \downarrow 0} \frac{C - \tilde{y}}{\lambda_j} - \tilde{y} = \lim_{\lambda_j \downarrow 0} \frac{C - (1 + \lambda_j)\tilde{y}}{\lambda_j}$
$\overset{(1+\lambda_j)C = C}{=} \lim_{\lambda_j \downarrow 0} \frac{(1+\lambda_j)(C - \tilde{y})}{\lambda_j} = \lim_{\lambda_j \downarrow 0} (1 + \lambda_j)\tilde{C} = \tilde{C}$. $\square$

Remark: $\exists$ singular minimal surface in $\mathbb{R}^8 \cong \mathbb{R}^4 \times \mathbb{R}^4$ via $\{|x|=|y| : x,y \in \mathbb{R}^4\}$
Two possible (but Minter doesn't know) because $\underset{n \in \mathbb{N}}{\arg\max}\, \mathcal{H}^n(B_1^n(0)) = 7$?

2/14 :

§ 3: Allard Regularity & Excess Decay

We go back to the usual setting:

- V is stationary integral n-varifold in $B_1^{n+k}(0)$
- We stratify $\operatorname{sing}(V) = \tilde{S}_n \sqcup \dots \sqcup \tilde{S}_0$

$\tilde{S}_n$ was problematic since there was no useful dimension bound. (Stability solves this, see the sheeting theorem). We know

$x \in \tilde{S}_n \Rightarrow$ (i) $\exists$ tangent cone of the form $\theta \cdot$ plane (ii) $\theta \in \{\cancel{1}, 2, \dots\}$, and so $\Theta_V(x) \in \mathbb{N}$

It turns out that if $\theta = 1$, then by Allard we know that
$\left(\begin{array}{l} x \in \operatorname{spt}\|V\| \text{ where a tangent cone} \\ \text{is a plane w/ mult. } 1 \end{array} \right) \Rightarrow x \notin \operatorname{sing}(V)$

In fact, Allard gives an ε-regularity theorem:
when V is ε-close to a multiplicity 1 plane, then
V is locally a $C^{1,\alpha}$ graph with estimates.

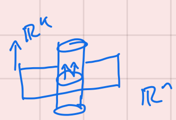
Theorem: (Allard Regularity)

Fix $\delta > 0$. Then $\exists\, \varepsilon(n,k,\delta)$ s.t. the following holds:

also works if non-integral if $\Theta_V(x) \geq 1$ a.e.

If V is a stationary integral varifold in $B_2^{n+k}(0)$ with

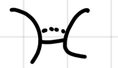
- $0 \in \operatorname{spt}\|V\|$ (V is nontrivial)
- $\dfrac{\|V\|(\mathbb{R}^k \times B_1^n(0))}{\omega_n} \leq 2 - \delta$ (V has multiplicity close to 1)
- $\hat{E}_V^2 := \displaystyle\int_{\mathbb{R}^k \times B_1^n(0)} \underbrace{\operatorname{dist}^2(x, \{0\} \times \mathbb{R}^n)}_{= \sum_{i=1}^k |x_i|^2} \, d\|V\|(x) < \varepsilon$ (close to planar)



Then, $\exists\, u \in C^{1,\alpha}(B_{\frac{1}{2}}^n(0), \mathbb{R}^k)$ s.t. $V \llcorner (\mathbb{R}^k \times B_{\frac{1}{2}}^n(0)) = \operatorname{graph}(u)$
with
$$\|u\|_{C^{1,\alpha}} \leq C(n,k)\, \hat{E}_V$$

can upgrade to smooth, etc. using elliptic PDE

Remarks: ① if V is graphical then $\hat{E}_V = \|u\|_{L^2}$, and we recover a classical PDE result $\|\cdot\|_{C^{1,\alpha}} \lesssim \|\cdot\|_{L^2}$.

② Very little is known for multiplicity ≥ 2. Consider the catenoid  which is minimal: by rescaling,

we may get  $\to \ldots$  $\to$ plane w/ mult 2, which certainly isn't graphical because of the neck.

③ An open question is: $\exists$ a minimal surface in $\mathbb{R}^3$ with an isolated singularity?

Now, some corollaries!

Corollary:

$\exists\, \alpha(n,k) \in (0,1)$ s.t. if V is stationary integral varifold, then

$$\Theta_V(x) < 1+\alpha \Rightarrow x \notin \operatorname{sing}(V)$$

Proof: First, suppose V is a cone.

Lemma: $\exists\, \alpha(n,k) > 0$ s.t. if C is a non-flat stationary integral cone, then $\Theta_C(0) \geq 1+\alpha$.

Proof of lemma: Suppose not. Then, $\exists\, (C_k)_k$ with $\Theta_{C_k}(0) \downarrow 1$, all non-flat.

C_k conical $\Rightarrow$ constant mass ratio $\Rightarrow \dfrac{\|C_k\|(B_1^n(0))}{\omega_n}$

$\Rightarrow \|C_k(B_1^n(0))\| = \omega_n \Theta_{C_k}(0)$.

Applying Schoen-Simon compactness, $\exists$ subseq $C_{k'} \to C$ (stationary integral cone). Varifold convergence implies convergence of mass,

$\|C\|(B_1^n(0)) = \lim_{k' \to \infty} \|C_{k'}\|(B_1^n(0)) = \omega_n \Rightarrow \underset{=1}{\text{mass ratio}} \Rightarrow \Theta_C(0) = 1$

Since C integral and so C integral, then $\Theta_C(x) \geq 1$ $\|C\|$-a.e., and so $S(C) = \operatorname{spt}\|C\| \Rightarrow C =$ multiplicity 1 plane

Applying Allard to each $C_{k'}$ and using varifold convergence* to the flat C, we find $\operatorname{sing}(C_{k'}) = \emptyset \Rightarrow C_{k'}$ flat $\Rightarrow$ ※. ◻

In general, if $x \in \operatorname{spt}\|V\|$ and $\Theta_V(x) < 1 + \alpha$ (from lemma), look at $C \in \operatorname{VarTan}_x(V)$. We have

$$\Theta_C(0) = \Theta_V(x) < 1+\alpha \overset{\text{lemma}}{\Rightarrow} C \text{ is flat with mult. } 1$$

Applying Allard, $x \in \operatorname{reg}(V)$. ◻

Lemma *:

Suppose $V_k \to V$ for V_k, V stationary integral n-varifolds in $B_1^{n+k}(0)$.
Then, $\forall\, K \subseteq B_1^{n+k}(0)$ compact,

$$d_H\left(\operatorname{spt}\|V_k\| \cap K,\ \operatorname{spt}\|V\| \cap K\right) \to 0$$

Hausdorff dist

In particular, $V_k \to \theta \cdot |\text{plane}|$ gives L^2 height excess $\to 0$.
From the height excess, we get L^2 tilt excess $\to 0$.

Proof: Unraveling definitions and forgetting subsequences since everything converges, we must show

① $x_k \in \operatorname{spt}\|V_k\| \cap K$ and $x_k \to x$ $\Rightarrow$ $x = \lim_{k\to\infty} x_k \in \operatorname{spt}\|V\| \cap K$. ($\in K$ by compactness)

Pf: Since $\Theta_{V_k}(x_k) \geq 1$, upper-semicontinuity of Θ $\Rightarrow$ $\Theta_V(x) \geq 1$
$\Rightarrow x \in \operatorname{spt}\|V\|$

② If $x \in \operatorname{spt}\|V\| \cap K$, then $\exists\, x_k \in \operatorname{spt}\|V_k\| \cap K$ with $x_k \to x$.

Pf: $\Theta_V(x) \geq 1 \overset{\text{monotonicity}}{\Rightarrow} \forall \rho > 0,\ \|V\|(B_\rho(x)) \geq \omega_n \rho^n > 0$
Varifold convergence $\Rightarrow \forall k$ large, $\|V_k\|(B_\rho(x)) \geq \frac{1}{2}\omega_n \rho^n > 0$
$\Rightarrow \operatorname{spt}\|V_k\| \cap B_\rho(x) \neq \emptyset \underset{\text{let } \rho = \frac{1}{k}}{\Rightarrow} x_k \in \operatorname{spt}\|V_k\|$ w/ $x_k \to x$.

Corollary:

$\operatorname{reg}(V) \subseteq \operatorname{spt}\|V\|$ is open and dense.

Proof: It's open by definition. Take $x \in \operatorname{spt}\|V\|$, fix $\rho > 0$.
Look at

$$\theta := \min\{ j \in \mathbb{N} : \Theta_V(y) = j \text{ for some } y \in B_\rho(x)\}$$

Then look at the varifold $\left(V \llcorner B_\rho(x),\ \frac{1}{\theta}\Theta_V\right)$ and apply the previous corollary.

$\square$

2/19 - Allard Proof for Lipschitz Minimal graphs

Proof: Let $u: B_1^n(0) \to \mathbb{R}$ be Lipschitz with $\mathrm{Lip}(u) < L$ and solving the functional minimal surface equation

$$\int_{B_1^n(0)} \frac{Du \cdot D\varphi}{\sqrt{1+|Du|^2}} = 0 \qquad \forall \varphi \in C_c^1(B_1^n)$$

Idea: We can characterize $C^{k,\alpha}$ regularity in terms of decay of integral quantities. Precisely,

"regularity $\Leftrightarrow$ decay estimates"

$$u \in C^{k,\alpha}(B_1^n(0) \to \mathbb{R}) \iff \sup_{\substack{x_0 \in B_1^n(0) \\ \rho \in (0,1)}} \inf_{\underset{\text{poly. of degree} \le k}{P \in P_k}} \underset{\text{natural decay rate for } C^{k,\alpha}}{\frac{1}{\rho^{n+2(k+\alpha)}}} \int_{B_\rho(x_0) \cap B_1^n(0)} |u-P|^2 < \infty$$

In the above, P is morally the k^{th} Taylor expansion of u.

⊛ If $k=1$ (as in our case), $C^{1,\alpha}$ reg. of a manifold has the geometric interpretation

$$\text{"}M \text{ is } C^{1,\alpha}\text{"} \overset{\text{by the above}}{\iff} \inf_{\text{planes}} \frac{1}{\rho^{n+2(1+\alpha)}} \int_M \mathrm{dist}^2(x, \text{Plane})\, d\mathcal{H}^n(x) < \infty$$

$$\iff \frac{1}{\rho^{n+2}} \int_M \mathrm{dist}^2(x, \text{plane}) \lesssim \rho^{2\alpha} \quad \text{for some plane}$$

⊛ In general, if u is "almost flat" then $|Du| \approx 0$, and so the MSE looks like Laplace equation $\Rightarrow$ u harmonic $\Rightarrow$ u smooth $\Rightarrow$ decay estimates

Step 1: Prove $W^{1,2}$ bound via a reverse Poincaré ineq.

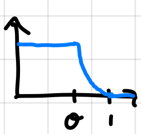
Take $\varphi^2 u$ in place of φ to get

$$\int \frac{|Du|^2 \varphi^2}{\sqrt{1+|Du|^2}} = -2 \int \frac{u \varphi\, Du \cdot D\varphi}{\sqrt{1+|Du|^2}}$$

Note that

$$\frac{1}{\sqrt{1+L^2}} \int |Du|^2 \varphi^2 \le 2 \int |u|\,|\varphi|\,|Du|\,|D\varphi| \overset{2ab \le a^2+b^2}{\le} \frac{1}{2\sqrt{1+L^2}} \int |Du|^2 \varphi^2 + C(L) \int |u|^2 |D\varphi|^2$$

$$\Rightarrow \int |Du|^2 \varphi^2 \le C(L) \int |u|^2 |D\varphi|^2$$

If $\varphi =$ 

$\Rightarrow |D\varphi| \le \frac{2}{1-\sigma}$, then $\underbrace{\int_{B_\sigma^n(0)} |Du|^2 \le \frac{C(L)}{(1-\sigma)^2} \int_{B_1^n(0)} |u|^2}_{\text{control } W^{1,2} \text{ norm by } L^2 \text{ norm on bigger ball}}$

$$\Rightarrow \|u\|_{W^{1,2}(B_\sigma(0))} \le C(L,\sigma) \|u\|_{L^2(B_1(0))}$$

Step 2: Linearize the equation via "blow-up"

Lipschitz weak solutions to MSE

Suppose BWOC we have $\varepsilon_k \downarrow 0$, and u_k as above with $\mathrm{Lip}(u_k) \le L$ and $\|u_k\|_{L^2(B_1(0))} \le \varepsilon_k$.

Set $$v_k := \frac{u_k}{\|u_k\|_{L^2(B_1(0))}} \Rightarrow \|v_k\|_{W^{1,2}(B_\sigma(0))} \le C(L,\sigma) \quad \forall \sigma \in (0,1) \quad \forall k$$

By Rellich compactness and a diagonal argument (to have $\sigma \uparrow 1$), then $\exists$ subsequence

$$v_{k'} \longrightarrow v \in W^{1,2}(B_1^n(0)) \quad \begin{array}{l} \text{strongly in } L^2_{loc}(B_1) \\ \text{weakly in } W^{1,2}_{loc}(B_1) \end{array}$$

Step 3: Does v satisfy any equation?

We know $\displaystyle\int_{B_1^n(0)} \frac{Dv_k \cdot D\varphi}{\sqrt{1+|Dv_k|^2}} = 0$. Since $\mathrm{Lip}(v_k) \le \varepsilon_k L \downarrow 0$, denominator doesn't matter.

We have

$$\int Dv_k \cdot D\varphi \overset{MSE}{=} \int Dv_k \cdot D\varphi \left(1 - \frac{1}{\sqrt{1+|Du_k|^2}}\right) = \int Dv_k \cdot D\varphi \cdot \frac{|Du_k|^2}{\sqrt{1+|Du_k|^2}\left(1+\sqrt{1+|Du_k|^2}\right)}$$

$$\le \sup|D\varphi| \int_{B_\sigma(0)} |Dv_k|\,|Du_k|^2 = \sup|D\varphi| \int_{B_\sigma(0)} |Du_k|^3 / \|u_k\|_{L^2(B_1)}$$

$\mathrm{spt}(\varphi) \subseteq B_\sigma(0)$ since $\varphi \in C_c$

$$\le \sup|D\varphi| \cdot L \cdot \frac{1}{\|u_k\|_{L^2(B_1)}} \|Du_k\|^2_{L^2(B_\sigma)}$$

$\|Du_k\| \le L$

$$\le \sup|D\varphi| \cdot L \cdot \frac{C(L)}{(1-\sigma)^2} \underbrace{\|u_k\|_{L^2(B_1)}}_{\le \varepsilon_k} \xrightarrow{k\to\infty} 0$$

Step 1

By $W^{1,2}_{loc}$ weak convergence, $\int Dv_k \cdot D\varphi \to \int Dv \cdot D\varphi$. Together, we get

$$\int Dv \cdot D\varphi = 0 \quad \forall \varphi \in C^1_c(B_1^n(0) \to \mathbb{R}) \Rightarrow v \text{ weakly harmonic} \xRightarrow{\text{Weyl lemma}} v \text{ smoothly harmonic}$$

Harmonic estimates give things like $\displaystyle\frac{1}{\rho^{n+2}}\int_{B_\rho(0)} |v-\ell|^2 \le C(n)\,\rho^2 \int_{B_1(0)} |v|^2$

linearization $\ell(x) = v(0) + x \cdot Dv(0)$, $\sup_{B_1}|\ell| \le C(n)$

L^2_{loc} convergence gives that $\forall k$ large (depends on ρ):

$$\frac{1}{\rho^{n+2}}\int_{B_\rho(0)} |v_k - \ell|^2 \le 2\,C(n)\,\rho^2 \underbrace{\int_{B_1(0)} |v_k|^2}_{=1}$$

same ℓ

$$\Rightarrow \frac{1}{\rho^{n+2}} \int_{B_\rho(0)} |u_k - \tilde{\ell}|^2 \le 2C(n)\rho^2 \int_{B_1(0)} |u_k|^2 \qquad \text{with } \tilde{\ell} = \ell \cdot \|u_k\|_{L^2(B_1)}$$

We have now proven an "excess decay" lemma.

Lemma: (Excess decay for Lipschitz minimal graphs)

Fix $L \in (0,\infty)$ and $\theta \in (0,1)$. Then, $\exists\, \varepsilon(n, L, \theta) \in (0,1)$ s.t. if

- $u : B_1^n(0) \to \mathbb{R}$
- $\mathrm{Lip}(u) \le L$
- u solution to MSE
- $\|u\|_{L^2(B_1)} < \varepsilon$

then, $\exists$ hyperplane ℓ s.t.

- $\displaystyle \frac{1}{\theta^{n+2}} \int_{B_\theta(0)} |u-\ell|^2 \le C(n)\,\theta^2 \int_{B_1(0)} |u|^2$ (kind of a Campanato estimate)
- $\displaystyle \sup_{B_1^n(0)} |\ell| \le C_*(n) \|u\|_{L^2(B_1(0))}$

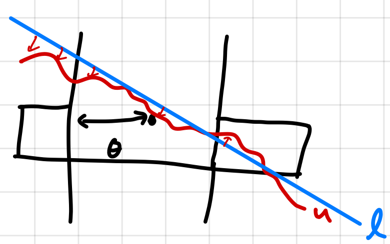
We still have the issue that the scale θ and ε are related.

Step 4: Iterate excess decay to all scales

Choose $\theta = \theta(n) \in (0, \frac{1}{4})$ s.t. $C_*(n)\,\theta^2 < \frac{1}{4}$. By excess decay lemma, $\exists\, \varepsilon(n,L) \in (0,1)$ s.t.

$$\|u\|_{L^2(B_1)} < \varepsilon \Rightarrow \frac{1}{\theta^{n+2}} \int_{B_\theta} |u-\ell|^2 \le \frac{1}{4} \int_{B_1} |u|^2$$

If we reparametrize $u|_{B_\theta}$ to view it as a function on the plane ℓ, it should still satisfy MSE. Since $\sup_{B_1} |\ell| \lesssim \|u\|_{L^2(B_1)} < \varepsilon$ we haven't tilted too much. So, we get $\tilde{u}$ defined on a subset of graph(ℓ) s.t.



- $\mathrm{Lip}(\tilde{u}) \le 2L$
- $\displaystyle \int_{B_1} |\tilde{u}|^2 \lesssim \frac{1}{\theta^{n+2}} \int_{B_\theta} |u-\ell|^2 \le \frac{1}{4} \int_{B_1} |u|^2 \le \frac{1}{4}\varepsilon$

Iterating this argument, we get: $\exists\, \varepsilon(n,L) \in (0,1)$ s.t. $\forall k \ge 1$, $\exists\, \ell_0, \ell_1, \dots, \ell_k$ s.t.

$\equiv 0$ the original plane

note that MSE for graphs is about other perturbations. When we reparametrize, need to check that we are still stationary. This doesn't matter in the general varifold case, where stationarity is defined ambiently.

(i) $\displaystyle \frac{1}{(\theta^k)^{n+2}} \int_{B_{\theta^{k+1}}} |u - \ell_{k+1}|^2 \le \frac{1}{4} \cdot \frac{1}{(\theta^k)^{(n+2)}} \int_{B_{\theta^k}} |u-\ell_k|^2$

(ii) $\displaystyle \sup_{B_1} |\ell_{k+1} - \ell_k|^2 \le C(n) \frac{1}{(\theta^k)^{n+2}} \int_{B_{\theta^k}} |u - \ell_k|^2$

We need to find a <u>single plane</u> for which this works.

By the triangle ineq., for $k_1 > k_2$,

$$\sup_{B_1} |\ell_{k_1} - \ell_{k_2}| \le C(n)\left(2^{-k_1} + \dots + 2^{-k_2}\right) \|u\|_{L^2(B_1)} \lesssim C(n)\, 2^{-k_2} \|u\|_{L^2(B_1)}$$

So, $(\ell_k)_k$ Cauchy $\Rightarrow$ $\ell_k \to \ell_*$ uniformly on B_1

$\underbrace{\text{with no subseq. nonsense!}}_{\Rightarrow \text{ unique tangent plane!}}$

Taking $k_1 \to \infty$, $\sup_{B_1} |\ell_k - \ell_*| \le C(n)\, 2^{-k} \|u\|_{L^2(B_1)} \quad \forall k \ge 1$

$$\overset{(:)}{\Longrightarrow} \frac{1}{(\theta^k)^{n+2}} \int_{B_{\theta^k}} |u - \ell_k|^2 \le \frac{1}{4^k} \int_{B_1} |u|^2$$

$$\overset{\text{tri. ineq.}}{\Longrightarrow} \frac{1}{(\theta^k)^{n+2}} \int_{B_{\theta^k}} |u - \ell_*|^2 \le \frac{1}{4^k} \int_{B_1} |u|^2 \qquad \forall k \in \mathbb{N}$$

Interpolating between scales, $\forall \rho \in (0,1)$, choose k s.t. $\theta^{k+1} \le \rho \le \theta^k$:

$\Rightarrow \theta^k \le \frac{\rho}{\theta}$

$$\frac{1}{\rho^{n+2}} \int_{B_\rho} |u - \ell_*|^2 \le \frac{1}{(\theta^{k+1})^{n+2}} \int_{B_{\theta^k}} |u - \ell_*|^2 \le \theta^{-n-2} \cdot \frac{1}{4^k} \int_{B_1} |u|^2$$

Since $\frac{1}{4^k} = \theta^{k \log_\theta(\frac{1}{4})} \le \left(\frac{\rho}{\theta}\right)^{\log_\theta(\frac{1}{4})} = \cancel{\theta^{-\log_\theta(\frac{1}{4})}}\,4\, \rho^{2\alpha}$ where $\alpha = \frac{1}{2} \log_\theta(\frac{1}{4}) \in (0,1)$,

$$\frac{1}{\rho^{n+2}} \int_{B_\rho} |u - \ell_*|^2 \le C(n)\, \rho^{2\alpha} \int_{B_1} |u|^2 \qquad \forall \rho \in (0,1)$$

This Campanato decay allows us to use the Campanato theory to get

$$\|u\|_{C^{1,\alpha}} \le C(n) \|u\|_{L^2(B_1)}$$

$\square$

2/21 - Proof of General Allard

Recall what we just did:

Step 1: Establish reverse Poincaré ineq. to get $W^{1,2}_{loc}$ control

Step 2: Use step 1 to "linearize" the problem via "blow-up" (L^2-rescaling)

This is where we needed that you're close to a plane with mult. 1! If mult. 2, there are 2 functions whose average is harmonic, but full regularity is tough to get after.

Step 3: Understand properties of the blown-up v (last time, v harmonic)

Step 4: Use blown-up v's regularity to get decay estimates for v and pass back to the nonlinear setting as "excess decay lemma"

Step 5: Iterate excess decay lemma to get Campanato estimate for nonlinear problem $\Rightarrow$ profit \$\$

We now prove the full Allard for varifolds, following these ideas. We will approximate by a nice graph and pass the error terms through.

Note: For Step 1, we have $\|\text{gradients}\|_{L^2} \lesssim \|\text{value}\|_{L^2}$, which in the geometric setting can be considered $\|\text{tilt}\|_{L^2} \lesssim \|\text{height}\|_{L^2}$. To get at this, we will again use tilt excess!

Tilt excess is
$$E_V^2 := \int_{\mathbb{R}^k \times B_1^n(0)} \|P_{T_xV} - P_{\mathbb{R}^n}\|^2 \, d\|V\|(x)$$

where $P_S : \mathbb{R}^{n+k} \to S$ is orthogonal proj. onto subspace S, $\mathbb{R}^n \cong \{0\}^k \times \mathbb{R}^n$, and $\|A\|^2 = \sum_{ij} |A_{ij}|^2$ is Frobenius norm. The height excess will then be denoted $\hat{E}_V^2$.

Step 1: - Reverse Poincaré

Lemma: (Reverse Poincaré for stationary varifolds)

Suppose V is a stationary integral n-varifold in $B_2^{n+k}(0)$. Then,
$$\int \|P_{T_xV} - P_{\mathbb{R}^n}\|^2 \varphi^2 \, d\|V\|(x) \leq 32 \int \operatorname{dist}^2(x, \mathbb{R}^n) \, |D\varphi|^2 \, d\|V\|(x)$$
for all test functions $\varphi \in C_c^1(B_2^{n+k}(0))$.

Proof: Take variation $Y_x := \varphi^2(x)\,(x^1,\dots,x^k,0,\dots)$ to be the analog to the upward test for $\varphi^2 u$ we used earlier. Then,

$$\begin{aligned}\mathrm{div}_{T_xV}(Y_x) &= \sum_{i=1}^{n+k} \nabla_i^{T_xV}(e_i\cdot Y_x) = \sum_{i=1}^{k} \nabla_i^{T_xV}(x^i\varphi^2)\\ &= \sum_{i=1}^{k} e_i\cdot P_{T_xV}\left[\nabla^{\mathbb{R}^{n+k}}(x^i\varphi^2)\right] = \sum_{i=1}^{k} e_i\cdot P_{T_xV}\left(2\varphi x^i D\varphi + \varphi^2 e_i\right)\\ &= \varphi^2\sum_{i=1}^{k}(P_{T_xV})_{ii} + \sum_{i=1}^{k}\sum_{j=1}^{n+k} 2(P_{T_xV})_{ij}\,\varphi\, x^i D_j\varphi\end{aligned}$$

Note that

equal by symmetry of P

$$\begin{aligned}\|P_{T_xV}-P_{\mathbb{R}^n}\|^2 &= \sum_{i,j=1}^{n+k}\left[(P_{T_xV})_{ij}-(P_{\mathbb{R}^n})_{ij}\right]^2 = \underbrace{\sum_{i,j=1}^{n+k}(P_{T_xV})_{ij}(P_{T_xV})_{ji}}_{=\mathrm{tr}(P_{T_xV}^2)\,=\,\mathrm{tr}(P_{T_xV})\,=\,n\ \text{since proj. op}} -2(P_{T_xV})_{ij}(P_{\mathbb{R}^n})_{ij} + (P_{\mathbb{R}^n})_{ij}^2\\ &= 2n - 2\sum_{i,j=1}^{n+k}(P_{T_xV})_{ij}(P_{\mathbb{R}^n})_{ij}\\ &= 2n - 2\sum_{i=k+1}^{n+k}(P_{T_xV})_{ii} = 2\sum_{i=1}^{k}(P_{T_xV})_{ii}\end{aligned}$$

same thing here

So,

$$\mathrm{div}_{T_xV}(Y_x) = \tfrac{1}{2}\varphi^2\|P_{T_xV}-P_{\mathbb{R}^n}\|^2 + 2\varphi\sum_{i=1}^{k}\sum_{j=1}^{n+k}\underbrace{(P_{T_xV})_{ij}}_{=(P_{T_xV})_{ij}-(P_{\mathbb{R}^n})_{ij}\ \text{for } i\le k \text{ since these coords of } P_{\mathbb{R}^n} \text{ are } 0}\, x^i D_j\varphi$$

Stationarity gives $\int \mathrm{div}_{T_xV}(Y_x)\, d\|V\| = 0$

$$\begin{aligned}\Rightarrow \int \varphi^2\|P_{T_xV}-P_{\mathbb{R}^n}\|^2\, d\|V\| &= -4\int\sum_{i=1}^{k}\sum_{j=1}^{n+k}\varphi x^i\left((P_{T_xV})_{ij}-(P_{\mathbb{R}^n})_{ij}\right)D_j\varphi\\ &\le 4\int|\varphi|\,\|P_{T_xV}-P_{\mathbb{R}^n}\|\,|(x^1,\dots,x^k,0,\dots,0)|\,|D\varphi|\\ {\scriptstyle 4ab\le \frac{a^2}{4}+4b^2}\quad &\le \frac{1}{4}\int\varphi^2\|P_{T_xV}-P_{\mathbb{R}^n}\|^2 + 4|D\varphi|^2|(x^1,\dots,x^k,0,\dots,0)|^2\end{aligned}$$

$$\Rightarrow \int\varphi^2\|P_{T_xV}-P_{\mathbb{R}^n}\|^2 \le 32\int\underbrace{|(x^1,\dots,x^k,0,\dots,0)|^2}_{=\mathrm{dist}^2(x,\mathbb{R}^n)}|D\varphi|^2$$

$\square$

$\boxed{\text{Step 2:}}$ - Blow-up & Lipschitz Approx.

Lemma: (Lipschitz approx)

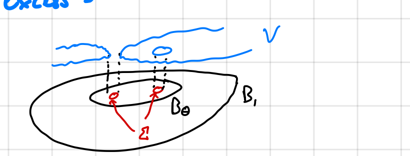
Fix $\delta,\theta\in(0,1)$. Then $\exists\, \varepsilon(n,k,\delta,\theta)>0$ s.t.

If V is a stationary integral n-varifold V in $B_2^{n+k}(0)$ obeying assumptions of Allard, then $\exists$ Lipschitz function $u: B_\theta^n(0)\to\mathbb{R}^k$, $\mathrm{Lip}(u)\le\frac{1}{2}$, and measurable $\Sigma\subseteq B_\theta^n(0)$ ← bad set s.t.

(i) $\sup_{B_\theta(0)}|u| \le C(n,k)\,\hat{E}_V^{\frac{1}{n+1}}$ (height of u ≲ height excess)

(ii) $V \llcorner (\mathbb{R}^k\times(B_\theta(0)\setminus\Sigma)) = \mathrm{graph}\left(u|_{B_\theta(0)\setminus\Sigma}\right)$

(iii) $\mathcal{H}^n(\Sigma) + \|V\|(\mathbb{R}^k\times\Sigma) \le C(n,k)\,\hat{E}_V^2$



Remark: Σ is explicit! It is the (projection of) the points in $\operatorname{spt}\|V\|$ where the tilt excess (and so the height excess by step 1) does not decay at <u>all</u> scales.

In the end, once we have shown excess decay at every point, we can come back and say $\Sigma = \emptyset \Rightarrow V$ is <u>entirely</u> a Lipschitz graph!

We will prove this later but use it now. Now, we construct our blow-ups. Consider a sequence $(V_k)_k$ of stationary int. n-varifolds in $B_2^{n+k}(0)$ s.t.

$$\bullet\ 0 \in \operatorname{spt}\|V_k\| \qquad \bullet\ \omega_n^{-1}\|V_k\|(\mathbb{R}^k \times B_1^n(0)) \le 2-\delta \qquad \bullet\ \hat{E}_{V_k}^2 < \varepsilon_k \downarrow 0$$

For any $\theta \in (0,1)$, $\forall k$ sufficiently large we can apply Lipschitz approx to V_k on $\mathbb{R}^k \times B_\theta^n(0)$ to get Lipschitz $u_k : B_\theta^n(0) \to \mathbb{R}^k$, $\operatorname{Lip}(u_k) \le \frac{1}{2}$, s.t.

$$\bullet\ \sup |u_k| \le C\, \hat{E}_{V_k}^{\frac{1}{n+1}} \qquad \bullet\ V_k \llcorner (\mathbb{R}^k \times (B_\theta \setminus \Sigma_k)) = \operatorname{graph}(u_k|_{B_\theta \setminus \Sigma_k})$$

$$\bullet\ \mathcal{H}^n(\Sigma_k) + \|V_k\|(\mathbb{R}^k \times \Sigma_k) \le C\, \hat{E}_{V_k}^2$$

So, $\forall k$ large,

$$\int_{B_\theta} |u_k|^2 = \underbrace{\int_{B_\theta \setminus \Sigma_k} |u_k|^2}_{\substack{= \int_{\mathbb{R}^k \times (B_\theta \setminus \Sigma_k)} \operatorname{dist}^2(x, \mathbb{R}^n) \cdot J^{-1}\, d\mathcal{H}^n \\ \text{area formula}}} + \underbrace{\int_{\Sigma_k} |u_k|^2}_{\le \sup_{B_\theta} |u_k|^2 \cdot \mathcal{H}^n(\Sigma_k) \le C\, \hat{E}_{V_k}^{2+\frac{2}{n+1}}} \le C\, \hat{E}_{V_k}^{2+\frac{2}{n+1}}$$

Jacobian $\frac{1}{\sqrt{1+|Du_k|^2}} \le 1$

Similarly,

$$\int_{B_\theta} |Du_k|^2 = \int_{B_\theta \setminus \Sigma_k} |Du_k|^2 + \int_{\Sigma_k} |Du_k|^2 \le \frac{1}{4}\mathcal{H}^n(\Sigma_k) \le C\, \hat{E}_{V_k}^2$$

On $\mathbb{R}^k \times (B_\theta \setminus \Sigma_k)$, viewing u as a function on ambient space independent of first k coords, $u_k^i \equiv x^i$ on $\operatorname{spt}\|V_k\| \cap (\mathbb{R}^k \times (B_\theta \setminus \Sigma))$

Note that

$$|\nabla^V u^i|^2 = \nabla^V u^i \cdot \nabla^V u^i = P_{T_xV}(\nabla^{\mathbb{R}^{n+k}} u^i) \cdot \nabla^{\mathbb{R}^{n+k}} u^i = \underbrace{|\nabla^{\mathbb{R}^{n+k}} u^i|^2}_{= |Du^i|^2} - \underbrace{P_{T_x^\perp V}}_{\substack{= P_{\mathbb{R}^n} P_{T_x^\perp V} P_{\mathbb{R}^n} \text{ for } u^i \\ = (P_{\mathbb{R}^n} - P_{T_xV})\, P_{T_x^\perp V}\, (P_{\mathbb{R}^n} - P_{T_xV})}}(\nabla^{\mathbb{R}^{n+k}} u^i) \cdot \nabla^{\mathbb{R}^{n+k}} u^i$$

$$\Rightarrow \left| |\nabla^V u^i|^2 - |Du^i|^2 \right| \le \|P_{\mathbb{R}^n} - P_{T_xV}\|^2 \underbrace{\|P_{T_xV}\|\, |Du^i|^2}_{\le \frac{1}{4}}$$

So, we may replace regular derivatives by ∇^{V_k}'s for absorbable error.

$$\int_{B_\theta \setminus \Sigma_k} |Du_k|^2 \le \int_{\mathbb{R}^k \times (B_\theta \setminus \Sigma_k)} \underbrace{|\nabla^{V_k}(x^1, \dots, x^k, 0, \dots, 0)|^2}_{= \sum (P_{T_xV})_{ii} = \frac{1}{2}\|P_{T_xV} - P_{\mathbb{R}^n}\|^2} + C \int_{\mathbb{R}^k \times B_\theta} \|P_{T_xV} - P_{\mathbb{R}^n}\|^2$$

$$\le C \int_{\mathbb{R}^k \times B_\theta} \|P_{T_xV} - P_{\mathbb{R}^n}\|^2 \underset{\text{rev. Poincaré}}{\le} C \int_{\mathbb{R}^k \times B_\theta} \operatorname{dist}^2(x, \mathbb{R}^n)\, d\|V_k\|$$

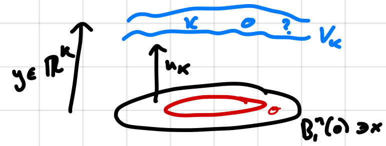
All in all, $\|u_k\|_{W^{1,2}(B_\theta)} \le C\, \hat{E}_{V_k}$. We will blow-up with this.

By reasoning like before, $v_k := \dfrac{u_k}{\hat{E}_{V_k}} \to v$ strongly in $L^2_{loc}(B_1)$, weakly in $W^{1,2}_{loc}(B_1)$

2/26 - Allard Continued

$\boxed{\text{Step 3:}}$ - Understand blow-up's properties (harmonic)

We will construct a variation s.t. stationarity yields a similar computation as before.



Take $\zeta \in C_c^1(B_1^n(0))$ and extend A to $\tilde{\zeta} \in C(\mathbb{R}^k \times B_1^n(0))$ via $\tilde{\zeta}(y,x) := \zeta(x)$. Let $\sigma > 0$ be s.t. $\operatorname{spt}(\zeta) \subseteq B_\sigma^n$.
Modify $\tilde{\zeta}$ to have compact support in $\mathbb{R}^k \times B_1^n$ via a vertical cutoff outside $\operatorname{spt}\|V_k\|$. Take $Z_x := \tilde{\zeta}(x) e^i$ ← $i \in [k]$ basis vectors

$$\Longrightarrow \underset{\substack{\uparrow \\ \text{drop for notation}}}{\operatorname{div}_{T_x V_k}} (Z_x) = \sum_{j=1}^{n+k} \nabla_j^{T_x V}(e_j \cdot Z_x) = \nabla_i^{T_x V} \tilde{\zeta} = e^i \cdot \nabla^{T_x V} \tilde{\zeta} = \nabla^{\mathbb{R}^{n+k}} x^i \cdot \nabla^{T_x V} \tilde{\zeta}$$

$$= \nabla^{T_x V} x^i \cdot \nabla^{T_x V} \tilde{\zeta}.$$

Stationarity of V_k gives

$$\int_{\mathbb{R}^k \times B_\sigma} \nabla^{T_x V} x^i \cdot \nabla^{T_x V} \tilde{\zeta} \, d\|V\| = 0 \Rightarrow \int_{\mathbb{R}^k \times (B_\sigma \setminus \mathcal{Z})} \underbrace{\nabla^{T_x V} x^i}_{= \nabla^{T_x V} u_i} \cdot \nabla^{T_x V} \tilde{\zeta} = - \underbrace{\int_{\mathbb{R}^k \times (B_\sigma \cap \mathcal{Z})} \nabla^{T_x V} x^i \cdot \nabla^{T_x V} \tilde{\zeta}}_{\substack{\leq \sup |D\zeta| \cdot C \|V\|(\mathbb{R}^k \times \mathcal{Z}) \\ \leq C \sup |D\zeta| \hat{E}_V^2}}$$

By the same computation as last time,

$$|\nabla^{T_x V} u^i \cdot \nabla^{T_x V} \tilde{\zeta} - Du^i \cdot D\zeta| \leq C \underbrace{\|P_{T_x V} - P_{\mathbb{R}^n}\|^2}_{\substack{\text{can control tilt excess} \\ \text{by height excess!}}}$$

So,

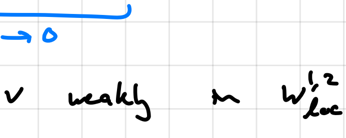
$$\int_{B_\sigma} Du_k \cdot D\zeta = O(1) \sup_{B_\sigma} |D\zeta| \, \hat{E}_{V_k}^2 \Rightarrow \int_{B_1} Dv_k \cdot D\zeta = O(1) \underbrace{\sup |D\zeta| \, \hat{E}_{V_k}}_{\to 0}$$

$$\Rightarrow \int Dv_k \cdot D\zeta \underset{k \to \infty}{\to} 0 \Rightarrow \int Dv \cdot D\zeta = 0 \quad \text{since } v_k \to v \text{ weakly in } W^{1,2}_{loc}(B_1).$$

Since $\int Dv \cdot D\zeta = 0$ $\forall \zeta \in C_c^1(B_1^n)$, we see that v is weakly harmonic, and so v is harmonic! By harmonic estimates again, $\forall \theta \in (0,1)$,

$$\frac{1}{\theta^{n+2}} \int_{B_\theta} |v - \underset{\substack{\uparrow \\ \ell = v(0) + x \cdot Dv(0)}}{\ell}|^2 \leq C(n,k) \theta^2 \int_{B_1} |v|^2 \overset{\substack{\text{fixed } \theta, \\ k \text{ large}}}{\Longrightarrow} \frac{1}{\theta^{n+2}} \int_{B_\theta} |u_k - \hat{E}_{V_k} \ell|^2 dx \leq C(n,k) \theta^2 \hat{E}_{V_k}^2.$$

Throwing away the "bad region" of our Lipschitz approx.,



$$\frac{1}{\theta^{n+2}} \int_{B_\theta \setminus \mathcal{Z}_k} |u_k - \hat{E}_{V_k} \ell|^2 dx \leq C \theta^2 E_{V_k}^2$$

So, letting $P_k = \operatorname{graph}(\hat{E}_{V_k} \ell) \subseteq \mathbb{R}^{n+k}$ be a plane, then $\operatorname{dist}(X, P_k) \leq |u_k(x) - \hat{E}_{V_k} \ell|$ by the picture.
$X = (u_k^1(x), \dots, u_k^k(x), x)$

Thus, (handling a Jacobian factor $1 \le J_s \le 1 + C|Du_k|^2$),

$$\frac{1}{\theta^{n+2}} \int_{\mathbb{R}^k \times (B_\theta \setminus \Sigma_k)} \operatorname{dist}^2(x, P_k)\, d\|V_k\| \le C\theta^2 \hat{E}_{V_k}^2 .$$

note u_k corresponds with spt$\|V_k\|$ since we ignore bad set Σ_k

We handle the bad set via

$$\frac{1}{\theta^{n+2}} \int_{\mathbb{R}^k \times (B_\theta \cap \Sigma_k)} \operatorname{dist}^2(x, P_k)\, d\|V_k\| \le \theta^{-n-2} \underbrace{\sup_{\operatorname{spt}\|V_k\| \cap (\mathbb{R}^k \times (B_\theta \cap \Sigma_k))} \operatorname{dist}^2(x, P_k)}_{\text{can be made arbitrarily small via Hausdorff distance convergence of } V_k \text{ to a plane. Say } \le \gamma} \underbrace{\|V_k\|(\mathbb{R}^k \times (B_\theta \cap \Sigma_k))}_{\le C\hat{E}_{V_k}^2}$$

Adding this back in,

$$\frac{1}{\theta^{n+2}} \int_{\mathbb{R}^k \times B_\theta} \operatorname{dist}^2(x, P_k)\, d\|V_k\| \le C\theta^2 \hat{E}_{V_k}^2 + C\theta^{-n-2}\gamma \hat{E}_{V_k}^2$$

Choose $\theta = \theta(n,k)$ s.t. $C\theta^2 < \frac{1}{8}$ and choose γ s.t. $C\theta^{-n-2}\gamma < \frac{1}{8}$, and so

$$\frac{1}{\theta^{n+2}} \int_{\mathbb{R}^k \times B_\theta} \operatorname{dist}^2(x, P_k)\, d\|V_k\| \le \frac{1}{4} \hat{E}_{V_k}^2 = \frac{1}{4} \int_{\mathbb{R}^k \times B_1} \operatorname{dist}^2(x, \mathbb{R}^n)\, d\|V_k\|$$

and $\operatorname{dist}_H(P_k \cap B_1, \mathbb{R}^n \cap B_1) \le C\hat{E}_{V_k}$.

We have now shown:

<u>Lemma (Allard Excess Decay):</u>

Fix $\delta \in (0,1)$ and $\theta \in (0,1)$. Then, $\exists\, \varepsilon(n,k,\theta,\delta)$ s.t.

If V is a stationary integral n-varifold in $B_2^{n+k}(0)$ and

- $0 \in \operatorname{spt}\|V\|$
- $\frac{1}{\omega_n}\|V\|(\mathbb{R}^k \times B_1^n) \le 2 - \delta$
- $\hat{E}_V < \varepsilon$

<u>Then</u>, $\exists$ affine n-plane $P \subseteq \mathbb{R}^{n+k}$ s.t.

(i) $$\frac{1}{\theta^{n+2}} \int_{\pi_P^{-1}(B_\theta)} \operatorname{dist}^2(x, P)\, d\|V\| \le C\theta^2 \int_{\pi_{\mathbb{R}^n}^{-1}(B_1)} \operatorname{dist}^2(x, \mathbb{R}^n)\, d\|V\|$$

(ii) $$\operatorname{dist}_H(P \cap B_1, \mathbb{R}^n \cap B_1) \le C\hat{E}_V$$

We would like P to be a subspace (i.e. go through 0) to make iteration easier.

<u>General principle</u>: "good density points" are inherited by the blow-up.

if $x \in$ plane of mult. q, then x good density if $\Theta_V(x) \ge q$. i.e. if all the density gets sent to the plane.

This is proved using the Hardt-Simon Inequality.

Lemma (Hardt-Simon): "blowups preserve Q-points" (basically a blow-up of monotonicity formula)

> If v is the blowup we constructed, since 0 is a good density point ($\Theta_v(0) \geq 1$ because integral varifold)
> then
> $$\int_{B_{\frac{1}{2}}(0)} \underset{R=|x|}{R^{2-n}} \left|\frac{\partial}{\partial R}\left(v/R\right)\right|^2 \leq C(n,k) < \infty$$

Proof: By the monotonicity formula, for all V_k (dropping subscript k),

$$\frac{1}{\omega_n}\int_{B^{n+k}_{3/4}(0)} \frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\|(x) \leq \frac{\|V\|(B^{n+k}_{3/4}(0))}{\omega_n(3/4)^n} - \underbrace{\Theta_V(0)}_{\geq 1 \text{ since int. var.}} \leq \frac{\|V\|(B^{n+k}_{3/4}(0)) - \omega_n(3/4)^n}{\omega_n(3/4)^n}$$

But,

$$\|V\|(B^{n+k}_{3/4}(0)) \leq \|V\|(\mathbb{R}^k \times B^n_{3/4}(0))$$
$$= \|V\|(\mathbb{R}^k \times (B^n_{3/4} \setminus \Sigma)) + \underbrace{\|V\|(\mathbb{R}^k \times (B^n_{3/4}(0) \cap \Sigma))}_{\leq C\hat{E}_V^2 \text{ by Lip. approx}}$$
$$\leq \int_{B^n_{3/4}(0)\setminus\Sigma} \overbrace{1 + C\underbrace{|Du|^2}_{\text{bounded by } \hat{E}_V^2 \text{ via reverse Poincaré}}}^{\text{Jacobian}} + C\hat{E}_V^2 \leq \omega_n\left(\tfrac{3}{4}\right)^n + C\hat{E}_V^2$$

So,

$$\int_{B^{n+k}_{3/4}(0)} \frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\| \leq C\hat{E}_V^2 \implies \int_{\mathbb{R}^k \times (B^n_{\frac{1}{2}}(0)\setminus\Sigma)} \frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\| \leq C\hat{E}_V^2$$

Since we are graphical over this set, consider the map

$$\Phi : \mathbb{R}^n \to \mathbb{R}^{k+n} \qquad x \mapsto (u^1(x), \dots, u^k(x), x) =: X$$

Clearly, $|X|^{n+2} = (|u|^2 + |x|^2)^{\frac{n+2}{2}}$ $\forall x \in \mathbb{R}^n$. Also, $\Phi(x) \in \operatorname{graph}(u)$ $\forall x \in \mathbb{R}^n$

$$\Rightarrow \frac{\partial}{\partial R}\Phi(x) \in T_x \operatorname{graph}(u) \Rightarrow \left[\frac{\partial}{\partial \underset{R=|x|,\, x\in\mathbb{R}^n \text{ part}}{R}}\left(\Phi/R\right)\right]^\perp = \left[\underbrace{\frac{\partial \Phi}{\partial R}}_{=0 \text{ since } \perp} \cdot \frac{1}{R} - \frac{\Phi}{R^2}\right]^\perp = -\frac{\Phi^\perp}{R^2} = -\frac{X^\perp}{R^2}$$

So,

$$X^\perp = -R^2\left[\frac{\partial}{\partial R}\left(\frac{(u(x),x)}{R}\right)\right]^\perp = -R^2\left[\frac{\partial}{\partial R}\left(\frac{(u(x),0)}{R}\right)\right]^\perp$$

$$\underset{\text{projection matrices, Lipschitz constant, ...}}{\Rightarrow \dots} \Rightarrow |X^\perp| \geq \frac{1}{2}R^2\left|\frac{\partial}{\partial R}\left(u/R\right)\right|$$

Thus,
$$\int_{B^n_{\frac{1}{2}}(0)\setminus\Sigma} \frac{R^4\left|\frac{\partial}{\partial R}\left(\frac{u}{R}\right)\right|^2}{(|u|^2+R^2)^{\frac{n+2}{2}}}\, dx \leq C\hat{E}_V^2.$$
Reintroducing the subscripts, $\forall k$:

$$\int_{B^n_{\frac{1}{2}}(0)\setminus\Sigma_k} \frac{R^4}{(|u_k|^2+R^2)^{\frac{n+2}{2}}}\left|\frac{\partial}{\partial R}\left(\frac{u_k}{R}\right)\right|^2 \leq C \xrightarrow[k\to\infty]{u_k \to 0 \text{ in } L^2 \text{ and so a.e.}} \int_{B^n_{\frac{1}{2}}(0)} \frac{R^4}{R^{n+2}}\left|\frac{\partial}{\partial R}\left(v/R\right)\right|^2 \leq C \qquad \square$$

Remarks:

① If multiplicity is Q and $\Theta_v(0) \geq Q$, same argument works with Q different u's and always handling sums of them.

② The same thing applies shifted by z.

2/28 -

Let's recall where we are. We've got a sequence $(V_k)_k$ of stationary integral varifolds with

$$V_k \to \text{plane with mult. } Q \ (Q=1),$$
$$\text{i.e. } \{0\}^k \times \mathbb{R}^n$$

We used Lip. approx to get $u_k : B^n_{\sigma_k}(0) \to \mathbb{R}^k$. Doing a blow-up,

$$v_k := \frac{u_k}{\hat{E}_{V_k}} \quad \text{has} \quad \underset{\text{the blowup}}{v_k \to v} \quad \begin{array}{l} \text{strongly in } L^2_{loc} \\ \text{weakly in } W^{1,2}_{loc} \end{array}$$

We showed v is harmonic, yielding decay estimates

$$\frac{1}{\theta^{n+2}} \int_{B_\theta} |v - \underset{\ell(x) = v(0) + x \cdot Dv(0)}{\ell}|^2 \leq C\theta^2 \int |v|^2 \Rightarrow \ldots \Rightarrow \text{excess decay of the varifolds } V_k$$

This basically completes the proof of Allard.

However, this is a good place to demonstrate a common theme: points of good density are preserved by blowups! We will see that in our blowup, if $0 \in \operatorname{spt}\|V_k\|$, and $\Theta_{V_k}(0) \geq Q$, then $v(0) = 0$ is anchored in the limit.

Prop:

$$v(0) = 0 \quad \text{for our blowup } v.$$

Proof: By Hardt-Simon, if $v(0) \neq 0$ then

$$\int_{B_{\frac{1}{2}}(0)} R^{2-n} \left(\frac{\partial v}{\partial R} \cdot \frac{1}{R} - \frac{v}{R^2} \right)^2 = \int_{B_{\frac{1}{2}}(0)} R^{2-n} \left(\frac{v^2}{R^4} - \frac{2}{R^3} v \frac{\partial v}{\partial R} + \frac{1}{R^2} \left(\frac{\partial v}{\partial R}\right)^2 \right) < \infty$$

$$\Rightarrow \int_{B_{\frac{1}{2}}(0)} R^{-2-n} \, dx < \infty \overset{\text{polar coords}}{\Longrightarrow} \int_0^{\frac{1}{2}} R^{-3} \, dR < \infty. \quad \times$$

$\square$

Remark: If $v(x) = CR^\alpha$ for some α, we see that Hardt-Simon implies $\alpha \geq 1$! So, blowups must decay sublinearly.

So, we know each plane ℓ is a subspace, and so we are doing rotations! Let's see how to rewrite the last bit of Allard using this.

Now, Allard excess decay reads:

$\exists$ rotation Γ s.t. $\displaystyle \frac{1}{\theta^{n+2}} \int_{\mathbb{R}^k \times B_\theta} \mathrm{dist}^2(x, \mathbb{R}^n)\, d\|\Gamma_\# V\| \le \frac{1}{4} \int_{\mathbb{R}^k \times B_1} \mathrm{dist}^2(x, \mathbb{R}^n)\, d\|V\|$

$\|\Gamma - \mathrm{Id}\| \le C \hat{E}_V$

$$\left(\Rightarrow \quad \hat{E}_{\Gamma_\# V / \theta} \le \frac{1}{2} \hat{E}_V \right)$$

Iterating in the same way, we get a limiting rotation Γ^* s.t.

$$\frac{1}{s^{n+2}} \int_{\mathbb{R}^k \times B_s} \mathrm{dist}^2(x, \underbrace{\Gamma^*(\mathbb{R}^n)}_{\text{unique tangent space at } 0})\, d\|V\| \le s^{2\alpha} \hat{E}_V^2 \qquad \left(\forall s \in (0, \tfrac{1}{2}) \right)$$

α came from same trick as in Lipschitz case

$$\left(\text{This is akin to the Campanato estimate} \quad \frac{1}{s^{n+2}} \int_{B_s} |u - \ell|^2 \le C s^{2\alpha} \Rightarrow u \in C^{1,\alpha} \right)$$

3/4 - Proving Lipschitz Approx Lemma

To fully wrap up Allard, we go back and prove Lip. approx lemma.

Lemma (Lipschitz Approx):

Fix $\delta, \theta \in (0,1)$. Then, $\exists\, \varepsilon(n,k,\theta,\delta)$ s.t.

If V is a stationary integral varifold in $B_2^{n+k}(0)$ with

- $0 \in \operatorname{spt}\|V\|$
- $\omega_n^{-1}\|V\|(\mathbb{R}^k \times B_1^n(0)) < 2-\delta$
- $\hat{E}_V < \varepsilon$

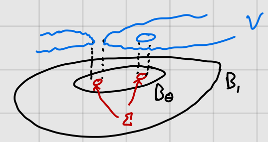
then $\exists$ Lipschitz $u: B_\theta^n(0) \to \mathbb{R}^k$ and measurable $\Sigma \subseteq B_\theta^n$ s.t.

↑ bad set

(i) $\operatorname{Lip}(u) \le \frac{1}{2}$, $\quad \sup_{B_\theta^n} |u| \le C \hat{E}_V^{\frac{1}{n+2}}$

(ii) $V \llcorner (\mathbb{R}^k \times (B_\theta \setminus \Sigma)) = \operatorname{graph}(u|_{B_\theta \setminus \Sigma})$

(iii) $\mathcal{H}^n(\Sigma) + \|V\|(\mathbb{R}^k \times \Sigma) \le C \hat{E}_V^2$



We will use a simple lemma:

Lemma:

Fix $\zeta \in (0, \frac{1}{2})$. Then $\exists\, \varepsilon(n,k,\delta,\theta,\zeta) > 0$ s.t. if V obeys the Lipschitz approximation lemma except instead of $\hat{E}_V < \varepsilon$ we require $E_V < \varepsilon$,
then:

L^2 tilt excess

(i) $\operatorname{spt}\|V\| \cap B_{\frac{1}{2}}^{n+k} \subseteq \zeta$-neighborhood of $\mathbb{R}^n$

(ii) $\dfrac{\|V\|(B_\rho(x))}{\omega_n \rho^n} \le 1 + \zeta \qquad \forall x \in B_{\frac{1}{4}}^{n+k}, \quad \forall \rho \in (0, \frac{1}{4})$

Proof: Suppose BWOC that $\exists (V_k)_k$ s.t. $0 \in \operatorname{spt}\|V_k\|$, $\omega_n^{-1}\|V_k\|(\mathbb{R}^k \times B_1(0)) < 2-\delta$, and $E_{V_k} \to 0$ but the results don't hold for V_k.

By compactness, we can take a convergent subsequence $V_k \rightharpoonup V$, and so $E_{V_k} \to 0 \Rightarrow E_V = 0 \Rightarrow V = Q$ planes parallel to $\mathbb{R}^n$. The mass upper bound means this plane has mult. 1, and since $0 \in \operatorname{spt}\|V_k\|$ (and so $0 \in \operatorname{spt}\|V\|$), we know $V = |\mathbb{R}^n|$. Thus, (i) must hold for V_k for k large enough.

If (ii) fails, $\exists x_k \in B_{\frac{1}{4}}$, $\rho_k \in (0, \frac{1}{4})$ s.t.

$$\frac{\|V_k\|(B_{\rho_k}(x_k))}{\omega_n \rho_k^n} \ge 1 + \zeta \quad \overset{\text{monotonicity}}{\Longrightarrow} \quad \frac{\|V_k\|(B_{\frac{1}{4}}(x_k))}{\omega_n (\frac{1}{4})^n} \ge 1 + \zeta.$$

Take $x_k \to x \in \overline{B_{\frac{1}{4}}}(0)$ and fix $r > \frac{1}{2}$. Then $\forall k$ large, $B_{\frac{1}{2}}(x_k) \subseteq B_r(x)$.
By varifold convergence,

$$\underbrace{\frac{\|V\|(B_r(x))}{\omega_n r^n}}_{= r^n/(\frac{1}{2})^n \text{ since } V = |\mathbb{R}^n|} = \lim_{k\to\infty} \frac{\|V_k\|(B_r(x))}{\omega_n r^n} \geq \lim_{k\to\infty} \frac{\|V_k\|(B_{\frac{1}{2}}(x_k))}{\omega_n r^n} \geq \frac{(\frac{1}{2})^n}{r^n}(1+\mathfrak{z})$$

Since V is the $\mathbb{R}^n$ plane, taking $r \downarrow \frac{1}{2}$ leads to a contradiction. $\square$

<u>Proof of Lip. Approx.</u>: As we have seen before (upper semicont. of density), if ε small then $\Theta_V \equiv 1$ a.e. in $\mathbb{R}^k \times B^n_\Theta$.

Fix $\lambda > 0$ to be chosen later, and set

if V were a $C^{1,\alpha}$ graph, this would just say that derivative $\leq \lambda$.

$$G := \left\{ y \in \operatorname{spt}\|V\| \cap B^{n+k}_{1/8}(0) : \frac{1}{\rho^n}\int_{B^{n+k}_\rho(y)} \|P_{T_xV} - P_{\mathbb{R}^n}\|^2 \, d\|V\|(x) \leq \lambda \quad \forall \rho \in (0,\tfrac{1}{2}) \right\}$$

note that when applying to Allard, we know by excess decay that $G = \operatorname{spt}\|V\| \cap B_{\frac{1}{8}}$

Pick $x \in G$ and $y \in \operatorname{spt}\|V\| \cap B_{\frac{1}{4}}(x)$ and $|x-y| < r < \min\{\frac{3}{2}|x-y|, \frac{1}{4}\}$.
By construction of G the tilt is small, i.e.

$$\frac{1}{(2r)^n}\int_{B_{2r}(x)} \|P_{T_zV} - P_{\mathbb{R}^n}\|^2 \, d\|V\|(z) \leq \lambda$$

Applying the above lemma to the shifted and scaled $(\eta_{x,2r})_\# V$, we get

$$\operatorname{spt}\|(\eta_{x,2r})_\# V\| \cap B_{\frac{1}{2}}(0) \subseteq \mathfrak{z}\text{-neighborhood of } \mathbb{R}^n \Rightarrow \operatorname{spt}\|V\| \cap B_r(x) \subseteq 2r\mathfrak{z}\text{-neighborhood of } x + \mathbb{R}^n$$

Also, the lemma gives $\frac{\|V\|(B_{2r}(x))}{\omega_n (2r)^n} \leq 1 + \mathfrak{z}$ for $\mathfrak{z} \leq \frac{1}{6}$ (which we may freely choose)

Since $y \in \operatorname{spt}\|V\| \cap B_r(x)$, $\quad \|P^\perp_{\mathbb{R}^n}(x) - P^\perp_{\mathbb{R}^n}(y)\| \leq 2\mathfrak{z}r \leq 3\mathfrak{z}|x-y| \leq \frac{1}{2}|x-y|$

By the triangle inequality, $\quad \Rightarrow \|P_{\mathbb{R}^n}(x) - P_{\mathbb{R}^n}(y)\| \geq \frac{1}{2}|x-y|$.

So, $P_{\mathbb{R}^n} : \operatorname{spt}\|V\| \cap B_{\frac{1}{4}}(x) \to \mathbb{R}^n$ is injective.

If $G \neq \emptyset$, then $\operatorname{spt}\|V\| \cap B_{\frac{1}{8}}(0) \subseteq \operatorname{spt}\|V\| \cap B_{\frac{1}{4}}(x)$ for $x \in G$, and so $P_{\mathbb{R}^n} : G \to \mathbb{R}^n$ is injective!

Letting $D := P_{\mathbb{R}^n}(G)$, then $\exists\, \tilde{u} : D \to \mathbb{R}^k$ with graph$(\tilde{u}) = G$, i.e. G is graphical.
In fact, $\tilde{u}$ is Lipschitz: if $v, w \in D$ then

$$|\tilde{u}(v) - \tilde{u}(w)| = |P^\perp_{\mathbb{R}^n}(\tilde{u}(v), v) - P^\perp_{\mathbb{R}^n}(\tilde{u}(w), w)| \leq 3\mathfrak{z}\,|(\tilde{u}(v), v) - (\tilde{u}(w), w)|$$

$$\underset{\substack{\text{since } (\tilde{u}(v),v) \in G \\ \text{and same for } w}}{\leq} 6\mathfrak{z}\,|P_{\mathbb{R}^n}((\tilde{u}(v), v)) - P_{\mathbb{R}^n}((\tilde{u}(w), w))| = 6\mathfrak{z}\,|v - w|$$

Note that we can make $\mathrm{Lip}(\tilde{u})$ as small as we like. Also, $\sup|\tilde{u}| \leq C E_V^{\frac{1}{n+2}}$ using earlier arguments (check notes).

Take $u : B^n_{1/8}(0) \to \mathbb{R}^k$ a Lipschitz extension of $\tilde{u}$. So, u has what we want, and we simply must bound the size of the bad set.

Set
$$\underbrace{\Sigma}_{\text{bad set}} := P_{\mathbb{R}^n}\left(B^{n+k}_{\frac{1}{8}}(0) \cap \left(\operatorname{spt}\|V\| \overset{\text{symmetric difference}}{\Delta} \operatorname{graph}(u)\right)\right) \quad \text{and} \quad F := \operatorname{spt}\|V\| \setminus G.$$

If $x \in F$, then by construction of G, $\exists \rho_x > 0$ s.t. $\frac{1}{\rho_x^n}\int_{B^{n+k}_{\rho_x}(x)} \|P_{T_z V} - P_{\mathbb{R}^n}\|^2 \, d\|V\|(z) > \lambda$

We know $F \subseteq \bigcup_{x\in F} B_{\rho_x}(x) \overset{5r \text{ covering lemma}}{\Longrightarrow} \exists$ countable disjoint subset $\{B_{\rho_i}(x_i)\}_i$ s.t. $F \subseteq \bigcup B_{5\rho_i}(x_i)$.

So,
$$\mathcal{H}^n(F) \le \omega_n 5^n \sum_i \rho_i^n \le \omega_n 5^n \cdot \frac{1}{\lambda} \underbrace{\sum_i}_{\text{disjoint}} \int_{B_{\rho_i}(x_i)} \|P_{T_z V} - P_{\mathbb{R}^n}\|^2 \, d\|V\|(z) \le \frac{C}{\lambda} E_V^2 \leftarrow \text{can be controlled by } E_V^2$$

Since $\Theta_V \ge 1$,
$$\|V\|(B^{n+k}_{\frac{1}{8}} \setminus G) \le \frac{C}{\lambda} E_V^2 \Rightarrow \|V\|(\mathbb{R}^k \times \Sigma) \le \frac{C}{\lambda} E_V^2.$$

Lastly, we need to bound the extra we got from the Lip. extension:
$$\mathcal{H}^n(\operatorname{graph}(u) \setminus G) \underset{\text{Jacobian bound}}{\le} C\, \mathcal{H}^n(P_{\mathbb{R}^n}(\operatorname{graph}(u) \setminus G)) = C\, \mathcal{H}^n(B^n_{\frac{1}{8}} \setminus P_{\mathbb{R}^n}(G)) = C\left(\frac{\omega_n}{8^n} - \mathcal{H}^n(P_{\mathbb{R}^n}(G))\right)$$
$$= C\left(\frac{\omega_n}{8^n} - \int_G J^{T_x V}_{P_{\mathbb{R}^n}} \, d\mathcal{H}^n\right) \le C\left(\frac{\omega_n}{8^n} - \mathcal{H}^n(G) + C E_V^2\right)$$
$$= C\left(\underbrace{\frac{\omega_n}{8^n} - \|V\|(B^{n+k}_{\frac{1}{8}})}_{\le 0 \text{ by monotonicity}} + \underbrace{\|V\|(B_{\frac{1}{8}} \setminus G)}_{\le \frac{C}{\lambda} E_V^2} + C E_V^2\right)$$
$$\le \frac{C}{\lambda} E_V^2$$

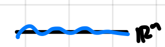
where we used that $|J^{T_x V}_{P_{\mathbb{R}^n}} - 1| \le C\|P_{T_x V} - P_{\mathbb{R}^n}\|^2$. Together, $\mathcal{H}^n(\Sigma) + \|V\|(\mathbb{R}^k \times \Sigma) \le \frac{C}{\lambda} E_V^2$. Since we chose ζ to make $\operatorname{Lip}(u)$ small, and we chose λ to satisfy the lemma with that choice of ζ, we are done.

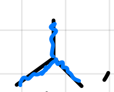
$\square$

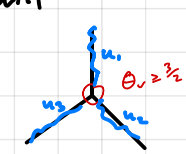
<u>Remark:</u> Note that in our entire Allard proof, the following things work even with being close to a mult. Q plane:

- Lip. approx
- reverse Poincaré
- constructing the blowup

What fails is understanding the regularity of the blowup.

<u>Plan:</u> What Allard has shown is that if you're close to  , then you're a $C^{1,\alpha}$ perturbation of the plane.

Next, we tackle the triple junction: if you're "close" to  , then you're a $C^{1,\alpha}$ perturbation of $\curlywedge$. Basically, remove the singular set o, apply Allard to each constituent plane, and <u>link them</u>. The linking step will require



① L^2 mass of u_i doesn't concentrate on o

② the constituent planes ℓ_i are related and together form a triple junction.

<u>3/6-</u>

§4 - Leon's Cylindrical Tangent Cones

Recall the stratification $\text{sing}(V) = \tilde{S}_1 \sqcup \dots \sqcup \tilde{S}_n$. $\tilde{S}_j$ = singular points where a tangent cone has $\dim(S(C)) = j$

- In Allard, we understood regularity around $x \in \text{spt}\|V\|$ where one tangent cone was a mult-1 plane – they are regular.

Since $\tilde{S}_n$ are the singular points where at least one tangent cone is a plane, Allard $\Rightarrow$ $\Theta_V|_{\tilde{S}_n} \gneq 1$

Cylindrical Tangent Cones

We may ask what can be said about more general tangent cones if mult is still 1.

Let $x \in \text{sing}(V)$, take $C \in \text{VarTan}_x(V)$ and assume C is mult-1 (i.e. $\Theta_C|_{\text{reg}(C)} = 1$). We may split

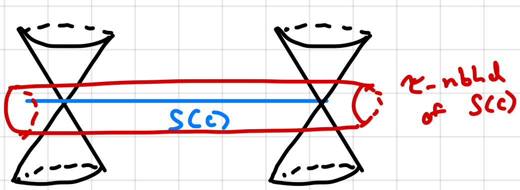
$$C = C_0 \times \mathbb{R}^k \quad k = \dim S(C)$$

Assume also that $\text{sing}(C) = S(C)$ (i.e. all singularities lie on the spine). This is referred to as C being cylindrical

So, $\text{sing}(C_0) = \{0\}$ is isolated (called C_0 being regular cone).
$\Rightarrow$ the link $\Sigma := C_0 \cap \mathbb{S}^\ell$ is smooth

Armed with a tangent cone $C = C_0 \times \mathbb{R}^k$ that is <u>cylindrical with mult 1</u>, let's try to follow Allard.

Following Allard



Take $(V_k)_k$ stationary integral varifolds with $V_k \to C$. $\forall \tau > 0$, for $k = k(\tau)$ large, we may apply Allard to express V_k on the the complement of τ-neighborhood of $S(C)$ (by cylindrical assumption) as smooth minimal graph $u_k \rightsquigarrow$ control $C^{k,\alpha}$ norm of u_k by $\|u_k\|_{L^2}$ via elliptic business (this removes the need for reverse Poincaré).

So, by Arzela-Ascoli, $\exists$ subsequence s.t. blowup $v_k \to v$ in $C^2_{loc}(B_1 \cap C \setminus S(C))$.
In Allard, v was harmonic: here, it satisfies a linearized MSE over C, i.e. the Jacobi equation over C: $\mathcal{L}_C v = 0$

Since $C = C_0 \times \mathbb{R}^k$,

polar coords since smooth on the link Σ — $r = |x|$, the C_0 coord

$$\mathcal{L}_C = \Delta_{\mathbb{R}^k} + \mathcal{L}_{C_0} = \Delta_{\mathbb{R}^k} + \Delta_{C_0} + |A_{C_0}|^2 = \Delta_{\mathbb{R}^k} + \frac{1}{r^{l-1}}\frac{\partial}{\partial r}\left(r^{l-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\underbrace{\left(|A_\Sigma|^2 + \Delta_\Sigma\right)}_{=:\mathcal{L}_\Sigma}$$

Note that $\mathcal{L}_\Sigma$ is S.A. and elliptic operators and Σ is smooth and compact.
So, eigenvalues of $-\mathcal{L}_\Sigma$ obey $\lambda_1 \le \lambda_2 \le \dots \to \infty$ with e'fns φ_k.

So, we can write v in the eigenfunction expansion $v = \sum_l \sum_k r^{\gamma_k} \varphi_k \psi_l$

e'fns of $\Delta_{\mathbb{R}^k}$ part

sum over e'fns in x and y parts — some powers depending on λ_k

We'd like a decay estimate for v; we'd need to subtract from v any piece of this expansion with r-homogeneity ≤ 1 (since the rest will decay).

If we had Hardt-Simon for v (i.e. if V_k has good density points), we can rule out homogeneities < 1 in this expansion. So, we'd only need to subtract pieces of homogeneity $= 1$. What we get is schematically

$$\int_{B_{1/2}} |v - (\text{homogeneity } 1 \text{ pieces})|^2 \le C r^{2\alpha} \int_{B_1} |v|^2$$

determined by first >1 homogeneity

(note the similarity to Allard decay of blowup, where {homo. 1} were planes)

If the 1-homogeneous solutions to the Jacobi operator don't look like the cone we started with, we're fucked since we can't pass excess decay back.

We need to understand geometrically what this piece is! It needs to be generated by a 1-parameter family of cones to get nice excess decay.

<u>Def:</u>

C_0 is integrable if every radially 1-homogeneous solution to $\mathcal{L}_{C_0} v = 0$ is generated by a 1-param family of cones.

With more assumptions, we can hope for excess decay iteration.

// if C_0 is flat (planes, bunches of planes, half-planes, etc.) then $\mathcal{L}_{C_0} = \Delta_{C_0}$ and 1-homo. solutions are also flat. So, ① doesn't happen, but ② still might //

- mult. 1 $\Rightarrow$ the cone won't split into multiple
- "no gaps" (i.e. good density points) $\Rightarrow$ no lower homogeneities $\Rightarrow \dots \Rightarrow$ fixes the space in place

To sum up, the things that go wrong:

① 1-homogeneous Jacobi solutions on C_0 not generated by cones (Simons cone does this)

② iteration messes up the cone

The Triple Junction

The triple junction will behave well under this argument.

- flat, so Jacobi operator is just Laplacian
- 1-homo parts of blow-ups should be linear, so some structure is preserved
- no gaps (i.e. good density points)
- integrability (i.e. no L^2 accumulation at singularity).

Lemma (triple junction has no gaps):

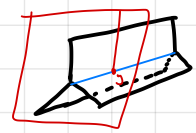
Suppose V is SIV in $B_2^{n+k}(0)$. Then $\exists\, \varepsilon(n,k)$ s.t.:

If V is ε-close to a mult. 1 triple-junction (i.e. the following hold)

- $\underset{\text{(nonempty)}}{0 \in \operatorname{spt}\|V\|}$ • $\underset{\text{(mult 1)}}{\omega_n^{-1}\|V\|(B_1) \le \frac{3}{2} + \frac{1}{4}}$ • $E_{V,C} < \varepsilon$

then in coordinates $(x,y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-1}$, $\forall y \in B_{\frac{1}{2}}^{n-1}(0)$ we have

$$\{\Theta_V \ge \tfrac{3}{2}\} \cap (\mathbb{R}^{k+1} \times \{y\}) \ne \emptyset$$ (all slices hit points of good density)



Also, $\forall \tau > 0$, if $\varepsilon = \varepsilon(n,k,\tau)$ is small, then $\Theta_V = 1$ outside $B_\tau(S(C))$.

Proof: Suppose BWOC that $\exists\, y \in B_{\frac{1}{2}}^{n-1}(0)$ s.t. $\{\Theta_V \ge \frac{3}{2}\} \cap (\mathbb{R}^{k+1} \times \{y\}) = \emptyset$.
Since $\{\Theta_V \ge \frac{3}{2}\}$ is closed by u.s.c. of density, $\exists\, \rho > 0$ s.t.

$$\{\Theta_V \ge \tfrac{3}{2}\} \cap (\mathbb{R}^{k+1} \times B_\rho^{n-1}(y)) = \emptyset$$

Look at $\operatorname{sing}(V)$ in $\mathbb{R}^{k+1} \times B_\rho^{n-1}(y)$:

$$\operatorname{sing}(V) = \tilde{S}_0 \cup \dots \cup \underbrace{\cancel{\tilde{S}_{n-2}}}_{\substack{C_0^2 \times \mathbb{R}^{n-2},\\ \text{density} \ge 3/2}} \sqcup \underbrace{\cancel{\tilde{S}_{n-1}}}_{\text{density} \ge \frac{3}{2}} \sqcup \underbrace{\cancel{\tilde{S}_n}}_{\substack{\text{density} \ge 2\\ \text{by Allard}}}$$

$\Rightarrow \operatorname{sing}(V) = S_{n-3} \Rightarrow \dim_{\mathcal{H}}(\operatorname{sing}(V)) \le n-3 \Rightarrow \exists\, \tilde{y} \in B_\rho^{n-1}(y)$ s.t. $\operatorname{sing}(V) \cap (\mathbb{R}^{k+1} \times \{\tilde{y}\}) = \emptyset$

$\overset{\operatorname{sing}(V)\text{ closed}}{\Longrightarrow} \exists\, \tilde{\rho} > 0$ s.t. $\operatorname{sing}(V) \cap (\mathbb{R}^{k+1} \times B_{\tilde{\rho}}^{n-1}(\tilde{y})) = \emptyset$

$\Rightarrow V$ is smooth submanifold in $\mathbb{R}^{k+1} \times B_{\tilde{\rho}}^{n-1}(\tilde{y})$

$\overset{\text{Sard's thm.}}{\Longrightarrow} \exists\, z \in B_{\tilde{\rho}}^{n-1}(\tilde{y})$ s.t. $\operatorname{spt}\|V\| \cap (\mathbb{R}^{k+1} \times \{z\})$ is smooth 1-manifold

$\overset{\text{Allard}}{\Longrightarrow} \operatorname{spt}\|V\| \cap (\mathbb{R}^{k+1} \times \{z\})$ has 3 boundary components. $\rightarrow\leftarrow$, # should be even.

$\square$

3/11 -

Last time, we saw that the triple junction. We have hope of proving something here, since it is <u>cylindrical</u> and <u>has no gaps</u>.

We just must show that the three planes don't get separated separately.

<u>Remark:</u> We have the following:

<u>Theorem (Simon '83 ⇝ Łojasiewicz ineq.):</u>

If C is a tangent cone with $\mathrm{Sing}(C) = \{0\}$ and C has multiplicity 1, <u>then</u> C is unique.

We can prove that if V is close to the triple junction $C = C_0 \times \mathbb{R}^{n-1}$, then density is always close to $\theta_C(0) = \frac{3}{2}$.

<u>Lemma:</u>

$\exists\, \varepsilon_0(n) \in (0,1)$ s.t. if V is SIV in $B_1^{n+k}(0)$ and $\omega_n^{-1}\|V\|(B_1^{n+k}(0)) \leq \frac{3}{2} + \frac{1}{8}$, <u>then</u> $\forall x \in B_{\varepsilon_0}(0)$ and all $\rho \in (0, 1-|x|)$, we have

$$\frac{\|V\|(B_\rho(x))}{\omega_n \rho^n} \leq \frac{3}{2} + \frac{1}{4} < 2$$

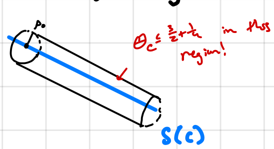
In particular, the density $= 1$ in a ball around the singularity.

<u>Proof:</u> By monotonicity, $\dfrac{\|V\|(B_\rho(x))}{\omega_n \rho^n} \leq \dfrac{\|V\|(B_{1-|x|}(x))}{\omega_n (1-|x|)^n} \leq \dfrac{\frac{3}{2}+\frac{1}{8}}{\omega_n (1-\varepsilon_0)^n} \leq \frac{3}{2} + \frac{1}{4}.$ ∎

<u>Remarks:</u> ① By translation and rescaling, if the original mass assumption holds on $B_2^{n+k}(0)$, then we get the same density bound conclusion on, say,

$$x \in B^{k+1}_{\rho_0(n)} \times B^{n-1}_{\frac{1}{2}}(0)$$

direction perpendicular to $S(C)$ — direction along $S(C)$



② If V is ε-close to C, we can apply Allard outside $B^{k+1}_{\rho_0}(0) \times B^{n-1}_{\frac{1}{2}}(0)$ to control <u>all mass ratios</u>.

After all, we were only worried about wrinkles at the spine.

Def:

For a given V close to C, let $\mathcal{M}$ be

in varifold topology

$$\mathcal{M} := \overline{\{\text{rotations} + \text{homothetic rescalings of } V\}}$$

This forms (by the above lemma) a multiplicity-1 class, i.e.

(i) if $V \in \mathcal{M}$ then $q_\#(\eta_{x,\rho})_\# V \in \mathcal{M}$ for $q \in SO(n+k)$ (rotations), $x \in B^{n+k}_{\frac{1}{2}}$, $\rho \in (0, \frac{1}{2})$.

(ii) if $(V_j)_j \subseteq \mathcal{M}$ with $\sup_j \|V_j\|(K) < \infty$ $\forall K \subseteq B^{n+k}_{\frac{1}{2}}(0)$ compact

then $\exists$ subseq. $V_j \to V \in \mathcal{M}$ and $\theta_V = 1$ a.e.

For a multiplicity-one class, we can prove a form of Allard without any mass or scale assumption.

Lemma:

mass upper bound

Fix $\Lambda > 0$ and let $\mathcal{M}$ be a mult-1 class.
Then $\exists \beta(\Lambda, \mathcal{M}) > 0$ s.t.:

if $V \in \mathcal{M}$, $\rho > 0$, $B^{n+k}_\rho(x_0) \subseteq B^{n+k}_{\frac{1}{2}}(0)$ with

- $\operatorname{spt}\|V\| \cap B^{n+k}_{3\rho/4}(x_0) \neq \emptyset$
- $\frac{1}{\omega_n \rho^n} \|V\|(B_\rho(x_0)) \leq \Lambda$
- $\frac{1}{\rho^{n+2}} \int_{B^{n+k}_\rho(x_0)} \operatorname{dist}^2(x, P)\, d\|V\|(x) < \beta$ for some P affine plane

then $\exists$ $u : P \cap B_{\rho/2}(x_0) \to P^\perp$ a C^2 map with $V \llcorner B_{\rho/2}(x_0) = \operatorname{graph}(u)$. The usual u estimates apply.

Remark: This is deceptively similar to Allard, but note that it works at all scales with the same β and Λ.

Proof: Suppose that this fails. For some contradicting sequence, and translate and rescale and rotate to assume WOLOG that $\rho_k = 1$, $(x_0)_k = 0$, and $P_k = \mathbb{R}^n$. This stays within the class $\mathcal{M}$, and so we have $V_k \in \mathcal{M}$ s.t.

$$\cdot\ \operatorname{spt}\|V_k\| \cap B^{n+k}_{\frac{3}{4}}(0) = \emptyset \qquad \cdot\ \omega_n^{-1} \|V_k\|(B_1(0)) \leq \Lambda \qquad \cdot\ \int_{B_1^{n+k}(0)} \operatorname{dist}^2(x, \mathbb{R}^n)\, d\|V_k\| \to 0$$

Since $\mathcal{M}$ is a compact class, we have a convergent subseq.

$V_k \xrightarrow{\text{weakly}} V \in \mathcal{M}$ and so $\Theta_V = 1$ a.e.. But V is a plane, and so $V = \mathbb{R}^n$ with mult.-1. So, we may apply Allard.

$\square$

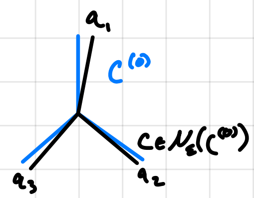
To state our result for the triple junction, use the following notation:

Write $C^{(0)} = C_0^{(0)} \times \mathbb{R}^{n-1}$ to be the (basic) triple junction.

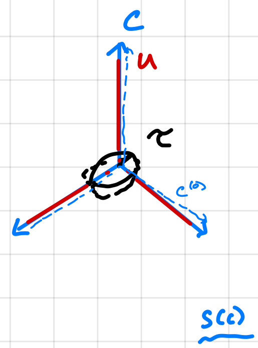
Write $\mathcal{N}_\varepsilon(C^{(0)})$ for the set of $V \in \mathcal{M}$ s.t.

$\bullet\ \omega_n^{-1} \|V\|(B_1) \le \frac{3}{2} + \frac{1}{4}$ $\quad \bullet\ \hat{E}_{V, C^{(0)}} < \varepsilon$

Write $\mathcal{C}_\varepsilon(C^{(0)})$ for the set of cones C with $S(C) = S(C^{(0)})$ allowing each half-plane in $C^{(0)}$ to rotate by some q_i with $|q_i - id| < \varepsilon$.



With this language,



Lemma: (Graphical Representation)

Fix $\tau \in (0, \frac{1}{100})$. Then, $\exists\, \varepsilon(n, k, \tau)$ s.t.:

if $C \in \mathcal{C}_\varepsilon(C^{(0)})$, $V \in \mathcal{N}_{\varepsilon_0}(C^{(0)})$,
then $\exists$ open $U \subseteq C \cap B_1$ satisfying

(i) U is rotationally symmetric about $S(C)$ and $\{(x,y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-1} : |x| > \tau\} \subseteq U$

(ii) $\exists\, u : U \to C^\perp$ that is C^2 and s.t.
$$V \llcorner B_{3/4}(0) \cap \{|x| > \tau\} = \operatorname{graph}\left(u|_{B_{3/4}(0) \cap \{|x| > \tau\}}\right)$$

(iii) $$\int_{B_{3/4} \setminus \operatorname{graph}(u)} |x|^2 \, d\|V\| + \int_{U \cap B_{3/4}} |x|^2 |\nabla u|^2 \, d\|V\| \le C(n,k)\, \hat{E}_{V,C}^2$$

integral over the places we're not graphical

Proof: For $\kappa \in (0, \frac{1}{8})$, $\rho \in (0,1)$, and $z \in S(C)$,
set
$$T_{\rho,\kappa}(z) := \{(x,y) : (|x| - \rho)^2 + |y - z|^2 < (\kappa \rho)^2\}$$

Let $U := \left(\bigcup T_{|\xi|, \frac{1}{4}}(z)\right) \cap C$ where the union is taken over all $(\xi, z) \in B_{3/4}$ s.t. over $T_{|\xi|, \frac{3}{32}}(z)$, V is graphical (with estimate).

If $(\xi, z) \in C \cap B_{\frac{3}{4}} \cap \partial U$, then by the lemma we must have
$$\int_{T_{|\xi|, \frac{1}{4}}(z)} \operatorname{dist}^2(x, C)\, d\|V\| \ge \omega_n |\xi|^{n+2} \beta^2$$

We know that $\int_{u\cap B_{10|\xi|}(0,\zeta)} |x|^2 \le (10)^2|\xi|^2 \cdot \omega_n (10|\xi|)^n \le C(n)|\xi|^{n+2}$

and $\int_{u\cap B_{10|\xi|}(\xi,\zeta)} |x|^2 \underbrace{|Du|^2}_{\le \beta \text{ by Allard}} \le C(n)|\xi|^{n+2}\beta^2 \le C(n)\int_{T_{|\xi|,1}(\xi)} \mathrm{dist}^2(x,C)\, d\|V\|$

We finish via a Vitali-style covering argument. $\square$

3/25 -

Recap: we wish to prove Allard regularity for the triple junction. So far, we've done the following:

(i) no gaps

(ii) reduce to "mult 1 class"

(iii) built a graphical representation away from $S(C^0)$ with error estimates

$$\int_{B^{n+k}_{3/4}(0)\setminus \mathrm{graph}(u)} r^2\, d\|V\| + \int_{u\cap B^{n+k}_{3/4}} r^2|Du|^2 \lesssim \hat{E}^2_{V,C}$$

To do iteration and pass L^2 estimates back, we need to investigate behavior near the spine. Morally, we can do this in regions where we have accumulation of good density points:

Theorem: (Simon's L^2 estimates)

Fix $\tau \in (0, \frac{1}{100})$. Then, $\exists\, \varepsilon_0(n,k,\tau)$ s.t.

If $\varepsilon \le \varepsilon_0$, $V \in \mathcal{N}_\varepsilon(C^{(0)})$, $C \in \mathcal{C}_\varepsilon(C^{(0)})$ (have same spine), and $u: U \to C^\perp$ is as above, then for any $Z = (\xi, \zeta) \in B^{n+k}_{3/4}(0)$ with $\Theta_V(Z) \ge \frac{3}{2}$ ($= \Theta_C(0)$), we have

(ξ: distance to spine $\in \mathbb{R}^{k+1}$; ζ: coordinate along spine $\in \mathbb{R}^{n-1} \cong S(C^0)$)

① $\mathrm{dist}(Z, S(C)) \equiv |\xi| \le C(n,k)\, \hat{E}_{V,C}$ (good density points remain bounded when we blow-up)

② $\int_{B^{n+k}_{3/4}(0)} \sum_{j=k+2}^{n+k} |e_j^\perp|^2\, d\|V\| \le C(n,k)\, \hat{E}^2_{V,C}$ (the spine directions) (Control derivative of blow-up parallel to spine)

③ $\int_{B^{n+k}_{3/4}(0)} \frac{\mathrm{dist}^2(x,C)}{|x-z|^{n-\frac{1}{2}}}\, d\|V\| \le C(n,k)\, \hat{E}^2_{V,C}$ (can be $n-\varepsilon$) (On a nbhd of z, L^2 distance is small $\Rightarrow$ L^2 nonconcentration)

④ $\int_{u\cap B^{n+k}_{3/4}(0)} R^{2-n} \left|\frac{\partial}{\partial R}(u/R)\right|^2 d\|V\| \le C(n,k)\, \hat{E}^2_{V,C}$ ($R = |x|$) (Hardt-Simon)

⑤ $\int_{C\cap B^{n+k}_{3/4}(0)\cap\{|x|\ge\tau\}} \frac{|u-\xi^\perp|^2}{|x-z|^{n+\frac{3}{2}}} \le C(n,k)\, \hat{E}^2_{V,c}$ (can be $n+2-\varepsilon$)

Remark: Note that (5) gives something like

$$\rho^{-n-2+\alpha}\int_{B_\rho(z)}\left|\frac{u}{\hat E}-\frac{\mathfrak{z}^\perp}{\hat E}\right|^2 \xrightarrow[\text{Campanato}]{} \text{blow-up is } C^{0,\alpha} \text{ up to the boundary.}$$

So, the blowup's boundary values are determined by the limiting projections of the good density points. Since these values (and their derivatives) are independent of V, the resulting linear approx. that we stitch together for Allard will have <u>the same spine</u>, and so we will be able to iterate.

Proof sketch: The main content will be controlling the error term in the monotonicity formula. First, we do so near the origin.

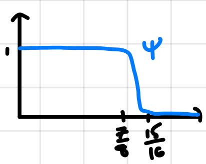
Lemma: Let V, C be ε-close to $C^{(0)}$ and $\Theta_V(0) \geq \frac{3}{2}$ (origin has good density). Then,

$$\int_{U\cap B_{3/4}} R^{2-n}\left|\frac{\partial}{\partial R}\left(\frac{u}{R}\right)\right|^2 + \int_{B_{3/4}}\sum_{j=k+2}^{n+k}|e_j^\perp|^2\, d\|V\| + \int_{B_{3/4}}\frac{dist^2(x,C)}{|x|^{n+\frac{3}{2}}}\, d\|V\| + \int_{B_{3/4}}\frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\| \leq C(n,k)\,\hat E_{V,C}^2$$

Proof of lemma: The monotonicity formula after differentiation gives

$$n\rho^{n-1}\int_{B_\rho}\frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\| = \frac{d}{d\rho}\underbrace{\left[\rho^n\int_{B_\rho}\frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\|\right]}_{\|V\|(B_\rho)-\omega_n\rho^n\Theta_V(0)} - \overbrace{\rho^n\frac{d}{d\rho}\int_{B_\rho}(\dots)}^{\geq 0}$$

$$\leq \frac{d}{d\rho}\Big(\|V\|(B_\rho) - \omega_n\rho^n\underbrace{\Theta_V(0)}_{\geq\frac{3}{2}=\Theta_C(0)}\Big) = \frac{d}{d\rho}\big(\|V\|(B_\rho)\big) - \underbrace{\omega_n n\rho^{n-1}\Theta_C(0)}_{=\frac{d}{d\rho}(\|C\|(B_\rho))}$$

Take $\psi(|x|) =$ 

as a cutoff fn. Multiply by $\psi^2(\rho)$ and take $\int_0^1 \dots d\rho$ to get

$$n\int_0^1\psi^2(\rho)\rho^{n-1}\int_{B_\rho}\frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\| \leq \int_{B_1}\psi^2(|x|)\, d\|V\| - \int_{B_1}\psi^2(|x|)\, d\|C\|$$

By construction, the LHS upper bounds $n\left(\frac{3}{4}\right)^{n-1}\left(\frac{7}{8}-\frac{3}{4}\right)\cdot\int_{B_{3/4}}\frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\|$

So,

$$C(n)\int_{B_{3/4}}\frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\| \leq \int_{B_1}\psi^2(|x|)\, d\|V\| - \int_{B_1}\psi^2(|x|)\, d\|C\|$$

Taking $\psi^2(|x|)(x,0)$ in the 1^{st} variation formula, ...

$\square$

Our first important corollary: use ③ to show that L^2 norm doesn't concentrate at $S(C)$.

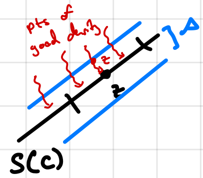
Corollary: (Non-concentration around spine)

Fix $\delta \in (0, \frac{1}{8})$. Then $\exists\, \varepsilon_0(n,k,\delta)$ s.t.:

If $\varepsilon \le \varepsilon_0$, $V \in \mathcal{N}_\varepsilon(C^{(0)})$, $C \in \mathcal{C}_\varepsilon(C^{(0)})$, then $\forall \Delta \in [\delta, \frac{1}{4})$,

$$\int_{B^{n+k}_{\frac{1}{2}}(0) \cap \{|x|<\Delta\}} \operatorname{dist}^2(\underbrace{X}_{X=(x,y)}, C)\, d\|V\| \le C(n,k)\, \Delta^{\overbrace{\alpha}^{\text{choose } \frac{1}{2}}}\, \hat{E}^2_{V,C}$$

Proof: Fix $\Delta \in [\delta, \frac{1}{4})$. Take $z \in B^{n-1}_{\frac{1}{2}}(0)$. If ε is small, all points of density $\ge \frac{3}{2}$ are in Δ-neighborhood of $S(C)$. So, choose $Z \in B^{n+k}_\Delta(z)$ with $\Theta_V(Z) \ge \frac{3}{2}$.



By ③,

$$\Delta^{-n+\frac{1}{2}} \int_{B_\Delta(z)} \operatorname{dist}^2(x, C)\, d\|V\| \le C\, \hat{E}^2_{V,C}$$

Now, cover $B^{k+1}_\Delta(0) \times B^{n-1}_{\frac{1}{2}}(0)$ by $N \le C(n,k)\Delta^{-(n-1)}$ balls $\{B_\Delta(z_i)\}_i$ with $z_i \in B^{n-1}_{\frac{1}{2}}(0)$. Summing the estimates,

$$\int_{B^{n+k}_{\frac{1}{2}}(0) \cap \{|x|<\Delta\}} \operatorname{dist}^2(X, C)\, d\|V\| \le \sum_{i=1}^{N} \int_{B_\Delta(z_i)} \operatorname{dist}^2(X,C)\, d\|V\|$$

$$\lesssim \sum_i \Delta^{n-\frac{1}{2}}\, \hat{E}^2_{V,C} \lesssim \Delta^{-(n-1)} \Delta^{n-\frac{1}{2}}\, \hat{E}^2_{V,C} = \Delta^{\frac{1}{2}}\, \hat{E}^2_{V,C}$$

$\square$

Remark: Now we know that the blow-ups will converge in L^2 all the way up to the spine.

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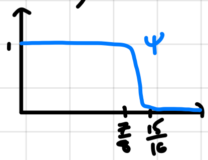
Lemma: (Initial L^2 Estimate)

Fix $\tau \in (0, \frac{1}{100})$. Then, $\exists\, \varepsilon_0(n,k,\tau) > 0$ s.t.

if $V \in \mathcal{N}_\varepsilon(C^{(0)})$, $C \in \mathcal{C}_\varepsilon(C^{(0)})$, and $\Theta_V(0) \geq \frac{3}{2}$, then

$$\int_{B_{3/4}(0)} \sum_{j=k+2}^{n+k} |e_j^\perp|^2 \, d\|V\| + \int_{B_{3/4}} \frac{\operatorname{dist}^2(x, C)}{|x|^{n+3/2}} \, d\|V\| + \int_{B_{3/4}} \frac{|x^\perp|^2}{|x|^{n+2}} \, d\|V\| \lesssim \hat{E}_{V,C}^2$$

Proof: $$\int_{B_{3/4}} \frac{|x^\perp|^2}{|x|^{n+2}} \, d\|V\| \leq C \left(\int_{B_1} \psi^2(|x|) \, d\|V\| - \int_{B_1} \psi^2(|x|) \, d\|C\| \right)$$

Consider the variation $\psi^2(|x|) \cdot (x, 0)$ with $\psi(|x|) =$ .

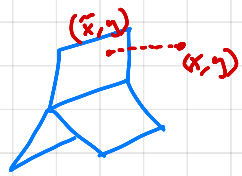
The first variation formula gives, with $C = \underset{x}{C_0} \times \underset{y}{\mathbb{R}^{m-1}}$,

$$\int_{B_1} \left(1 + \frac{1}{2} \sum_{j=k+2}^{n+k} |e_j^\perp|^2 \right) \psi^2(|x|) \, d\|V\| \leq C(n,k) \int |(x,0)^\perp|^2 \left(\psi^2(|x|) + [\psi'(|x|)]^2\right)$$
$$- 2 \int |x|^2 |X|^{-1} \psi(|x|) \psi'(|x|) \, d\|V\|$$

The non-graphical piece of the 1^{st} term on RHS is
$$\leq C \int_{B_{15/16} \setminus \operatorname{graph}(u)} r^2 \, d\|V\| \leq C \hat{E}_{V,C}^2$$ by the graphing lemma.

For the graphical part, if $(x,y) \in \operatorname{graph}(u)$ then $(x,y) = (\tilde{x}, y) + u(\tilde{x}, y)$ for some $(\tilde{x}, y) \in \operatorname{spt}\|C\|$. So,

$$(x,0)^\perp = \Pi^\perp_{T_xV}((x,0)) = \underbrace{\left(\Pi^\perp_{T_xV} - \Pi^\perp_{T_{(x,0)}C}\right)}_{\|\cdot\| \leq C|Du|} (x,0) + \overbrace{\Pi^\perp_{T_{(\tilde{x},0)}C}(x,0)}^{\equiv u(\tilde{x},0)}$$



So, $|(x,0)^\perp|^2 \leq C(r^2 |Du|^2 + |u|^2)$, and so the graphical part of the 2^{nd} term on the RHS is controlled by

$$C \int_{U \cap B_{15/16}} |u|^2 + r^2 |Du|^2 \, d\|V\| \leq C \hat{E}_{V,C}^2$$

Look at

integrate along bdy. of half-plane, then along rays

$R^2 = r^2 + |y|^2$

$$\int_{H \cap B_1} \psi^2(|X|) = \int_{B_1 \cap \partial H} \int_0^1 \psi^2(R) \, dr \, dy$$

half-plane

$= \frac{r}{R}$

$$\overset{\text{IBP}}{=} \int_{B_1 \cap \partial H} \left[-2 \int_0^1 r \psi(R) \psi'(R) \frac{dR}{dr} \, dr \, dy \right.$$

$$= -2 \int_{B_1 \cap \partial H} \int_0^1 r^2 R^{-1} \psi(R) \psi'(R) \, dr \, dy$$

$$= -2\int_{H\cap B_1} r^2 R^{-1}\,\psi(R)\,\psi'(R)\,dr\,dy$$

We rewrite the LHS with the above substituted to get

$\psi' \le 0$, so we may throw nongraphical piece

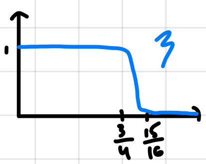
$$\int_{B_{3/4}} \sum_{j>k^2}^{\infty} |e_j^\perp|^2 + \int_{B_1} \psi^2(|x|)\,d\|V\| - \int_{B_1} \psi^2(|x|)\,d\|C\| \le C\hat{E}_{V,C}^2 + 2\int_{B_1} r^2 R^{-1}\psi(R)\psi'(R)\,dr\,dy - 2\int_{B_1} r^2 R^{-1}\psi(R)\psi'(R)\,d\|V\|$$

Since $\psi \equiv 1$ near the origin, this integral is away from the origin ($R^{-1} \le \frac{8}{7}$). So, where this nongraphical, we are bounded and so nongraphical piece controlled by $\int_{B_{15/16}\setminus \mathrm{graph}(u)} r^2\,d\|V\| \le C\hat{E}_{V,C}^2$

So,
$$\int_{\mathrm{graph}(u)\cap B_{15/16}} r^2 R^{-1}\psi(R)\psi'(R)\,d\|V\| = \int_{u\cap B_{15/16}} \underbrace{r_u^2}_{|x|^2+|u(x,y)|^2}\, \underbrace{R_u^{-1}}_{|x|^2+|y|^2+|u(x,y)|^2}\,\psi(R_u)\,\psi'(R_u)\,\underbrace{J_u}_{1+C|Du|^2}\,d\|C\|$$

So, the RHS is $\le C\hat{E}_{V,C}^2 + C\int_{u\cap B_{15/16}} |u|^2 + r^2|Du|^2 \le C\hat{E}_{V,C}^2$

We also get control of the Hardt-Simon as before. The last bit we need is control of the $\int \frac{\mathrm{dist}^2}{|x|^{n+3/2}}$ term. This is another 1^{st} variation argument, with the radial variation:

Let ① $\zeta =$  $|D\zeta| \le 16$ and take the variation $\underbrace{\zeta^2}_{\text{cutoff}}\,\underbrace{R^{-n+\frac12}}_{\text{grows @ origin}}\,\overbrace{\left(\frac{\tilde d}{R}\right)^2}^{\tilde d \text{ homo, and so this is bdd @ origin}}X$

② $\tilde d$ is homo degree 1, smoothing of $\mathrm{dist}(\cdot, C)$ with
$$c^{-1}\mathrm{dist}(x,C) \le \tilde d(x) \le c\,\mathrm{dist}(x,C) \quad\text{and}\quad \mathrm{Lip}(\tilde d) \le C.$$

With this variation, the 1^{st} variation formula gives

"the mass drop"

$$\int_{B_{3/4}} \frac{\mathrm{dist}^2(x,C)}{|x|^{n+3/2}}\,d\|V\| \le C\underbrace{\underbrace{\int \zeta^2 \frac{|x^\perp|^2}{|x|^{n+3/2}}}_{\le C\hat{E}_{V,C}^2} + \underbrace{\underbrace{\frac{\mathrm{dist}^2(x,C)}{|x|^{n-1/2}}}_{\sim \text{LHS}\cdot|x|^2}\,\underbrace{|\nabla\zeta|^2}_{\text{supported away from } B_{3/4}}\,d\|V\|}_{\le 16^2\left(\frac{4}{3}\right)^{n-\frac12}\mathrm{dist}^2(x,C)}}$$

$\square$

To get the last needed estimates for translations, we proceed.

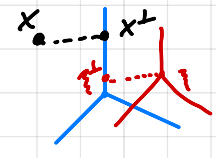
Lemma: (Cone Shifting)

$\exists \varepsilon_0(n,k)$ s.t. if $\varepsilon \le \varepsilon_0$, $V \in \mathcal{N}_\varepsilon(C^{(0)})$, $C \in \mathcal{C}_\varepsilon(C^{(0)})$, then for any $z \in B_{1/2}$ with $\Theta_V(z) \ge \frac{3}{2}$, we have

$$\operatorname{dist}^2(z, S(C)) + \int_{B_1} \operatorname{dist}^2(x, C+z)\, d\|V\|(x) \le C\, \hat{E}^2_{V,C}$$

Remarks: ① We first observe that if $z = (\overset{S(C)^\perp}{\xi}, \overset{S(C)}{\zeta})$, then $|d(x,C) - d(x, C+z)| \le |\xi|$

② Consider $X = (x,y)$ and let $X^\perp = (\tilde{x}, y)$.



for enough away to not switch half-planes

If $|x| \ge \theta^{-1}(|\xi| + \operatorname{dist}(X,C))$ for suitable θ, then

$$\operatorname{dist}(x, C+z) = |\underbrace{(x,y) - (\tilde{x},y)}_{= d(x,C)} - \xi^\perp| + R, \quad |R| \le \frac{C|\xi|^2}{|x|}$$

$$\Rightarrow \operatorname{dist}(x, C+z) = |\operatorname{dist}(x,C) - \xi^\perp| + R$$

Proof: $\forall a \in \mathbb{R}^{k+1}$, $s \in (0, \frac{1}{4})$, $z \in \operatorname{spt}\|V\| \cap B_{3/4}$, we have that

$|a^{\perp_{T_x C}}| \ge \delta |a|$ on a set of measure $\ge \delta s^n$ in $\operatorname{spt}\|V\| \cap B_s(z)$.

For a fixed a and s, integrating gives

$$s^n |\xi|^n \le C \int_{B_s(z)} |\xi^{\perp_{T_{(\tilde{x},y)}C}}|^2\, d\|V\|$$

By above,

$$s^n |\xi|^2 \le C \int_{B_s(z)} \operatorname{dist}^2(x, C+z)\, d\|V\| + \int_{B_s(z)} \operatorname{dist}^2(x,C)\, d\|V\| + \frac{C|\xi|^4}{s^2}$$

Applying the previous lemma to $\frac{V-z}{1/2}$,

$$s^{-n-3/2} \int_{B_s(z)} \operatorname{dist}^2(x, C+z)\, d\|V\| \le \hat{E}^2_{\frac{V-z}{1/2}, C} \equiv c \int_{B_1} \operatorname{dist}^2(x, C+z)\, d\|V\|$$

$$\overset{\text{Remark ①}}{\le} c \int_{B_1} \operatorname{dist}^2(x,c)\, d\|V\| + C|\xi|^2$$

$$\Rightarrow s^n |\xi|^2 \le C \hat{E}^2_{V,C} + C s^{n+3/2} |\xi|^2 + \frac{C|\xi|^4}{s^2} \overset{\text{choice of } s,\, \varepsilon}{\Longrightarrow} |\xi|^2 \le C \hat{E}^2_{V,C}$$

$\square$

4/1 -

Recap: We showed that at good density points $z = (\xi, \zeta)$:

(i) $\int_{B_{3/4}} \sum_{j=k+2}^{n+k} |e_j^{\perp}|^2 \, d\|V\| \leq C \hat{E}_{V,C}^2$ (tilt excess at spine)

(ii) $\int_{B_{3/4}} \frac{\text{dist}^2(x, C)}{|x|^{n+3/2}} \, d\|V\|(x) \leq C \hat{E}_{V,C}^2$ if $z = 0$

we'd like this to be $x - z$

(iii) $|\xi| \leq C \hat{E}_{V,C}$ (distance to spine)

(iv) $\int_{B_1} \text{dist}^2(x, C + z) \, d\|V\| \leq C \hat{E}_{V,C}^2$ (shifted cone)

(v) $|\text{dist}(x, C+z) - \text{dist}(x, C)| \leq |\xi|$ (triangle ineq)

We can now prove the rest of Simons' L^2 estimates.

Proof of remaining estimates: Apply (ii) to $\tilde{V} := (\eta_{z, 1/4})_{\#} V$, getting

$$\int_{B_{3/4}(0)} \frac{\text{dist}^2(x, C)}{|x|^{n+3/2}} \, d\|\tilde{V}\| \leq C \hat{E}_{\tilde{V},C}^2 = C \int_{B_1} \text{dist}^2(x, C) \, d\|\tilde{V}\|$$

undo homothety

$$\Rightarrow \int_{B_{3/16}(z)} \frac{\text{dist}^2(x, C+z)}{|x-z|^{n+3/2}} \, d\|V\| \leq C \int_{B_{1/4}} \text{dist}^2(x, c+z) \, d\|V\| \overset{(iv)}{\leq} C \hat{E}_{V,C}^2$$

$$\Rightarrow \int_{B_{3/16}(z)} \frac{\text{dist}^2(x, C+z)}{|x-z|^{n-1/2}} \, d\|V\| \leq C \hat{E}_{V,C}^2$$

$$\overset{(v)}{\Rightarrow} \int_{B_{3/16}(z)} \frac{\text{dist}^2(x, C)}{|x-z|^{n-1/2}} \, d\|V\| \leq \int_{B_{3/16}(z)} \frac{\text{dist}^2(x, C+z) + |\xi|^2}{|x-z|^{n-1/2}} \, d\|V\|$$

$$\leq C \hat{E}_{V,C}^2 + C|\xi|^2 \overset{(iii)}{\leq} C \hat{E}_{V,C}^2$$

③ in Simons' L^2 estimates

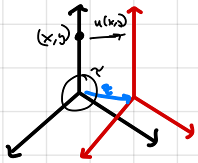
Lastly, we need

$$\int_{C \cap B_{3/2}^{n+k}(0) \cap \{|x| \geq \tau\}} \frac{|u(x,y) - \xi^{\perp}|^2}{|(x,y) + u(x,y) - z|^{n+3/2}} \leq \hat{E}_{V,C}^2$$

③ in Simons' L^2 estimates

Fix $\tau > 0$. For a shift by z, specifically for the triple junction, if $\varepsilon(n, k, \tau)$ is small,

$$\text{dist}(\underbrace{(x,y) + u(x,y)}_{X}, C + z) = |u(x,y) - \xi^{\perp}|$$



Using the earlier estimate,

$$\int_{C\cap B^{n+k}_{\frac{3}{4}}(0)\cap\{|x|\geq\tau\}} \frac{|u(x,y)-\xi^{\perp}|^2}{|(x,y)+u(x,y)-z|^{n+\frac{3}{2}}} \leq C\int_{B_{\frac{1}{16}}(z)\cap\{|x|>\tau\}} \frac{\operatorname{dist}^2(x, C+z)}{|x-z|^{n+\frac{3}{2}}}\, d\|V\| \overset{③}{\leq} C\,\hat{E}_{V,C}^2.$$

$\square$

<u>**Remark:**</u> By a similar argument done to $\tilde{V}_s := (\eta_{z,s})_{\#} V$ instead, we can do the same provided ε depends on s, getting

$$\int_{B_{s/2}(z)} \frac{\operatorname{dist}^2(x, C+z)}{|x-z|^{n+\frac{3}{2}}} \lesssim s^{-n-\frac{3}{2}} \int_{B_s(z)} \operatorname{dist}^2(x, C+z)$$

Fixing $\sigma\in(0, s/2]$, we get

$$\sigma^{-n-\frac{3}{2}}\int_{B_\sigma(z)} \operatorname{dist}^2(x, C+z)\, d\|V\| \lesssim s^{-n-\frac{3}{2}}\int_{B_s(z)} \operatorname{dist}^2(x, C+z)$$

Morally, this says

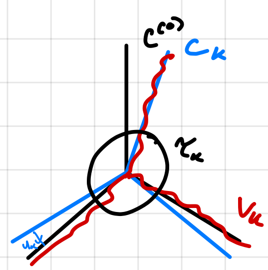
$$\sigma^{-n-\frac{3}{2}}\int_{B_\sigma} |u-\xi^{\perp}|^2 \lesssim s^{-n-\frac{3}{2}}\int_{B_s(z)} |u-\xi^{\perp}|^2$$

which is basically a $C^{0,\frac{3}{4}}$ estimate at the spine, which is where the boundary regularity comes from.

"good density points come with a Holder estimate for free"

To finish <u>ε-regularity of triple junction</u>, we will construct the blow-up.

<u>ε-Regularity of Triple Junction</u>



Take $\varepsilon_k \downarrow 0$ and $V_k\in\mathcal{N}_{\varepsilon_k}(C^{(0)})$
$C_k\in\mathcal{C}_{\varepsilon_k}(C^{(0)})$
and $\tau_k\downarrow 0$ as slow as we want.

Since we are graphical on $B^c_{\tau_k}$, take u_k to be the graphical representation of V_k over C_k in the region $B_{\frac{3}{4}}(0)\cap\{|x|>\tau_k\}\cap C_k$. To remove domain dependence on K, in general we have to reparameterize u_k to be relative to $C^{(0)}$. Since these are half planes, we can rotate all half-planes to a fixed H and so u_k is a triple of functions on $H\cap\{|x|>\tau_k\}$.

Define the blow-up $v_k := u_k / \hat{E}_{v_k, C_k}$, which have bdd L^2 norm and have good regularity away from the spine. Pass to a subsequence to get

① $v_k \to v \in C^2(C^{(0)} \cap \{|x|>0\} \to C^{(0)\perp})$

convergence in C^2_{loc}

② $v_k \to v$ in $L^2(C^{(0)} \cap B_{3/4})$ by L^2 nonconcentration

③ v is harmonic on $C^{(0)} \cap \{|x|>0\}$.

What about at the boundary? Take $(0,y) \in \{|x|=0\} \in S(C^{(0)})$. By no gaps, we may take some $z_k = (\xi_k, \zeta_k) \to (0,y)$ with $\Theta_{V_k}(z_k) \geq 3/2$.

For fixed $\theta > 0$, we know $\forall k$ large (depending on θ & ρ):

$$B_{\theta/2}((0,y)) \subseteq B_\theta(z_k) \qquad \text{and} \qquad B_\rho(z_k) \subseteq B_{2\rho}((0,y))$$

L^2 estimates

$$\Rightarrow \quad \theta^{-n-3/2} \int_{B_{\theta/2}((0,y))} |u_k - \xi_k^\perp|^2 \leq C \rho^{-n-3/2} \int_{B_{2\rho}((0,y))} |u_k - \xi_k^\perp|^2$$

Divide by $\hat{E}_{v_k, C_k}$, noting $\left| \frac{\xi_k}{\hat{E}_{v_k,C_k}} \right| \leq C$ (and so converges up to s/seq), getting

$$\theta^{-n-3/2} \int_{B_{\theta/2}(0,y) \cap C^{(0)}} |v - \kappa(y)^\perp|^2 \leq C \rho^{-n-3/2} \int_{B_{2\rho}(0,y) \cap C^{(0)}} |v - \kappa(y)|^2$$

where $\frac{\xi_k}{\hat{E}_{v_k,C_k}} \to \kappa(y)$ and $0 < \theta \leq \rho/2 < \frac{1}{16}$.

This is precisely a Campanato-ish estimate at the boundary (uniform in ρ, θ) for any fixed $(0,y) \in S(C^{(0)})$. Together with interior harmonic estimates,

$$v \in C^{0,3/4}\left(\overline{C^{(0)} \cap B_{1/2}}\right)$$

(each of the three is $C^{0,3/4}$ up to the spine)

There is a basic fact about harmonic functions, in which on a half-plane H,

$$\left.\begin{array}{l} \Delta u = 0, \quad u \in C^2(H) \cap C^{0,\alpha}(\overline{H}), \\ u|_{\partial H} \in C^{1,\alpha}(\partial H) \end{array}\right. \implies u \in C^{1,\alpha}(\overline{H})$$

So, we need to show $\kappa \in C^{1,\alpha}(\underbrace{\{|x|=0\}}_{S(C^{(0)})})$ to get blow-up regularity up to the spine.

Take a test fn $\varphi = \varphi(x,y) = \varphi(|x|,y)$ smooth s.t.

- $\varphi \equiv 0$ outside $B^{n+l}_{3/8}(0)$
- $\frac{\partial \varphi}{\partial r} = 0$ on a nbhd of $\{|x|=0\}$ such as $\{|x| < 2\tau_*\}$.

Fix a direction $i \in \{1, \dots, k+1\}$ orthogonal to the spine and a derivative direction $j \in \{1, \dots, n-1\}$ and consider $e_i \frac{\partial \varphi}{\partial y_j}$ in 1st variation formula to get:

$$\int \nabla^{V_k} x^i \cdot \nabla^{V_k}\left(\frac{\partial \varphi}{\partial y_j}\right) d\|V_k\| = 0$$

Splitting this into graphical and nongraphical pieces.

<u>non-graphical:</u> $$\left| \int_{\text{non-graphical}} \nabla^{V_k} x^i \cdot \nabla^{V_k}\left(\frac{\partial \varphi}{\partial y_j}\right) \right| \le C \left(\int \sum_{j=k+2}^{n+k} |e_j^{\perp}|^2 \right)^{\frac{1}{2}} \cdot \tau_k^{\frac{1}{2}} \lesssim \tau_k^{\frac{1}{2}} \hat{E}_{V_k, C_k} \qquad (*)$$

<u>graphical:</u> Sum over possible i, getting

$$\int_{\{|x| \wedge \tau_k\} \wedge C^{(0)}} \nabla u_k \cdot \nabla\left(\frac{\partial \varphi}{\partial y_j}\right) = o(\hat{E}_{V_k, C_k})$$

We know derivatives converge weakly up to the spine by $(*)$, and so we blowup and pass to the limit

$$\int_{C^{(0)}} \nabla v \cdot \nabla\left(\frac{\partial \varphi}{\partial y_j}\right) = 0$$

We can do a reflection argument to show that the sum v of the components is harmonic on the whole plane, and so its boundary values are smooth (i.e. K is smooth).

From this, we follow Allard: get excess decay, find new cone which turns out to be $\approx$ triple junction, and iterate.

$$\Rightarrow \dots \Rightarrow \underline{\varepsilon\text{-regularity at triple junction!}}$$

$\square$

<u>Remark:</u> Note that this only gives regularity of the <u>sum</u>. We don't actually learn much about the individual pieces, but since they all agree at the boundary. So, in theory we can do this with arbitrary # of planes, as long as

(i) no gaps (which holds for λ) (ii) all mult-1 planes (Minter said so)

4/3 -

Let's recap the whole course, since now it will all come together

① tangent cones & stratification : $\mathrm{sing}(V) = \bigsqcup_{j=0}^{n} \tilde{S}_j$, $\dim_{\mathcal{H}}(S_j) \le j$

② Schoen-Simon regularity & compactness : stable, stationary, $\mathcal{H}^{n-2}(\mathrm{sing}(V)) = 0$
$\Rightarrow$ • Sheeting theorem (close to plane)
• compactness theory w/ codim-7 singular set via Simons' classification

③ Allard regularity : close to mult-1 plane $\Rightarrow$ $C^{1,\alpha}$ pert. of plane

④ Simon's $\curlywedge$ regularity : close to $\curlywedge$ $\Rightarrow$ $C^{1,\alpha}$ pert of $\curlywedge$

§ 5 - Wickramasekera's Regularity Theory

Neshan's regularity theory is a significant (optimal) strengthening of the Schoen-Simon stuff from §2.

We consider the class $\mathcal{S}_\infty$ of integral n-dim varifolds in $B_2^{n+1}(0)$ (codim-1) with $0 \in \mathrm{spt}\|V\|$ and $\|V\|(B_2^{n+1}(0)) < \infty$ and obeying:

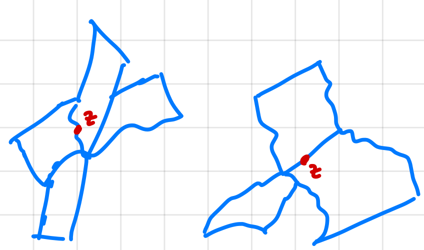
(S1) stationary (for area)

(S2) reg(V) is stable (i.e. if $\Omega \subseteq B_2^{n+1}(0)$ open with $\dim_{\mathcal{H}}(\mathrm{sing}(V) \cap \Omega) \le n-7$, then $\int_{\mathrm{reg}(V) \cap \Omega} |A|^2 \varphi^2 \, d\mathcal{H}^n \le \int_{\mathrm{reg}(V) \cap \Omega} |\nabla \varphi|^2 \, d\mathcal{H}^n$

(S3) V has no classical singularities

Def: (Classical Singularity)

A point $z \in \mathrm{sing}(V)$ is a classical singularity if $\exists \rho > 0$ s.t. $\mathrm{spt}\|V\| \cap B_\rho^{n+1}(z)$ is the union of a finite number of $C^{1,\alpha}$ submanifolds-with-boundary in $B_\rho^{n+1}(z)$, all with a common $C^{1,\alpha}$ boundary containing z, and they do not intersect other than at their common boundary.



Remark: Note that a classical singularity cannot be isolated, and so

$$\mathcal{H}^{n-1}(\operatorname{sing}(V)) \Longleftarrow \text{no classical singularities}$$

$$\Downarrow$$

$$\dim_{\mathcal{H}}(\operatorname{sing}(V)) \leq n-7$$

Neshan's result proves the blue for stationary, stable sets.

In fact, the assumption can be weakened to $\tilde{S}_{n-1} = \emptyset$. If $x \in \tilde{S}_{n-1}$, then near x we are close to a $\lambda \Rightarrow$, which is a classical singularity and cannot happen.

Theorem: (Regularity & Compactness)

Let $(V_k)_k \subseteq \mathcal{S}_\infty$ be s.t. $\limsup_{k\to\infty} \|V_k\|(B_2^{n+1}(0)) < \infty$.

Then, $\exists$ subseq k' of k and $V \in \mathcal{S}_\infty$ with $\dim_{\mathcal{H}}(\operatorname{sing}(V)) \leq n-7$ and $V_{k'} \to V$ as varifolds in $B_2^{n+1}(0)$ and smoothly in $B_2^{n+1}(0) \setminus \operatorname{sing}(V)$

In particular, $V \in \mathcal{S}_\infty \Rightarrow \dim_{\mathcal{H}}(\operatorname{sing}(V)) \leq n-7$ and $\operatorname{reg}(V)$ is orientable.

The main parts of the proof are ruling out $\tilde{S}_{n-1}$ and $\tilde{S}_n$. This is done via the following, which rules out $\tilde{S}_n$ (basically general-mult. Allard):

Theorem (Sheeting Theorem):

Fix $\Lambda \in [1,\infty)$. Then, $\exists\, \varepsilon(n,\Lambda) > 0$ s.t.:

If $V \in \mathcal{S}_\infty$, $\frac{1}{\omega_n 2^n}\|V\|(B_2^{n+1}(0)) \leq \Lambda$, and

$\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|V\| \cap (\mathbb{R}\times B_1^n(0)), \{0\}\times B_1^n(0)) < \varepsilon$, then

$$V \llcorner (\mathbb{R}\times B_{\frac{1}{2}}^n(0)) = \sum_{j=1}^{Q} |\operatorname{graph}(u_j)|$$

for some $Q \in \mathbb{N}$, where $u_j \in C^\infty(B_{\frac{1}{2}}^n(0))$ minimal graphs with $u_1 < u_2 < \dots < u_Q$ and $\|u_j\|_{C^{1,\alpha}(B_{\frac{1}{2}})} \leq C\hat{E}_V$ $= \int_{\mathbb{R}\times B_1^n} \operatorname{dist}^2(x, \{0\}\times\mathbb{R})\, d\|V\|(x)$

Note that if we know a-priori that $\operatorname{sing}(V)$ is small, then this is just Schoen-Simon.

For $\tilde{S}_{n-1}$, we have:

Theorem (Minimum distance theorem):

Let C be an n-dim (stationary) cone with $\dim(S(C)) = n-1$ (= half-hyperplanes with same boundary). Then, $\exists\, \varepsilon(n, C)$ s.t. if $V \in S_\infty$ with $\Theta_V(0) \geq \Theta_C(0)$ and $\frac{1}{\omega_n 2^n}\|V\|(B_2^{n+1}(0)) \leq \Theta_C(0) + \frac{1}{4}$,

<u>then</u> $\operatorname{dist}_H(\operatorname{spt}\|V\| \cap B_1^{n+1}(0),\ \operatorname{spt}\|C\| \cap B_1) \geq \varepsilon$.

So, it holds all the way down to $n-2$, from which Schoen-Simon kicks in.

Overview of strategy:

In Schoen-Simon, multiplicity was irrelevant. Here, it matters. We "stratify" by density as follows:

1 inductive step

$$\begin{tikzcd}
\text{Assumptions} \arrow[r, Rightarrow] & \{\Theta_V \leq 1\} \arrow[r, Rightarrow] & \{\Theta_V < \frac{3}{2}\} \arrow[d, Rightarrow] \\
\{\Theta_V \leq 2\} \arrow[d, Rightarrow] & \{\Theta_V < 2\} \arrow[l, Rightarrow] & \{\Theta_V \leq \frac{3}{2}\} \arrow[l, Rightarrow] \\
\{\Theta_V < \frac{5}{2}\} \arrow[r, Rightarrow] & \{\Theta_V \leq \frac{5}{2}\} \arrow[r, Rightarrow] & \{\Theta_V < 3\} \ \ldots
\end{tikzcd}$$

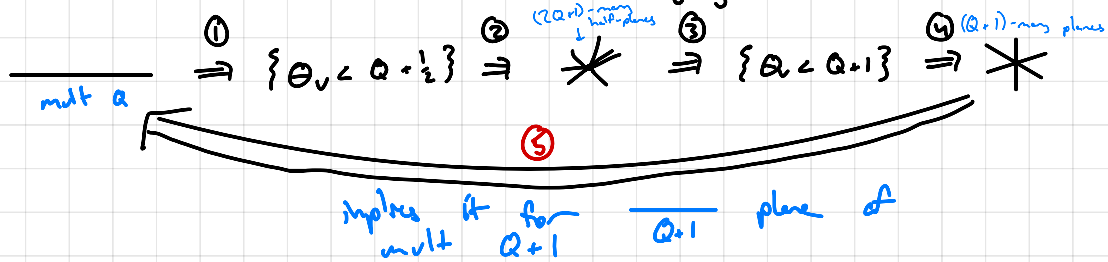
- using stratification + Schoen-Simon-type argument
- $\approx$ Leon's argument for triple junctions (need to show we have no gaps, but points of lower density are already covered via induction)

The last pieces are going from

which is the heart of the proof with new ideas needed.

4/10 -

Recall from last time that we are going for an induction argument



Implications ① – ④ are pretty much what we've done so far in the course. We focus on ⑤.

Proceed to understand the situation when:

- close to a hyperplane of mult. Q
- sheeting theorem holds for planes of mult. $< Q$ <u>and</u>
- minimal distance theorem holds for classical cones of density $\leq Q$

The game plan is as always:

① Take Lipschitz approx. with understanding of the "bad set"

things get tough here → ② blow-up, and understand behavior of blow-ups, ideally showing a $C^{1,\alpha}$ integral estimate

ideally, its Q harmonic functions

③ pass estimate back to manifold via an excess decay lemma

proof by contradiction, suppose we could keep zooming in and get closer to the linear picture, and then use the linear decay

④ iterate to conclude

Theorem: (General Lipschitz Approx)

Fix $Q \in \mathbb{N}$ and $\sigma \in (0,1)$. Then $\exists\, \varepsilon_0(n, Q, \sigma) > 0$ s.t.:

If V is a SIV (n-dim) in $B_2^{n+1}(0)$ s.t.

$$(*) \quad \bullet \ \frac{1}{\omega_n 2^n} \|V\|(B_2^{n+1}(0)) < Q + \overset{\text{can be } 1-\delta}{\widetilde{\tfrac{1}{2}}} \quad \text{and} \quad Q - \tfrac{1}{2} \le \frac{\|V\|(\mathbb{R} \times B_1^n)}{\omega_n} < Q + \tfrac{1}{2}$$

$$\bullet \ \hat{E}_V^2 \equiv \int_{\mathbb{R} \times B_1^n(0)} |x'|^2 \, dV(x) < \varepsilon_0 \quad \text{(L^2 height excess)}$$

then $\exists\, \Sigma \subseteq B_\sigma^n(0)$ (explicit) s.t.

(a) $\mathcal{H}^n(\Sigma) + \|V\|(\mathbb{R} \times \Sigma) \le C \hat{E}_V^2$

(b) $\exists$ Lipschitz $u^1, \dots, u^Q : B_\sigma^n(0) \to \mathbb{R}$ with $\mathrm{Lip}(u^j) \le \frac{1}{2}$, $\sup_{B_\sigma} |u^j| \le C \hat{E}_V^{\frac{2}{n+1}}$, $\underbrace{u^1 \le \dots \le u^Q}_{\text{no higher codim analogue}}$, and

$$V \llcorner \left(\mathbb{R} \times (B_\sigma^n(0) \setminus \Sigma)\right) = \sum_{j=1}^{Q} |\mathrm{graph}(u^j)|$$

Here, C depends on n, Q, σ.

Proof: omitted :).

$\square$

Blow-ups

The above takes care of ①. So, let's construct the blow-ups.

Let $(V_k)_k \subseteq \mathcal{S}_\infty$ be s.t. $(*)$ holds for all k and $\hat{E}_{V_k} \to 0$. Fix $\sigma \in (0,1)$; $\forall k$ large we get from the theorem that $\exists$ Lip. functions $u_k^j : B_\sigma(0) \to \mathbb{R}$ with $u_k^1 \le \dots \le u_k^Q$, $\mathrm{Lip}(u_k^j) \le \frac{1}{2}$, for which

$$V_k \llcorner \left(\mathbb{R} \times (B_\sigma(0) \setminus \Sigma_k)\right) = \sum_{j=1}^{Q} |\mathrm{graph}(u_k^j)|$$

and $\|V_k\|(\mathbb{R} \times \Sigma_k) + \mathcal{H}^n(\Sigma_k) \le C \hat{E}_{V_k}^2$

As before, we seek $W^{1,2}$ estimates:

$$\int_{B_\sigma} \underbrace{|u_k|^2}_{:= \sum_{j=1}^{Q} |u_k^j|^2} = \underbrace{\int_{B_\sigma \setminus \Sigma_k} |u_k|^2}_{\le \text{Jacobian factor} \cdot \int_{\mathbb{R} \times (B_\sigma \setminus \Sigma_k)} |x'|^2 \, d\|V_k\| \ \lesssim \hat{E}_{V_k}^2} + \overbrace{\int_{\Sigma_k} \underbrace{|u_k|^2}_{\le 1}}^{\le 1 \cdot \mathcal{H}^n(\Sigma_k) \lesssim \hat{E}_{V_k}^2}$$

and $$\int_{B_\sigma}|Du_k|^2 = \int_{B_\sigma\setminus\Sigma_k}|Du_k|^2 + \underbrace{\int_{\Sigma_k}\underbrace{|Du_k|^2}_{\le 1}}_{\lesssim \mathcal{H}^n(\Sigma_k)\lesssim \hat{E}_{V_k}^2}$$

Note that when V_k agrees with u_k^j and the tangent spaces coincide,

Here, the unit normal to graph(u_k^j) is $\dfrac{1-Du_k^j}{\sqrt{1+|Du_k^j|^2}}$

$(x\mapsto(x,u(x)))$
$(1,0,\dots,0,D_1u)$
$(0,1,\dots,0,D_2u)$

$$\Rightarrow \nabla^{V_k}x^1 = P_{T_xV}(\nabla^{\mathbb{R}^{n+1}}x^1) = \nabla^{\mathbb{R}^{n+1}}x^1 - P^\perp_{T_xV}(\underbrace{\nabla^{\mathbb{R}^{n+1}}x^1}_{e_1})$$

$$\Rightarrow |\nabla^{V_k}x^1|^2 = \frac{|Du_k^j|^4+|Du_k^j|^2}{(1+|Du_k^j|^2)^2} = \frac{|Du_k^j|^2}{1+|Du_k^j|^2}$$

tilt excess ↓

So, $$\int_{B_\sigma\setminus\Sigma_k}|Du_k|^2 = \int_{B_\sigma\setminus\Sigma_k}\frac{|Du_k|^2}{\sqrt{1+|Du_k|^2}}\cdot\sqrt{1+|Du_k|^2}\lesssim 2E_{V_k}^2$$

Reverse Poincaré allows us to get that $\|u_k\|_{W^{1,2}(B_\sigma)}\le C\hat{E}_{V_k}$

So, set $v_k := u_k/\hat{E}_{V_k}$. By Rellich compactness and a diagonal argument to take $\sigma\uparrow 1$, $\exists$ subseq $v_{k'}\to v\in W^{1,2}_{loc}(B_1)\cap L^2(B_1)$, where the convergence is strongly in $L^2_{loc}(B_1)$ and weakly in $W^{1,2}_{loc}(B_1)$.

As usual, we call $v\equiv(v^j)_{j=1}^Q$ a blow-up. Write $\mathfrak{B}_Q$ for the class of all such blow-ups attainable in this way given the assumptions.

What can be said about $\mathfrak{B}_Q$?

Prop:

(B1) $\mathfrak{B}_Q\subseteq W^{1,2}_{loc}(B_1)\cap L^2(B_1;\mathbb{R}^Q)$

(B2) if $v\in\mathfrak{B}_Q$, then $v^1\le\dots\le v^Q$ a.e.

(B3) if $v\in\mathfrak{B}_Q$, then $\Delta v_{av}=0$, where $v_{av}:=\frac{1}{Q}\sum_{j=1}^{Q}v^j$

closure properties (B5) if $v\in\mathfrak{B}_Q$, then

- if $v\not\equiv 0$ on $B_\sigma(z)$ for $z\in B_1$, $\sigma\in(0,\frac{3}{8}(1-|z|))$, then $$\tilde{v}_{z,\sigma}(\cdot):=\frac{v(z+\sigma(\cdot))}{\|v(z+\sigma(\cdot))\|_{L^2(B_1)}}\in\mathfrak{B}_Q$$ translating and rescaling creates another blowup
- $v\circ\gamma\in\mathfrak{B}_Q$ for all orthogonal rotations γ of $\mathbb{R}^n$.
- if $v\not\equiv\ell_v$ in B_1, where $\ell_v(x):=v_{av}(0)+\langle x, Dv_{av}(0)\rangle$, then $$\frac{v-\ell_v}{\|v-\ell_v\|_{L^2(B_1)}}\in\mathfrak{B}_Q$$

(compactness) (B6) if $(v_k)_k\subseteq\mathfrak{B}_Q$, then $\exists\,(k')\subseteq(k)$ subseq. and $v\in\mathfrak{B}_Q$ s.t. $v_{k'}\to v$ strongly in $L^2_{loc}(B_1)$ and weakly in $W^{1,2}_{loc}$.

None of the above depend on stability or the lack of classical singularities. There are also:

(B4)

(no classical sing. in blow-up) (B7)

Proof: As in Allard, get $\int_{\mathbb{R}\times B_\sigma(0)} \langle \nabla^{V_k} x', \nabla^{V_k} \tilde{\xi} \rangle \, d\|V_k\| = 0$, where $\tilde{\xi}$ is extension of some $\xi \in C^1_c(B^n_\sigma(0))$. As before, we get

$$\sum_{j=1}^{Q} \int_{B_\sigma} \langle Du_k^j, D\xi \rangle = o(\hat{E}_{V_k}) \xRightarrow{k\to\infty} V_{av} \text{ weakly harmonic} \Rightarrow V_{av} \text{ harmonic}$$

For (v), $\tilde{V}_k := (\eta_{(0,z),\sigma})_\# V_k$ blows up to the desired $\tilde{v}_{z,\sigma}$.
Same with rotations.
The last part requires fiddling.

For (vi), take $(V_k^\ell)_{\ell=1}^\infty \subseteq S_\infty$ with blow-up v_k. Choose ℓ_k large s.t. $\left\| \hat{E}_{V_k^{\ell_k}}^{-1} u_k - v_k \right\|_{L^2(B_{1-\frac{1}{k}})} \leq \frac{1}{2}$. This states that the two sequences share a limit.

more detailed proofs in the notes

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Last time, we constructed blowups in a more general setting and used stationarity to prove global properties of the blow-up. Now, we use stability and the lack of classical singularities to derive <u>local</u> properties:

(Hardt-Simon Dichotomy) (B4) Let $v \in \mathfrak{B}_Q$. Then, $\forall z \in B_1$, at least <u>one</u> of the following holds:

(B4 I) The Hardt-Simon inequality:

$\ell_{v_{av},z}(x) := v_{av}(z) + \langle x-z, Dv(z) \rangle$

can be replaced with the linear piece $\ell_{v_{av},z}$

$$\sum_{j=1}^{Q} \int_{B_{\rho/2}} R_z^{2-n} \left| \frac{\partial}{\partial R_z}\left(\frac{v^j - v_{av}(z)}{R_z}\right)\right|^2 \leq C \rho^{-n-2} \int_{B_\rho(z)} |v - \ell_{v_{av},z}|^2$$

$R_z(x) := |x-z|$

holds $\forall \rho \in (0, \frac{3}{8}(1-|z|)]$, $C = C(n, Q)$.

(B4 II) $\exists\, \sigma_1 = \sigma_1(z) \in (0, 1-|z|)$ s.t. v is harmonic on $B_{\sigma_1}(z)$.

$\Delta v^j = 0 \ \forall j$

The heuristic is that having good density points yields (B4 I), (proved by blowing up monotonicity formula) whereas if there are gaps, then we can use the inductive results about places of density $< Q$, apply Schoen-Simon and sheeting, and prove harmonicity. This uses stability.

The final property uses the notion of classical singularities (and also stability):

classical singularities in blow-ups induce classical singularity in V

(B7) If $v \in \mathfrak{B}_Q$ is s.t. graph(v) is a classical cone, then in fact $v^1 = v^2 = \ldots = v^Q = L$ for some linear L.

<u>Theorem:</u>

If $v \in \mathfrak{B}_Q$, <u>then</u> $v^1, \ldots, v^Q$ are harmonic. Moreover, if (B4 I) holds anywhere, then in fact $v^1 = \ldots = v^Q$ coincide.

<u>Remark:</u> Using this and the density dichotomy, either (B4 I) holds somewhere and the linear pieces coincide and so we can iterate and stay close to planes, <u>OR</u> (B4 II) holds everywhere, there are no points of Q-density, and so we are in the $\{\Theta_V < Q\}$ regime, which we understand by induction.

This + B4 + B7 $\Rightarrow$ excess decay dichotomy $\Rightarrow$ sheeting theorem

Let's first look at proving B4.

Lemma: (height at good density points)

Fix $Q \in \mathbb{N}$. Then, $\exists\, \varepsilon_1(n,Q) \in (0,1)$ s.t.:

if V is a SIV on $B_2^{n+1}(0)$ obeying

- $\frac{1}{\omega_n 2^n} \|V\|(B_2^{n+1}(0)) < Q + \frac{1}{2}$
- $Q - \frac{1}{2} \le \frac{1}{\omega_n} \|V\|(\mathbb{R} \times B_1^n(0)) < Q + \frac{1}{2}$
- $\hat{E}_V < \varepsilon_1$,

then $\forall\, z = (z', \tilde{z}) \in \operatorname{spt}\|V\| \cap (\mathbb{R} \times B_{\frac{1}{16}}^n(0))$ with $\Theta_V(z) \ge Q$ we have that $|z'| \le C\hat{E}_V$.

Proof: Monotonicity formula gives:

$$\frac{1}{\omega_n} \int_{B_{\frac{3}{8}}^{n+1}(z)} \frac{|(x-z)^\perp|^2}{|x-z|^{n+2}}\, d\|V\|(x) = \frac{\|V\|(B_{\frac{3}{8}}^{n+1}(z))}{\omega_n (\frac{3}{8})^n} - \Theta_V(z)$$

Provided ε_1 small,

$$\begin{aligned}
\|V\|(B_{3/8}^{n+1}(z)) &\le \|V\|(\mathbb{R} \times B_{3/8}^n(\tilde{z})) \le \|V\|(\mathbb{R} \times (B_{3/8}^n(\tilde{z}) \setminus \Sigma)) + \|V\|(\mathbb{R} \times (B_{\frac{3}{8}}^n(\tilde{z}) \cap \Sigma)) \\
&= \sum_{j=1}^{Q} \int_{B_{\frac{3}{8}}^n(\tilde{z}) \setminus \Sigma} \sqrt{1 + |Du_j|^2}\, dx + \|V\|(\mathbb{R} \times \Sigma) \\
&\le \sum_{j=1}^{Q} \int_{B_{\frac{3}{8}}^n(\tilde{z})} \sqrt{1 + |Du_j|^2}\, dx + C\hat{E}_V^2
\end{aligned}$$

So, as $\Theta_V(z) \ge Q$,

$$\begin{aligned}
\frac{\|V\|(B_{3/8}^{n+1}(z))}{\omega_n (3/8)^n} - \Theta_V(z) &\le \sum_{j=1}^{Q} \int_{B_{3/8}^n(\tilde{z})} \overbrace{\left(\sqrt{1 + |Du_j|^2} - 1\right)}^{= \frac{|Du_j|^2}{1 + \sqrt{1+|Du_j|^2}} \le \frac{1}{2}|Du_j|^2} dx + C\hat{E}_V^2 \\
&\le \int_{\underbrace{B_{3/8}^n(\tilde{z})}_{\subseteq B_{7/16}^n(0)}} |Du|^2\, dx + C\hat{E}_V^2 \le C\hat{E}_V^2
\end{aligned}$$

$$\Longrightarrow \int_{B_{\frac{3}{8}}^{n+1}(z)} \frac{|(x-z)^\perp|^2}{|x-z|^{n+2}}\, d\|V\| \le C\hat{E}_V^2.$$

We may also bound the LHS:

$$\begin{aligned}
\int_{B_{\frac{3}{8}}^{n+1}(z)} \frac{|(x-z)^\perp|^2}{|x-z|^{n+2}}\, d\|V\| &\ge 4^{n+2} \int_{B_{\frac{1}{4}}^{n+1}(z)} \underbrace{|(x-z)^\perp|^2}_{= (x^1 - z^1)e_1^\perp + \sum_{j=2}^{n+1} (x^j - z^j) e_j^\perp} d\|V\| \\
&\overset{\text{using } |a|^2 \ge \frac{1}{2}|a+b|^2 - |b|^2}{\ge} 4^{n+2} \int_{B_{\frac{1}{4}}^{n+1}(z)} \frac{1}{2} \underbrace{|x^1 - z^1|^2}_{\ge \frac{1}{2}|z^1|^2 - |x^1|^2} |e_1^\perp|^2 - \underbrace{4^{n+2} \int_{B_{\frac{1}{4}}^{n+1}(z)} \sum_{j=2}^{n+1} \underbrace{|x^j - z^j|^2}_{\le \frac{1}{4}} |e_j^\perp|^2}_{\le \text{ tilt excess } \lesssim \hat{E}_V^2} \\
&\ge 4^n \int_{B_{\frac{1}{4}}^{n+1}(z)} |z^1|^2 |e_1^\perp|^2\, d\|V\| - 4^{n+1} \int_{B_{\frac{1}{4}}^{n+1}(z)} |x^1|^2 \underbrace{|e_1^\perp|^2}_{\le 1}\, d\|V\| - C\hat{E}_V^2
\end{aligned}$$

$$\geq 4^n |z'|^2 \int_{B^{n+1}_{1/4}(z)} |e_1^\perp|^2 \, d\|V\| - C \hat{E}_V^2$$

We can throw away the bad set Σ with error $\lesssim \hat{E}_V^2$, and so

$$\geq 4^n |z'|^2 \int_{B^n_{1/8}(\tilde{z}) \setminus \Sigma} \overbrace{\frac{1}{\sqrt{1+|Du_j|^2}}}^{\geq \frac{1}{\sqrt{1+\frac{1}{4}}}} \, dx - C \hat{E}_V^2$$

$$\geq C |z'|^2 \mathcal{H}^n(B^n_{1/8}(\tilde{z})) - C \hat{E}_V^2$$

$$\Longrightarrow \quad |z'|^2 \lesssim \hat{E}_V^2.$$

$\square$

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i showed up late, go over how to use previous lemma to show the following:

Prop:

For any $z = (z', \tilde{z}) \in \operatorname{spt}\|V\| \cap (\mathbb{R} \times B_{1/8}(0))$ with $\Theta_V(z) \geq Q$, we have

$C_* = C_*(n, Q)$

$$\sum_{j=1}^{Q} \int_{B_{\frac{1}{2}}^{n}(\tilde{z}) \setminus \Sigma} \left(\frac{R_{\tilde{z}}^2}{|u_j - z'|^2 + R_{\tilde{z}}^2} \right)^{\frac{n+2}{2}} R_{\tilde{z}}^{2-n} \left| \frac{\partial}{\partial R_{\tilde{z}}} \left(\frac{u^j - z'}{R_{\tilde{z}}} \right) \right|^2 d\tilde{x} \leq C_* \hat{E}_V^2 \qquad (*)$$

Using this, we can prove:

(Hardt-Simon Dichotomy) (B4) Let $v \in \mathfrak{B}_Q$. Then, $\forall z \in B_1$, at least one of the following holds:

(B4 I) The Hardt-Simon inequality:

$\ell_{v_{av},z}(x) := v_{av}(z) + \langle x - z, Dv(z) \rangle$

can be replaced with the linear piece $\ell_{v_{av},z}$

$$\sum_{j=1}^{Q} \int_{B_{\rho/2}} R_z^{2-n} \left| \frac{\partial}{\partial R_z} \left(\frac{v^j - v_{av}(z)}{R_z} \right) \right|^2 \leq C \rho^{-n-2} \int_{B_\rho(z)} |v - \ell_{v_{av},z}|^2$$

$R_z(x) := |x - z|$

holds $\forall \rho \in (0, \frac{3}{8}(1 - |z|)]$, $C = C(n, Q)$.

(B4 II) $\exists \sigma_1 = \sigma_1(z) \in (0, 1 - |z|)$ s.t. v is harmonic on $B_{\sigma_1}(z)$.

$\Delta v^j = 0 \ \forall j$

Proof of (B4): Let $v \in \mathfrak{B}_Q$ and let $\tilde{z} \in B_1^n(0)$ be s.t. (B4 I) fails for v at $\tilde{z}$. By (B5 II), $\tilde{v} := \dfrac{v - \ell_{v_{av},z}}{\| v - \ell_{v_{av},z} \|_{L^2(B_1)}} \in \mathfrak{B}_Q$ (if $v = \ell_{av,z}$ then done).

Let $(V_k)_k \subseteq \mathcal{S}_\infty$ be s.t. $\tilde{v}$ is the blow-up of $(V_k)_k$.

Claim: $\exists \sigma_1 = \sigma_1(\tilde{z}) > 0$ s.t. $\forall k$ suff. large,

$$z \in \operatorname{spt}\|V_k\| \cap (\mathbb{R} \times B_{\sigma_1}^n(\tilde{z})) \implies \Theta_{V_k}(z) < Q$$

(if not (B4 I) @ z, then $\exists$ open set of gaps)

Proof of claim: If not, then we have (up to subseq.) $\exists z_k \in \operatorname{spt}\|V_k\|(\mathbb{R} \times B_{1/k}(\tilde{z}))$ with $\Theta_{V_k}(z_k) \geq Q$.
Now fix $\rho \in (0, \frac{3}{8}(1 - |\tilde{z}|)]$, and consider $\tilde{V}_k := (\eta_{z_k,\rho})_\# V_k$, still having $\tilde{V}_k \to Q|\{0\} \times \mathbb{R}^n|$

mult. Q n-plane

Now, apply $(*)$ to $(\tilde V_k)_k$ with $z=0$ and change variables to get

$$\sum_{j=1}^{Q} \int_{B_{\rho/2}(\tilde z_k)\setminus \Sigma_k} \left(\frac{R_{\tilde z_k}^2}{|u_k^j - z_k'|^2 + R_{\tilde z_k}^2}\right)^{\frac{n+2}{2}} R_{\tilde z_k}^{2-n}\left|\frac{\partial}{\partial R_{\tilde z_k}}\left(\frac{u_k^j - z_k'}{R_{\tilde z_k}}\right)\right|^2 \qquad (\dagger)$$
$$\le C_* \rho^{-n-2}\int_{B_\rho(\tilde z_k)} |x'-z'|^2 \, d\|V_k\|$$

Dividing both sides by $\hat E_{V_k}^2$: $\forall k$ large, we know $\underbrace{|z_k'| \le C}_{\text{depends on } \tilde z_k}\hat E_{V_k}$, and so (up to subseq.), $\frac{z_k'}{\hat E_{V_k}} \to y \in \mathbb{R}$

Taking $k\to\infty$ (and being careful about domains $B^n_{\rho/2}(\tilde z_k)\setminus\Sigma_k$),

$$\sum_{j=1}^{Q}\int_{B^n_{\rho/2}(\tilde z)} R_{\tilde z}^{2-n}\left|\frac{\partial}{\partial R_{\tilde z}}\left(\frac{\tilde v^j - y}{R_{\tilde z}}\right)\right|^2 \le C_*\rho^{-n-2}\int_{B_\rho(z)}|\tilde v^j - y|^2$$

$$\Rightarrow \int_{B^n_{\rho/2}(\tilde z)} R_{\tilde z}^{2-n}\left|\frac{\partial}{\partial R_{\tilde z}}\left(\frac{\tilde v_{av}-y}{R_{\tilde z}}\right)\right|^2<\infty$$

Next, $\tilde v_{av}$ smooth $\Rightarrow$ $\overset{=0 \text{ by def. of } \tilde v}{\tilde v_{av}(\tilde z)} = y$ $\Rightarrow$ $y=0$.
Since this limit is independent of ρ, we have shown that $\forall \rho\in(0,\frac{3}{4}(1-|z|)]$,

$$\sum_{j=1}^{Q}\int_{B^n_{\rho/2}(\tilde z)} R_{\tilde z}^{2-n}\left|\frac{\partial}{\partial R_{\tilde z}}\left(\frac{\tilde v^j}{R_{\tilde z}}\right)\right|^2\le C_*\rho^{-n-2}\int_{B^n_\rho(\tilde z)}|\tilde v|^2$$

This is Hardt-Simon, and so (B4 I) holds at $\tilde z$, $*$. □

The claim, together with Schoen-Simon, gives that (B4 II) holds at $\tilde z$ with $\theta=\theta_1/2$. So, (B4). □

We have proven (B1) – (B6). We will now show that all the properties (B1) – (B7) together show that v^j harmonic $\forall j$, after which we will go over (B7). The main prop. is

Note: One can prove two facts about blow-ups:

Fact 1: $v\in\mathcal{B}_Q \Rightarrow v\in \overset{\text{Hölder}}{C^{0,\alpha}}$ $\forall \alpha\in(0,1)$ with estimates as $\alpha\uparrow 1$.

Fact 2: If $v\in\mathcal{B}_Q$ is homogeneous of degree 1 on an annulus $B_1(0)\setminus\overline{B_r(0)}$, then v is homo. of degree 1 on $B_1(0)$.

These facts can be used $\overset{\text{see Neshan's pf.}}{\text{(but aren't needed)}}$ to show the following:

Proposition:

recall that this is what's required to push Leon's triple junction stuff through

Suppose $v \in \mathcal{B}_Q$ is homogeneous of degree 1. Then, $v^1 \equiv v^2 \equiv \dots \equiv v^Q \equiv L$ ← linear!

Proof: Let $v \in \mathcal{B}_Q$ be homo. of degree 1. Then, since v_{av} is harmonic, it is harmonic + homo of deg. 1 $\Rightarrow$ linear $\Rightarrow v_{av} = \ell_{v_{av},0}$

If $v^j = \ell_{v_{av},0}$ $\forall j$, done. Otherwise, (B5 III) gives

$$v_* := \frac{v - \ell_{v_{av},0}}{\|v - \ell_{v_{av},0}\|_{L^2(B_1)}} = \frac{v - v_{av}}{\|v - v_{av}\|_{L^2(B_1)}}$$

Thus, it suffices to prove the result when $v_{av} \equiv 0$ and $\|v\|_{L^2(B_1)} = 1$.
So, we look at $\tilde{\mathcal{B}}_Q := \left\{ v \in \mathcal{B}_Q : \begin{array}{l} v_{av} = 0, \quad \|v\|_{L^2(B_1)} = 1, \\ v \text{ homo deg. } 1 \end{array} \right\}$ ← if this is empty, we're done

Let $\tilde{v}$ be a homo. deg. 1 extension of $v \in \tilde{\mathcal{B}}_Q$ to $\mathbb{R}^n$.
Recall from cone stratification that homogeneous structures are translation-invariant under subspaces. Write $S(\tilde{v})$ for the set of $z \in \mathbb{R}^n$ for which $\tilde{v}$ is invariant under translation by z.

$\tilde{v}$ homo deg. 1 $\Rightarrow$ $S(\tilde{v})$ is a subspace $\Rightarrow$ $\tilde{\mathcal{B}}_Q = \bigcup_{j=0}^{n} H^j$ where $H^j := \left\{ v \in \tilde{\mathcal{B}}_Q : \dim(S(\tilde{v})) \leq n-j \right\}$

<u>The goal is to show $\tilde{\mathcal{B}}_Q = \emptyset$,</u> since then $v^j = \ell_{v_{av},0}$ $\forall j$.

Note:
- $H_0 = \emptyset$ since $v \in H^0 \Rightarrow v \equiv 0$, $\nsim$ to $\|v\|_{L^2} = 1$
- $H_1 = \emptyset$ exactly by (B7) ← only place (B7), i.e. classical singularities, appears at all.

We now claim $H_j = \emptyset$ $\forall j$; if we can prove this then we are done. If not, let $d \in \{2, 3, \dots, n\}$ be minimal s.t. $H_d \neq \emptyset$, and fix $v \in H_d$. For notation, set

closed in B_1 → $\Gamma_v := \left\{ z \in B_1 : \begin{array}{l} \text{(B4 I) holds at } z \text{ and} \\ v \not\equiv 0 \text{ on a nbhd of } z \end{array} \right\}$

The main claim to get C^1 reg. away from $S(\tilde{v})$ is a reverse Hardt-Simon inequality.

Claim: Fix $K \subseteq B_1(0) \setminus S(\tilde{v})$ compact. Then, $\exists\, \varepsilon(v, K, n, Q) \in (0, d(K, S(\tilde{v}) \cup \partial B_1))$ s.t. the following holds:

$\forall z \in \underbrace{K \cap \Gamma_v}_{\text{compact}}$ and every $\rho \in (0, \varepsilon]$, we have:

(reverse Hardt-Simon)
$$\sum_{j=1}^{Q} \int_{B_\rho(z) \setminus B_{\rho/2}(z)} R_z^{2-n} \left| \frac{\partial}{\partial R_z}\left(\frac{v^j}{R_z}\right)\right|^2 \geq \varepsilon \rho^{-n-2} \int_{B_\rho(z)} |v|^2$$

Remark: Where Hardt-Simon holds too, we see that $\int_{B_{\rho/2}(z)} \cdots \lesssim \rho^{n-2} \int_{B_\rho(z)} \cdots$, a Campanato-type estimate.

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Proof of claim: Suppose BWOC false. Then $\forall i \geq 1$, $\exists\, \varepsilon_i \downarrow 0$ and points $z_i \in \Gamma_v' \cap K$ (WLOG with $z_i \to z \in \Gamma_v' \cap K$) and radii $\rho_i \downarrow 0$ with

$$(*) \quad \sum_{j=1}^{Q} \int_{B_{\rho_i}(z_i) \setminus B_{\rho_i/2}(z_i)} R_{z_i}^{2-n} \left| \frac{\partial}{\partial R_{z_i}} \left(\frac{v^j}{R_{z_i}} \right) \right|^2 < \varepsilon_i \rho_i^{-n-2} \int_{B_{\rho_i}(z_i)} |v|^2$$

Set $w_i := \dfrac{v(z_i + \rho_i(\cdot))}{\|v(z_i+\rho_i(\cdot))\|_{L^2(B_1)}} \in \mathcal{B}_Q$, and so

recentering and rescaling

by (BSI)

$$(*) \Rightarrow \int_{B_1 \setminus B_{1/2}} R^{2-n} \left| \frac{\partial}{\partial R}\left(\frac{w_i}{R}\right)\right|^2 < \varepsilon_i \qquad (**)$$

By (B6) and the apriori $C^{0,\alpha}$ estimate of blow-ups, we can find subsequence via the compactness property s.t.

Fact 1

$$w_i \to w_* \in \mathcal{B}_Q \quad \begin{array}{l}\text{locally uniformly and}\\ \text{locally weakly in } W^{1,2}(B_1)\end{array}$$

Uniform convergence implies that $(w_*)_{av} = 0$. So, we need to show that $w_* \neq 0$ and is 1-homogeneous to get that $w_* \in \tilde{\mathcal{B}}_Q$.

Subclaim: $w_* \not\equiv 0$

Proof: Observe that if $u \in C^1$, then $\forall r, s \in [\frac{1}{2}, 1]$ and $\omega \in S^{n-1}$ we have

$$\left| \frac{u(r\omega)}{r} - \frac{u(s\omega)}{s} \right| \leq \int_{\frac{1}{2}}^{1} \left| \frac{d}{dt}\left(\frac{u(t\omega)}{t}\right)\right| dt \quad \text{by FTOC}$$

Triangle ineq. and Cauchy-Schwarz gives

$$|u(r\omega)|^2 \leq C(n) \left(|u(s\omega)|^2 + \int_{\frac{1}{2}}^{1} t^{n-1} \left|\frac{d}{dt}\left(\frac{u(t\omega)}{t}\right)\right|^2 dt \right)$$

Integrating over the unit sphere,

$$\int_{S^{n-1}} |u(r\omega)|^2 \, d\omega \leq C \left(\int_{S^{n-1}} |u(s\omega)|^2 \, d\omega + \int_{B_1 \setminus B_{1/2}} \left| \frac{d}{dR}\left(\frac{u}{R}\right)\right|^2 \right)$$

To get integrals over balls, we multiply by r^{n-1} and take $\int_{1/2}^{1} \ldots dr$, then multiply by s^{n-1} and take $\int_{1/2}^{3/4} \ldots ds$ to get (after adding $\int_{B_{1/2}} |u|^2$ to both sides),

$$\int_{B_1} |u|^2 \leq C \int_{B_{3/4}} |u|^2 + C \int_{B_1 \setminus B_{1/2}} \left| \frac{\partial}{\partial R}\left(\frac{u}{R}\right)\right|^2$$

This holds for $u \in C^1$: by approximation, holds for $W^{1,2}$.

Apply this with $u = (w_i^1, \dots, w_i^Q)$ and sum over directions to get

$$\underbrace{\int_{B_1} |w_i|^2}_{\substack{=1\ \forall i \\ \text{by construction}}} \le \underbrace{C\int_{B_{3/4}} |w_i|^2}_{\to \int_{B_{3/4}} |w_*|^2} + \underbrace{C\int_{B_1 \setminus B_{1/2}} \left|\frac{\partial}{\partial R}\left(\frac{w_i}{R}\right)\right|^2}_{\to 0 \text{ by } (**)}$$

So, $1 \le C(n)\int_{B_{3/4}} |w_*|^2 \Rightarrow w_* \not\equiv 0$ on $B_{3/4}$.

$\square$

<u>Subclaim:</u> w_* is homogeneous of degree 1

<u>Proof:</u> $(**)$ gives $\int_{B_1 \setminus B_{1/2}} R^{2-n}\left|\frac{\partial}{\partial R}\left(\frac{w_*}{R}\right)\right|^2 = 0$

$\Rightarrow w_*$ is homo. of deg 1 on $B_1 \setminus B_{1/2}$.

Foot 2 $\Rightarrow w_*$ is homo. of deg 1 on B_1

$\square$

So, (up to normalizing $w_* \leftarrow \frac{w_*}{\|w_*\|_{L^2(B_1)}}$), we get $w_* \in \tilde{B}_Q$.

Each w_i is translation-invariant along $S(\tilde{v})$ by construction, and since $w_i \to w_*$ locally uniformly, then $S(\tilde{v}) \subseteq S(\tilde{w}_*)$. But, since $z \notin S(\tilde{v})$ by construction and w_* is translation-invariant in direction z, $\dim(S(\tilde{v})) = n-d < \dim(S(\tilde{w}_*))$. Contradicts minimality of d.

$\square$

Note that if $z \in \Gamma_v$, Hardt-Simon holds by construction. So, $\forall z \in K \cap \Gamma_v$ and all $\rho \in (0, \varepsilon]$,

$$\int_{B_\rho(z) \setminus B_{\rho/2}(z)} R_z^{2-n}\left|\frac{\partial}{\partial R_z}\left(\frac{v}{R_z}\right)\right|^2 \ge \frac{\varepsilon}{C}\int_{B_{\rho/2}(z)} R_z^{2-n}\left|\frac{\partial}{\partial R_z}\left(\frac{v}{R_z}\right)\right|^2$$

By a technique called "hole filling" (Nestor, i.e. $+ \int_{B_{\rho/2}(z)}(\dots)$),

$$\int_{B_{\rho/2}(z)} R_z^{2-n}\left|\frac{\partial}{\partial R_z}\left(\frac{v}{R_z}\right)\right|^2 \le \xi \int_{B_\rho(z)} R_z^{2-n}\left|\frac{\partial}{\partial R_z}\left(\frac{v}{R_z}\right)\right|^2$$

$\xi = \frac{1}{1+\frac{\varepsilon}{C}} \in (0,1)$

This is a decay of the integral! Now, we can iterate this + interpolate between scales (just like Allard) to get $\forall\ 0 < \sigma \le \rho/2 < \varepsilon/2$,

$$\int_{B_\sigma(z)} R_z^{2-n}\left|\frac{\partial}{\partial R_z}\left(\frac{v}{R_z}\right)\right|^2 \le \beta\left(\frac{\sigma}{\rho}\right)^{2\mu}\int_{B_\rho(z)} R_z^{2-n}\left|\frac{\partial}{\partial R_z}\left(\frac{v}{R_z}\right)\right|^2$$

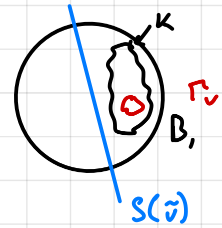
where $\beta = \beta(v, K, n, Q)$ and $\mu = \mu(v, K, n, Q)$. Using Hardt-Simon and reverse Hardt-Simon at z, we in fact get: $\forall\ 0 < \sigma \le \rho/4 \le \varepsilon/8$,

$$\sigma^{-n-2}\int_{B_\sigma(z)} |v|^2 \le C\left(\frac{\sigma}{\rho}\right)^{2\mu}\rho^{-n-2}\int_{B_\rho(z)} |v|^2$$

This looks like a Campanato $C^{1,\mu}$ estimate at z! Usually, we would have decay of $v - \ell_{v,z}$ (recall Hardt-Simon $< \infty \Rightarrow$ linear approx in Allard) so this also tells us that $\ell_{v,z} = 0$ for such z. This makes sense, since by the (BW) dichotomy and choice of Γ_v, these are the good density points and glue everything together.

Using harmonic estimates away from Γ_v, we find $v \in C^{1,\mu}(K)$ by Campanato theory. As $K \subseteq B_1 \setminus S(\tilde{v})$ arbitrary,

$$\underline{v \in C^{1,\mu}(B_1 \setminus S(\tilde{v}))}$$



To finish our contradiction, two more claims:

<u>Claim:</u> $\Gamma_v \subseteq S(\tilde{v})$

<u>Proof:</u> If not, take $z \in \Gamma_v \setminus S(\tilde{v})$ and consider $u^j := v^j - v^{j-1}$. We know $u^j \geq 0$ and u^j is C^1 about z. But $u^j(z) = 0 \Rightarrow \underbrace{Du^j(z) = 0}_{\text{the sheets touch}} \Rightarrow \ldots \Rightarrow \times$ to Hopf boundary point lemma.

$\square$

Since v is translation-invariant along Γ_v, v is determined by some function $f: \mathbb{R}^d \to \mathbb{R}^Q$ $(d \geq 2)$ (quotient out the spine)

where

- $f \in C^1(B_1 \setminus \{0\})$ (as $v \in C^1(B_1 \setminus S(\tilde{v}))$)
- $f \in C^{0,\alpha}(B_1)$ (by Fact 1)
- f is homo of deg. 1
- f is harmonic on $B_1 \setminus \{0\}$

Removable singularity of harmonic functions $\Rightarrow f$ harmonic on B_1.
So, f^j is linear $\forall j$ (sheeting theorem implies $f^1 \equiv \ldots \equiv f^Q \equiv L$) linear on B_1
Furthermore, f avg.-free $\Rightarrow f \equiv 0 \Rightarrow v \equiv 0$, which contradicts that $\|v\|_{L^2(B_1)} = 1$.

$\square$

Finally, we've shown that homogeneous blowups are linear.
We now aim to show that <u>all</u> blowups are harmonic.
It suffices to prove $\mathcal{B}_Q \subseteq C^1(B_1)$ (then we can make the same Hopf boundary point argument to get $\Gamma_v = \emptyset \Rightarrow$ locally harmonic on Γ_v $\Rightarrow$ harmonic)

To prove this, it suffices to prove that $\exists \beta = \beta(n,Q)$ and $\mu = \mu(n,Q)$ s.t. $\forall v \in \mathcal{B}_Q$, $z \in \Gamma_v \cap B_1$, we have the Campanato estimate

$$\sigma^{-n-2} \int_{B_\sigma(z)} |v - \ell_{v_\sigma, z}|^2 \leq \beta \left(\frac{\sigma}{\rho}\right)^{2\mu} \rho^{-n-2} \int_{B_\rho(z)} |v - \ell_{v_\rho, z}|^2 \quad (\forall\, 0 < \sigma \leq \rho/2 < \tfrac{1}{8})$$

Last time, we did this by proving a reverse Hardt-Simon and iterating.
More precisely, we can show in a similar way to last time that

$$\int_{B_1 \setminus B_{1/2}} R^{2-n} \left| \frac{\partial}{\partial R} \left(\frac{v}{R} \right) \right|^2 < \varepsilon_i \downarrow 0$$

$\square$

So, we've shown that

$(B1) - (B7) \implies$ all blow-ups are harmonic!

Next class (the final one :;), we will investigate (B7).

4/29 -

Recall $\underline{(B7):}$ If $v \in \mathcal{B}_Q$ has graph(v) a classical cone, $\underline{\text{then}}$ $v^1 \equiv \dots \equiv v^Q = L$ is linear.

There are a couple cases that could happen:

note that if we have a density Q point, by Hardt-Simon it will "pin" the blowup and so it must all agree

$\underline{\text{Case 1:}}$ If all half-planes on at least one side coincide.

In this case, unique continuation $\implies$ v_{av} is linear $\implies$ WOLOG, $v \equiv 0$ on the half-space $x^2 < 0$, ↙ 2nd coord, not $x \cdot x$ with points $(x^1, \underbrace{x^2, \dots, x^{n+1}}_{B_1^n(0)})$.

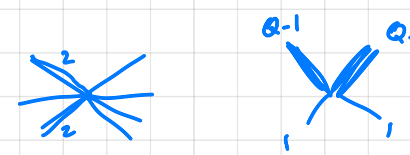
Take the test function ζe^2 in 1^{st} variation formula for

↑ test fn ↖ basis vec

$(V_k)_k \subseteq S_\infty$ with blow-up v. Then, $\sum_{j=1}^{Q} |Dv_j|^2$ is constant across the interface. If $v^1 = a_j x^2$ on the other side, we would need $\sum_{j=1}^{Q} |a_j|^2 = 0$ $\implies a_i \equiv 0 \implies v = 0$.

$\underline{\text{Case 2:}}$ If v splits on both sides.

e.g.

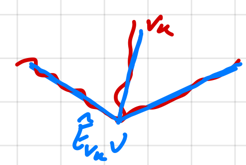


In this case, Hardt-Simon gives in fact that $\forall k$ large,

$$\mathbb{R} \times \left(B^n_{3/4}(0) \setminus \{ |x^2| < \tfrac{1}{16} \} \right) \subseteq \{ \Theta_{V_k} < Q \}$$ (otherwise it wouldn't split)

Now we are in a situation where we can use inductive information to apply Schoen-Simon in this region.

We know that $V_k \sim \hat{E}_{V_k} v$, and $C_k := \text{graph}(\hat{E}_{V_k} v)$ is a classical cone. One can show that V_k is $\underline{\text{much closer}}$ to C_k than it is to the plane, in the sense that



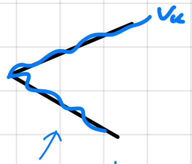
$$\int_{\mathbb{R}\times B_1^n(0)} \mathrm{dist}^2(x, C_k)\, d\|V_k\| + \int_{\mathbb{R}\times(B^n_{1/2}\setminus\{|x^2|<t_0\})} \mathrm{dist}^2(x, \mathrm{spt}\|V_k\|)\, d\|C_k\| \le o(\hat{E}^2_{V_k})$$

two-sided height excess $Q^2_{V_k, C_k}$

i.e. $\overbrace{Q_{V_k, C_k} \ll \hat{E}_{V_k}}^{\text{"hypothesis (H) in Neshan"}}$ ← height to plane. One can also show that the planar approximation of $\{x^1 = 0\}$ is optimal in the sense

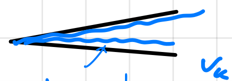
$$\hat{E}_{V_k} \le \underbrace{M(n,Q)}_{\text{explicit!}} \cdot \inf_P \hat{E}_{V_k, P}$$

"Hypothesis (*)" in Neshan



should parameterize as graphs over half-planes

$Q_{V_k, C_k} \ll Q_{V_k, C_k'}$



should parameterize over flattened plane

$Q_{V_k, C_k} \sim Q_{V_k, C_k'}$

To parameterize V_k over C_k, need something to know that C_k is a "good core" to parameterize over.

the only hypothesis we "don't get for free" → "Hypothesis (**)":

Either

(i) C_k consists of exactly 4 distinct half-hyperplanes (no collapsing can occur)

or

(ii) C_k has $p \ge 5$ (distinct) half-hyperplanes and

$$Q_{V_k C_k} < \beta(n,Q) \inf_{\tilde{C}} Q_{V_k, \tilde{C}}$$

← classical cores with $< p$ hyperplanes

Under these hypotheses, one can show that V_k is graphical over C_k and the graphs $\tilde{u}_k$ over C_k obey good L^2 estimates analogous to Leon's L^2 estimates for the triple junction.

Now, we blow-up the reparameterized $\tilde{u}_k$ via $v_k := \dfrac{\tilde{u}_k}{\hat{E}_{V_k, C_k}}$: this is called a fine blow-up.

These $\tilde{u}_k$ are all minimal functions over half-hyperplanes: so, they blow-up to harmonic functions. The fine blow-up is then:

- Q harmonic functions on $\{x^2 < 0\}$
- Q harmonic functions on $\{x^2 > 0\}$

If we can show a boundary regularity statement at $x^2=0$ (such as $C^{1,\alpha}$ up to boundary), then we could run excess decay. This is more complicated than, but similar to, the triple junction case in which we showed the sum was harmonic up to boundary and then split it into v^j harmonic.

Given all the Hypotheses (H, *, **), we can connect the harmonic parts in a $C^{1,\alpha}$ way and we are done. (H) and (*) come freely, and so we must just work with Hypothesis (**).
To accomplish this, we just iterate arguments for when Hypothesis (**) doesn't hold ← what?

So, (B7) is proven (-ish).

$\square$

This concludes the proof of Neshan's paper on stable minimal hypersurfaces. In the last 15 minutes, let's look at some corollaries of Neshan's work.

Corollaries

① Unique Continuation Principle for Singular Minimal Hypersurfaces

Theorem: (Neshan)

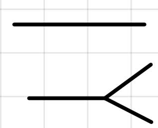
Let V_1, V_2 be stationary integral n-varifolds on a smooth Riemannian manifold (M^{n+1}, g) s.t. $\operatorname{spt}\|V_j\|$ connected and $\mathcal{H}^{n-1}(\operatorname{sing}(V_j)) = 0$. Then,

$$\operatorname{spt}\|V_1\| \neq \operatorname{spt}\|V_2\| \implies \dim_{\mathcal{H}}\left(\operatorname{spt}\|V_1\| \cap \operatorname{spt}\|V_2\|\right) \leq n-1$$

This is "optimal", seen by considering

$V_1 =$

$V_2 =$



Note: Minter mentioned a target cone to a stationary varifold must have directions summing to 0

② Strong Maximum Principle for Singular Minimal Hypersurfaces

Theorem: (Neshan)

Suppose V_1, V_2 are stationary integral n-varifolds on smooth (M^{n+1}, g) with $\operatorname{spt}\|V_j\|$ connected. If
(i) $\operatorname{spt}\|V_2\|$ lies locally on one side of $\operatorname{reg}(V_1)$
(ii) $\mathcal{H}^{n-1}(\operatorname{sing}(V_1)) = 0$ ← no conditions on V_2!

then either $\operatorname{spt}\|V_1\| \equiv \operatorname{spt}\|V_2\|$ or $\operatorname{spt}\|V_1\|, \operatorname{spt}\|V_2\|$ disjoint

③ Min-Max Theory via Allen-Cahn (codim 1)

Define the functional

$$E_\varepsilon(u) := \int_M \varepsilon^2 |\nabla u|^2 + W(u)/\varepsilon^2$$

Use PDE min-max theory for each ε, take the limit $\varepsilon \downarrow 0$. Can use the Morse index to show stability of level sets of the limit. There isn't enough extra structure to use Schoen-Simon, but it is enough for Wickramasekera's work.

This is because you can use a sheeting and stability argument to rule out classical singularities.