MAT 526: Problem Set 2

Due on December 17, 2023

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Problem 1

Let $[n]$ denote the set $\{1, \ldots, n\}$ for $n \in \mathbb{N}$. Consider

(i) a sequence of integers $\{m_j\}_{j\in\mathbb{N}}$

(ii) families of nonempty compact subsets $E_{i_1,\ldots,i_k}$ where each index i_j ranges from 1 to m_j, and k is arbitrary

(iii) a nonnegative real number s and a positive constant c

such that

(a) $d(k) := \max\{\operatorname{diam}(E_{i_1,\ldots,i_k}) : \ i_j \in [m_j]\}$ converges to 0 as $k \to \infty$

(b) $E_{i_1,\ldots,i_k} \subseteq E_{i_1,\ldots,i_{k-1}}$ for every choice of $i_1, \ldots, i_k$

(c) $\sum_{j=1}^{m_{k+1}} \operatorname{diam}(E_{i_1,\ldots,i_k,j})^s = \operatorname{diam}(E_{i_1,\ldots,i_k})^s$

(d) $\sum_{B\cap E_{i_1,\ldots,i_k}\neq\emptyset} \operatorname{diam}(E_{i_1,\ldots,i_k})^s \leq c \operatorname{diam}(B)^s$ for every ball B with $\operatorname{diam}(B) \geq d(k)$

Let

$$E^{(k)} := \bigcup_{\substack{i_1,\ldots,i_k \\ i_j\in[m_j]}} E_{i_1,\ldots,i_k} \quad \text{and} \quad E := \bigcap_{k\in\mathbb{N}} E^{(k)}$$

Show that

$$0 < \mathcal{H}^s(E) < \infty$$

Solution

Proof. We begin by observing some initial facts. By property (b), $E^{(k+1)} \subseteq E^{(k)}$ for all k; to see this, note that every $x \in E^{(k+1)}$ belongs to some $E_{i_1,\ldots,i_k,i_{k+1}}$, which means $x \in E_{i_1,\ldots,i_k}$ by property (b) and so $x \in E^{(k)}$. Next, from property (c) we have that for all $k \in \mathbb{N}$,

$$\sum_{\substack{i_1,\ldots,i_k \\ i_j\in[m_j]}} \operatorname{diam}(E_{i_1,\ldots,i_k})^s = \sum_{\substack{i_1,\ldots,i_k,i_{k+1} \\ i_j\in[m_j]}} \operatorname{diam}(E_{i_1,\ldots,i_k,i_{k+1}})^s$$

By induction, the quantity $C := \sum_{\substack{i_1,\ldots,i_k \\ i_j\in[m_j]}} \operatorname{diam}(E_{i_1,\ldots,i_k})^s > 0$ is preserved as k is varied. We will show that

$$0 < \frac{\omega_s C}{2^{2s} c} \leq \mathcal{H}^s_\delta(E) \leq \frac{\omega_s}{2^s} C < \infty$$

for all $\delta > 0$, from which the main result will follow. So, let $\delta > 0$ be arbitrary.

($\leq 2^{-s}\omega_s C$) Choose k large enough that $d(k) < \delta$, which we know we can do by property (a). Then, the collection $\{E_{i_1,\ldots,i_k} : \ i_j \in [m_j]\}$ is a cover of $E^{(k)}$ with $\operatorname{diam}(E_{i_1,\ldots,i_k}) \leq d(k) < \delta$, and so

$$\mathcal{H}^s_\delta(E^{(k)}) \leq \frac{\omega_s}{2^s} \sum_{\substack{i_1,\ldots,i_k \\ i_j\in[m_j]}} \operatorname{diam}(E_{i_1,\ldots,i_k})^s = \frac{\omega_s C}{2^s}$$

Since $E \subseteq E^{(k)}$, monotonicity of measure gives the desired result.

($\geq 2^{-2s}\omega_s C/c$) Let $\varepsilon > 0$ be arbitrary, and let $\{A_n\}_{n\in\mathbb{N}}$ be a cover of E such that $\operatorname{diam}(A_n) \leq \delta$ for all n and

$$\frac{\omega_s}{2^s} \sum_{n\in\mathbb{N}} \operatorname{diam}(A_n)^s \leq \mathcal{H}^s_\delta(E) + \frac{2^{-2s}}{\omega_s c}\varepsilon$$

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For each A_n, select a ball B_n of diameter $\text{diam}(B_n) = 2\,\text{diam}(A_n)$ around any point of A_n. Then, we certainly have that $A_n \subseteq B_n$ and so $\{B_n\}_{n\in\mathbb{N}}$ covers E. Since E is compact (it is an intersection of finite unions of compact sets), then there must be a finite subcover of balls $\{B_{n_\ell}\}_{\ell=1}^L$ with $E \subseteq \bigcup_{\ell=1}^L B_{n_\ell}$. We have that

$$\sum_{\ell=1}^{L} \text{diam}(B_{n_\ell})^s \leq \sum_{n\in\mathbb{N}} \text{diam}(B_n)^s = 2^s \sum_{n\in\mathbb{N}} \text{diam}(A_n)^s$$
$$\leq \frac{2^{2s}}{\omega_s}\mathcal{H}^s_\delta(E) + \frac{1}{c}\varepsilon$$

Fix k large enough that $d(k) \leq \min_{\ell\in[L]}\{\text{diam}(B_{n_\ell})\}$, which we know we can do by property (a). For each $\ell \in [L]$ define

$$\mathcal{S}_\ell := \{E_{i_1,\dots,i_k} : \; i_j \in [m_j] \text{ and } E_{i_1,\dots,i_k} \cap B_{n_\ell} \neq \emptyset\}$$

Then, by property (d) we know that for each ℓ,

$$\text{diam}(B_{n_\ell})^s \geq \frac{1}{c} \sum_{E_{i_1,\dots,i_k}\in S_\ell} \text{diam}(E_{i_1,\dots,i_k})^s$$

Letting

$$\mathcal{S} := \bigcup_{\ell\in[L]} \mathcal{S}_\ell = \left\{ E_{i_1,\dots,i_k} : \; E_{i_1,\dots,i_k} \cap \left(\bigcup_{\ell\in[L]} B_{n_\ell} \right) \neq \emptyset \right\}$$

and summing over ℓ, we see that

$$\frac{2^{2s}}{\omega_s}\mathcal{H}^s_\delta(E) + \frac{1}{c}\varepsilon \geq \frac{1}{c}\sum_{\ell\in[L]}\sum_{E_{i_1,\dots,i_k}\in S_\ell} \text{diam}(E_{i_1,\dots,i_k})^s \geq \frac{1}{c}\sum_{E_{i_1,\dots,i_k}\in S} \text{diam}(E_{i_1,\dots,i_k})^s$$

We claim that each possible $E_{i_1,\dots,i_k}$ is actually in S. To do so, we will show that for any fixed $E_{i_1,\dots,i_k}$ with $i_j \in [m_j]$, the infinite collection $\mathcal{C} := \{E_{i_1,\dots,i_k}, E^{(1)}, E^{(2)}, \dots, E^{(k)}, \dots\}$ of compact sets has the finite intersection property. Since $E^{(N+1)} \subseteq E^{(N)}$ for each N, this amounts to showing that $E_{i_1,\dots,i_k} \cap E^{(N)} \neq \emptyset$ for every $N \in \mathbb{N}$. However, this follows trivially since if $N \leq k$ then $E_{i_1,\dots,i_k} \cap E^{(N)} = E_{i_1,\dots,i_k}$ and if $N > k$, then for any $E_{i_1,\dots,i_k,\dots,i_N}$ we know

$$\emptyset \neq E_{i_1,\dots,i_k,\dots,i_N} = E_{i_1,\dots,i_k,\dots,i_N} \cap E^{(N)} \subseteq E_{i_1,\dots,i_k} \cap E^{(N)}$$

where $\subseteq$ follows from property (b). So, $\mathcal{C}$ is a collection of compact subsets of $E^{(1)}$ (which is itself a compact set) with the finite intersection property, and therefore the infinite intersection

$$\bigcap \mathcal{C} = E_{i_1,\dots,i_k} \cap \left(\bigcap_{N\in\mathbb{N}} E^{(N)} \right) = E_{i_1,\dots,i_k} \cap E$$

is nonempty. Since $E \subseteq \bigcup_{\ell=1}^L B_{n_\ell}$ by construction, we find that $E_{i_1,\dots,i_k} \in S$. We may therefore say that

$$\frac{2^{2s}}{\omega_s}\mathcal{H}^s_\delta(E) + \frac{1}{c}\varepsilon \geq \frac{1}{c}\sum_{\substack{i_1,\dots,i_k \\ i_j\in[m_j]}} \text{diam}(E_{i_1,\dots,i_k})^s = \frac{C}{c}$$

with $C = \sum_{\substack{i_1,\dots,i_k \\ i_j\in[m_j]}} \text{diam}(E_{i_1,\dots,i_k})^s$ independent of k as described earlier. So,

$$\frac{c2^{2s}}{\omega_s}\mathcal{H}^s_\delta(E) + \varepsilon \geq C$$

Since none of the above quantities depend on ε anymore, taking $\varepsilon \to 0$ gives the desired result. ■

Problem 2

Give an example of a purely 2-unrectifiable set $E \subseteq \mathbb{R}^3$ which is compact, connected, and such that $0 < \mathcal{H}^2(E) < \infty$. Show that E can be even made contractible.

Solution

Proof. We will construct E as a 3-dimensional analog to the purely 1-unrectifiable set made via the $\frac{1}{4}$-Cantor set that we did in lecture.

Let $\alpha := \frac{1}{\sqrt{8}}$. Start with the unit cube $F^{(0)} := [0,1] \times [0,1] \times [0,1]$. Now, let i_j run from 1 to 8 and denote which corner of a cube we are looking at, and let F_{i_1} be the cube of side length α placed in the i_1^{th} corner of the unit cube. Define $F^{(1)} := \bigcup_{i_1=1}^{8} F_{i_1}$. Then, $F^{(1)}$ is a disjoint union of 8 cubes of side length α. Repeat this construction where each F_{i_1,i_2} is a cube of side length α^2 located at the i_2^{th} corner of the cube F_{i_1}, and define $F^{(2)} := \bigcup_{i_1,i_2=1}^{8} F_{i_1,i_2}$. Then, $F^{(2)}$ is a disjoint union of 64 cubes of side length α^2. Repeating indefinitely for $k \in \mathbb{N}$, we see that each $F^{(k)}$ is a disjoint union of 8^k cubes, each of side length α^k. Define $F := \bigcap_{k \in \mathbb{N}} F^{(k)}$.

To ensure contractibility (which implies connectedness), we will add 1-dimensional lines to our construction and connect everything. Specifically, for every new cube $F_{i_1,\ldots,i_k}$ at step k, let $G_{i_1,\ldots,i_k}$ be a line segment connecting the two cubes $F_{i_1,\ldots,i_k}$ and $F_{i_1,\ldots,i_{k-1},1}$. In other words, each $G_{i_1,\ldots,i_k}$ connects **FINISH** ∎

Problem 3

Suppose that E is Borel with $0 < \mathcal{H}^\alpha(E) < \infty$. Define the convex density of a point $x \in E$ as

$$D_c^\alpha(E,x) := \frac{2^\alpha}{\omega_\alpha} \lim_{r\to 0} \left[\sup \left\{ \frac{\mathcal{H}^\alpha(U \cap E)}{\operatorname{diam}(U)^\alpha} :\ x \in U,\ U \text{ is convex, and } \operatorname{diam}(U) < r \right\} \right]$$

Show that $D_c^\alpha(E,x) \geq 1$ for $\mathcal{H}^\alpha$-a.e. $x \in E$.

Solution

Proof. For notation define $\gamma := 2^{-\alpha}\omega_\alpha$ and

$$\Psi(x,r) := \sup \left\{ \frac{\mathcal{H}^\alpha(U \cap E)}{\gamma \operatorname{diam}(U)^\alpha} :\ x \in U,\ U \text{ is convex, and } \operatorname{diam}(U) < r \right\}$$

Observe that $\Psi(x,r)$ is monotonically non-increasing in r for each fixed x since the set we are taking the supremum over shrinks and $D_c^\alpha(E,x) = \lim_{r\to 0} \Psi(x,r)$. Define the sets

$$S := \{x \in E :\ D_c^\alpha(E,x) < 1\}$$

and

$$S_k := \left\{ x \in E :\ \Psi\left(x, \frac{1}{k}\right) < \frac{k}{k+1} \right\} \quad (k \in \mathbb{N})$$

Lemma 1.

$$S = \bigcup_{k\in\mathbb{N}} S_k$$

Proof of Lemma. For each $k \in \mathbb{N}$, note that $S_k \subseteq S$ since for each $x \in S_k$ we have

$$D_c^\alpha(E,x) = \lim_{r\to 0} \Psi(x,r) \leq \Psi\left(x, \frac{1}{k}\right) < \frac{k}{k+1} < 1,$$

where the first inequality is by monotonicity of $\Psi(x,\cdot)$, the second inequality follows by construction of S_k, and the third follows since $\frac{k}{k+1} < 1$ trivially. Now, let $x \in S$ be arbitrary. Then, there is some $N \in \mathbb{N}$ such that $D_c^\alpha(E,x) < \frac{N}{N+1}$. Let $\varepsilon := \frac{N}{N+1} - D_c^\alpha(E,x) > 0$. Then, by definition of a limit there is some $r > 0$ such that

$$\Psi(x,r) - D_c^\alpha(E,x) < \varepsilon \implies \Psi(x,r) < \frac{N}{N+1}$$

If we take $k \geq \max\{N, \frac{1}{r}\} \implies \frac{1}{k} < r$, monotonicity of $\Psi(x,\cdot)$ grants that

$$\Psi\left(x, \frac{1}{k}\right) \leq \Psi(x,r) < \frac{N}{N+1} \leq \frac{k}{k+1},$$

where the last inequality follows since $k \geq N$. So, $x \in S_k$ and therefore $S \subseteq \bigcup_{k\in\mathbb{N}} S_k$. ■

The above lemma shows that it suffices to show $\mathcal{H}^\alpha(S_k) = 0$ for all k, as this will reveal that $\mathcal{H}^\alpha(S) = 0$ and therefore that $D_c^\alpha(E,x) \geq 1$ for $\mathcal{H}^\alpha$-a.e. $x \in E$. To this end, fix $k \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary, and let $\{E_i\}_{i\in\mathbb{N}}$ be a cover of S_k such that $\operatorname{diam}(E_i) \leq \frac{1}{k}$ and $E_i \cap S_k \neq \emptyset$ for each i, and

$$\gamma \sum_{i\in\mathbb{N}} \operatorname{diam}(E_i)^\alpha \leq \mathcal{H}^\alpha_{1/k}(S_k) + \varepsilon$$

Problem 3 continued on next page...

For each i, set U_i to be the convex hull of E_i, and so $E_i \subseteq U_i$ and $\operatorname{diam}(U_i) = \operatorname{diam}(E_i)$. Since $U_i \cap S_k \neq \emptyset$, U_i is convex, and $\operatorname{diam}(U_i) = \operatorname{diam}(E_i) \leq \frac{1}{k}$, we have the estimate

$$\mathcal{H}^\alpha(U_i \cap E) < \gamma \operatorname{diam}(U_i)^\alpha \frac{k}{k+1} = \gamma \operatorname{diam}(E_i)^\alpha \frac{k}{k+1}$$

By monotonicity of measure,

$$\mathcal{H}^\alpha(S_k) \leq \sum_{i \in \mathbb{N}} \mathcal{H}^\alpha(S_k \cap E_i) \leq \sum_{i \in \mathbb{N}} \mathcal{H}^\alpha(E \cap U_i),$$

where the first inequality is because $\{E_i\}_i$ covers S_k and the second is because $S_k \subseteq E$ and $E_i \subseteq U_i$. Plugging in our estimate,

$$\mathcal{H}^\alpha(S_k) \leq \frac{k}{k+1}\gamma \sum_{i \in \mathbb{N}} \operatorname{diam}(E_i)^\alpha \leq \frac{k}{k+1}\left(\mathcal{H}^\alpha_{1/k}(S_k) + \varepsilon\right)$$

where the last inequality follows by selection of the E_i's. Since none of the resulting quantities depend on ε, we may take $\varepsilon \to 0$ and get that

$$\mathcal{H}^\alpha(S_k) \leq \frac{k}{k+1}\mathcal{H}^\alpha_{1/k}(S_k) \leq \frac{k}{k+1}\mathcal{H}^\alpha(S_k)$$

where we used that $\mathcal{H}^\alpha_\delta(S_k)$ is monotone non-decreasing in δ. Since $\frac{k}{k+1} < 1$, the only way for this to hold is if $\mathcal{H}^\alpha(S_k) = 0$. Therefore, $\mathcal{H}^\alpha(S) = 0$, and we are done. ∎

Problem 4

Identify $O(m)$ with the space of orthogonal matrices on $\mathbb{R}^{m\times m}$. Show that it is a smooth compact $\frac{m(m-1)}{2}$-dimensional submanifold and that the restriction of the $\mathcal{H}^{m(m-1)/2}$ Hausdorff measure on it is a multiple of the Haar measure θ_m.

Identify each n-dimensional plane of $\mathbb{R}^m$ with the linear map $L \in \mathbb{R}^{m\times m}$ which gives the orthogonal projection onto it. Show that the set of such maps is an $n(m-n)$-dimensional compact submanifold of $\mathbb{R}^{m\times m}$. Let μ be the restriction of the measure $\mathcal{H}^{m(n-m)}$ on the latter set and define the measure γ as in Mattila's book:

$$\gamma(E) = \theta_m\left(\{O : O(V) \in E\}\right)$$

for some fixed V. Prove that μ and γ differ by a multiplicative constant.

Solution

Proof. Let $O(m) \subseteq \mathbb{R}^{m\times m}$ be the set of orthogonal matrices. We have

$$O(m) = \{A \in \mathbb{R}^{m\times m} : \ A^T A = I_m\} = \{A \in \mathbb{R}^{m\times m} : \ A = [v_1, \dots, v_m] \text{ and } \{v_j\}_j \text{ forms an ONB for } \mathbb{R}^m\},$$

where we use $A = [v_1, \dots, v_m]$ to denote that $v_j \in \mathbb{R}^m$ is the j^{th} column of A. To see compactness, we will show $O(m)$ is closed and bounded. For closure, note that the map $g : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ sending $A \mapsto A^T A$ is smooth, and we may write $O(m) = g^{-1}(\{I_m\})$. So, $O(m)$ is the preimage of a singleton under a continuous function, which means that it is closed. For boundedness, we note that

$$A \in O(m) \implies A = [v_1, \dots, v_m] \text{ with } \|v_j\| = 1 \implies \|A\|^2_{\mathbb{R}^{m\times m}} = \sum_{j=1}^m \|v_j\|^2 = m$$

and so $O(m)$ is bounded. Next, we note that $O(m)$ is a level set of a smooth map from $\mathbb{R}^{m^2} \to \mathbb{R}^{m^2}$, and so if we can show that this map has nonvanishing Jacobian over $O(m)$ then it follows that $O(m)$ is a smooth submanifold. Letting A_{ij} be the matrix elements, we have that for $i \neq j$,

$$g(A)_{ij} = \sum_{k=1}^m A_{ki}A_{kj} \implies \frac{\partial g(A)_{ij}}{\partial A_{k\ell}} = \begin{cases} A_{kj} & \ell = i \\ A_{ki} & \ell = j \\ 0 & \text{else} \end{cases}$$

So, the only way for the Jacobian to be equal to 0 is if A itself is 0, which is not an element of $O(m)$. Therefore, $O(m)$ is a smooth submanifold of $\mathbb{R}^{m\times m}$. To compute its dimensionality, note that we may identify $O(m)$ with the space of orthonormal bases of $\mathbb{R}^m$. The first vector in our ONB may be any unit vector, the second may be any unit vector orthogonal to the first, and so on. So, we find that $O(m) \cong \mathbb{S}^{m-1} \times \left(\mathbb{S}^{m-1} \cap \mathbb{R}^{m-1}\right) \times \left(\mathbb{S}^{m-1} \cap \mathbb{R}^{m-2}\right) \times \ldots \times \left(\mathbb{S}^{m-1} \cap \mathbb{R}^{1}\right)$. Since topological dimension is additive under taking product spaces and each $\mathbb{S}^{m-1} \cap \mathbb{R}^k$ has topological dimension $k-1$, we see that $O(m)$ has topological dimension $(m-1) + (m-2) + (m-3) + \ldots + 1 = \frac{m(m-1)}{2}$. This is the manifold dimension of $O(m)$ as well.

Lastly, we must show that $\mu_m := \mathcal{H}^{m(m-1)/2} \llcorner O(m)$ is a multiple of the Haar measure θ_m. To do so, it suffices to show that μ_m is invariant under the group action of $O(m)$ and $\mu_m(O(m) < \infty$ since the Haar measure is defined as the unique invariant probability measure on $O(m)$. To see this, consider any fixed orthogonal matrix $A \in O(m)$. We note that the map from $O(m) \to O(m)$ of left multiplication by A can be expressed as

$$O(m) \ni [v_1, \dots, v_m] \mapsto [Av_1, \dots, Av_m] \in O(m)$$

So, we may express this elementwise as a block diagonal $m^2 \times m^2$ matrix

$$\begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix}$$

We note that this represents an orthogonal matrix in $O(m^2)$ since

$$\begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix}^\top \begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix} = \begin{bmatrix} A^\top A & & \\ & \ddots & \\ & & A^\top A \end{bmatrix} = \begin{bmatrix} I_m & & \\ & \ddots & \\ & & I_m \end{bmatrix} = I_{m^2}$$

So, since the $\mathcal{H}^{m(m-1)/2}$ measure on $\mathbb{R}^{m^2}$ is $O(m^2)$ invariant (Hausdorff measures are always invariant under linear isometries), we see that μ_m is invariant under the group action. Therefore, $\frac{1}{\mu_m(O_m)}\mu_m$ is the Haar measure.

We identify the Grassmannian $G(m, n)$ with the space of n-dimensional orthogonal projection matrices, i.e. matrices $P \in \mathbb{R}^{m \times m}$ such that

$$P^2 = P^\top = P \quad \text{and} \quad \dim \text{im}(P) = n$$

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Problem 5

This example, taken from a paper of Martin and Mattila, shows a closed set $E \subseteq \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$ such that $\mathcal{H}^1(\mathbf{p}_\ell(E)) = 0$ for every 1-dimensional line $\ell \subseteq \mathbb{R}^2$. The set E is reached as $\bigcap_k E_k$. E_0 is the closed unit disk. E_1 consists of the two closed disks of radius $\frac{1}{2}$ centered at the points $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$. We apply the same operation to each disk of E_1 to produce four closed disks of radius $\frac{1}{4}$ whose union gives E_2: however, these new disks are centered on parallel diameters of the old disks, which form an angle α_1 with the horizontal line. At each iteration k, the set E_k consists of 2^k closed disks of radii 2^{-k} obtained from the previous set E_{k-1} by replacing each disk of E_k with the same operation leading from E_0 to E_1. However, each pair of "new" disks lie on the diameter of the corresponding old disk, which forms an angle $\sum_{i=1}^{k-1} \alpha_i$ with the horizontal line. The sequence $\alpha_i > 0$ is chosen to be infinitesimal and such that $\sum_i \alpha_i = \infty$.

Solution

Proof. E is certainly closed, as it is an intersection of finite unions of closed disks. We first show that $0 < \mathcal{H}^1(E) < \infty$ by an application of Problem 1. Namely, each m_j is 2, each $E_{i_1,\dots,i_k}$ denotes a closed disk of radius 2^{-k} which is either the first or second (depending on the value of i_k) disk generated from $E_{i_1,\dots,i_{k-1}}$. Certainly, $E = \bigcap_{k\in\mathbb{N}} \bigcup_{\substack{i_1,\dots,i_k \\ i_j \in \{1,2\}}} E_{i_1,\dots,i_k}$, and so to apply Problem 1 we must verify the 4 properties.

(a) We see that $d(k) = 2^{-k}$, and so $d(k) \to 0$

(b) For any fixed disk indices $i_1, \dots, i_{k-1}$, the two next disks $E_{i_1,\dots,i_{k-1},1}$ and $E_{i_1,\dots,i_{k-1},2}$ are both subsets of $E_{i_1,\dots,i_{k-1}}$.

(c) We note that

$$\text{diam}(E_{i_1,\dots,i_k}) = 2^{-k} = 2(2^{-(k+1)}) = \sum_{j=1}^{2} \text{diam}(E_{i_1,\dots,i_k,j})$$

and so property (c) holds.

(d) We will show this with $c = 8$. For $\text{diam}(B) \geq \frac{1}{8}$ the result holds trivially since the sum of disk diameters over all the disks of a certain scale is 1. Otherwise, let $m := \lceil -\log_2(\text{diam}(B)) \rceil$ be the integer such that $d(m-1) \geq \text{diam}(B) \geq d(m)$. Then, we know that B is only able to intersect two $E_{i_1,\dots,i_{m-1}}$'s since it is not large enough to span more disks in the construction. Then, the number of $E_{i_1,\dots,i_k}$'s that B can intersect is no more that $4 \cdot 2^{k-m}$. So, we have that

$$\sum_{B \cap E_{i_1,\dots,i_k} \neq \emptyset} \text{diam}(E_{i_1,\dots,i_k}) \leq 4 \cdot 2^{k-m} d(k) = 4 \cdot 2^{-m} \leq 8\, \text{diam}(B)$$

Therefore, by Problem 1 we know that $0 < \mathcal{H}^1(E) < \infty$. We now show that the projection onto every line has 0 measure. Let $\ell \subseteq \mathbb{R}^2$ be an arbitrary 1-dimensional line, say with counterclockwise angle α_0, and so the horizontal lies at an angle of α_0 above ℓ. So, we may suppose without loss of generality that ℓ is the horizontal line if we say that each E_k is constructed from disks whose diameters form angles of $\sum_{i=0}^{k-1} \alpha_i$; that is, if we rotate the entire plane clockwise by α_0. We proceed.

Denote by $\mathbf{p}(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ the projection onto the horizontal line (i.e. $\mathbf{p}((x, y)) = x$).

Lemma 2. *There exists a subsequence $\{k_n\}_{n\in\mathbb{N}}$ (depending on the α_k's) and a constant $C < 1$ such that for every $n \in \mathbb{N}$,*

$$\mathcal{H}^1(\mathbf{p}(E_{k_n})) \leq C^n$$

Problem 5 continued on next page. . .

The result that $\mathcal{H}^1(\mathbf{p}(E)) = 0$ follows clearly from this lemma. To see this, let $\delta > 0$ be arbitrary, and let $N \in \mathbb{N}$ be such that $C^N < \delta$ and so $\mathcal{H}^1(\mathbf{p}(E_{k_N})) \leq C^N < \delta$. Then, since $E \equiv \bigcap_k E_k \subseteq E_{k_N}$ by construction, we see that $\mathbf{p}(E) \subseteq \mathbf{p}(E_{k_N})$, and so $\mathcal{H}^1(\mathbf{p}(E)) \leq \mathcal{H}^1(\mathbf{p}(E_{k_N})) < \delta$. Since this holds for all $\delta > 0$ it must be that $\mathcal{H}^1(\mathbf{p}(E)) = 0$ as desired. So, we conclude by proving the lemma.

Proof of Lemma. For notation, let $\beta_k := \left(\sum_{i=0}^{k-1} \alpha_i\right) \pmod{\pi}$ be the cumulative sums of the α_k's and $r_k := 2^{-k}$. At each step $k+1$, from the center of each disk in E_k, we form two disks: one a distance r_k at angle β_k and the other a distance r_k in the opposite direction. As such, we can index each of the 2^k disks formed at step k with the sequence $\mathbf{s} = (s_1, \ldots, s_k) \in \{-1, 1\}^k$, where s_j denotes whether we moved in the β_j direction or against it at step j. Then, we have that the center of disk $\mathbf{s}$ is located at the point

$$\sum_{j=1}^{k} s_j \left(r_j \cos(\beta_j), r_j \sin(\beta_j)\right)$$

As such, denoting the closed disk of radius $r > 0$ about $x \in \mathbb{R}^2$ by $B_r(x) \subseteq \mathbb{R}^2$, we see that

$$\mathbf{p}(E_k) = \bigcup_{\mathbf{s} \in \{-1,1\}^k} \mathbf{p}\left(B_{r_k}\left(\sum_{j=1}^{k} s_j (r_j \cos(\beta_j), r_j \sin(\beta_j))\right)\right) =: \bigcup_{\mathbf{s} \in \{-1,1\}^k} A^{(\mathbf{s})},$$

where we let $A^{(\mathbf{s})} := \mathbf{p}\left(B_{r_k}\left(\sum_{j=1}^{k} s_j (r_j \cos(\beta_j), r_j \sin(\beta_j))\right)\right) \subseteq \mathbb{R}$ for notation. Note that for any two sequences $\mathbf{s}, \mathbf{t} \in \{-1, 1\}^k$, the distance between the centers of $A^{(\mathbf{s})}$ and $A^{(\mathbf{t})}$ is precisely

$$\sum_{j \in J(\mathbf{s},\mathbf{t})} 2 r_j \cos(\beta_j), \quad \text{where} \quad J(\mathbf{s}, \mathbf{t}) := \{j \in [k] : s_j \neq t_j\}$$

Since $\sum_i \alpha_i = \infty$ but α_i's are infinitesimal, we can select a subsequence $\{k_n\}_{n \in \mathbb{N}}$ such that each $|\cos(\beta_{k_n})| < 2^{-k_n} = r_{k_n}$. Fix an $N \in \mathbb{N}$ and pay attention to step k_N. For all $\mathbf{s}, \mathbf{t}$ with $J(\mathbf{s}, \mathbf{t}) \subseteq \{k_1, k_2, \ldots, k_N\}$, it holds that the distance between the centers is less than $\sum_{n=1}^{N} 2 r_{k_n} r_{k_n} = 2 \sum_{n=1}^{N} 2^{-2k_n}$. We may partition $\{-1, 1\}^{k_N}$ into equivalence classes, where $\mathbf{s} \sim \mathbf{t}$ iff $J(\mathbf{s}, \mathbf{t}) \subseteq \{k_1, k_2, \ldots, k_n\}$ (it is an equivalence relation since J is symmetric and transitive). Then, we have that for each $\mathbf{s}$,

$$\mathcal{H}^1\left(\bigcup_{\substack{\mathbf{t} \in \{-1,1\}^{k_N} \\ \mathbf{s} \sim \mathbf{t}}} A^{(\mathbf{t})}\right) \leq 2 r_{k_N} + 2 \sum_{n=1}^{N} 2^{-2k_n} \leq 4 \sum_{n=1}^{N} 2^{-2k_n}$$

since all such $A^{(\mathbf{t})}$ have centers lying at most $2 \sum_{n=1}^{N} 2^{-2k_n}$ away from the center of $A^{(\mathbf{s})}$. There are $2^{k_N - N}$ such equivalence classes, since each equivalence class is uniquely determined by the shared values on the indices in $\mathbb{N} \setminus \{k_1, \ldots, k_N\}$. As such, we see that

$$\mathcal{H}^1(\mathbf{p}(E_{k_N})) = \mathcal{H}^1\left(\bigcup_{\mathbf{s} \in \{-1,1\}^{k_N}} A^{(\mathbf{s})}\right) \leq 4 \cdot 2^{k_N - N} \sum_{n=1}^{N} 2^{-2k_n} = \sum_{n=1}^{N} 2^{2 + k_N - 2k_n - N}$$

■

The main result follows. ■