MAT 526: Problem Set 1

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Problem 1

Denote by $\dim_H$ the Hausdorff dimension of subsets of $\mathbb{R}^n$. Prove that, if $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^n$, then

$$\dim_H(A \times B) \leq \min\{\dim_H(A) + n,\ \dim_H(B) + k\}$$

Solution

Proof. Note that it suffices to show that $\dim_H(A \times B) \leq \dim_H(A) + n$, since we would then be able to apply identical logic with A and B switched to see the other bound. To accomplish this task, we will show that $\mathcal{H}^{\dim_H(A)+n+\epsilon}(A \times B) = 0$ for all $\epsilon > 0$. In fact, we will show that $\mathcal{H}^{\dim_H(A)+n+\epsilon}(A \times B_R) = 0$ for every ball $B_R \subseteq \mathbb{R}^n$ of diameter R (radius $\frac{R}{2}$), from which the result will follow via a countable union. We proceed.

Let $R > 0$ be arbitrary. Let $\epsilon > 0$ be arbitrary. Let $d_A := \dim_H(A)$ for notation. By definition of the Hausdorff dimension,

$$\mathcal{H}^{d_A+\epsilon}(A) = 0 \implies \mathcal{H}^{d_A+\epsilon}_\delta(A) = 0 \quad \forall \delta > 0$$

Let $\delta > 0$ be arbitrary. Let $\gamma > 0$ be arbitrary. Then, by definition of the infimum there exists an efficient countable cover $\{E_i\}_{i\in\mathbb{N}}$ of A consisting of sets of diameter $< \delta$ such that

$$\frac{\omega_{d_A+\epsilon}}{2^{d_A+\epsilon}} \sum_{i\in\mathbb{N}} \operatorname{diam}(E_i)^{d_A+\epsilon} < \gamma$$

For each i we will attempt to cover $B_R \subseteq \mathbb{R}^n$ with sets of diameter at most $\operatorname{diam}(E_i)$. The following lemma helps us understand how large such a cover must be.

Lemma 1. *Let $0 < r < R$, and let $B_R \subseteq \mathbb{R}^n$ be the closed ball around the origin of diameter R. Then, there exists a cover $\{F_j\}_{j=1}^N$ of B_R such that $\operatorname{diam}(F_j) = r$ for all j and $B_R \subseteq \bigcup_{j=1}^N F_j$. The size of this cover is $N = \left\lceil \frac{R\sqrt{2}}{r} \right\rceil^n$.*

Proof of Lemma 1. Let $C_R := \left[-\frac{R}{2}, \frac{R}{2}\right]^n$ be the n-dimensional box of width R centered at the origin in $\mathbb{R}^n$. For each $x \in B_R$, we know that $|x_i| \leq \frac{R}{2}$ for each coordinate i, and so $x \in C_R$; thus, $B_R \subseteq C_R$. We may efficiently tile C_R by cubes of width $\frac{r}{\sqrt{2}}$ (and thus diameter r) by placing them edge to edge. To do so will require $\left\lceil \frac{R}{\frac{r}{\sqrt{2}}} \right\rceil = \left\lceil \frac{R\sqrt{2}}{r} \right\rceil$ cubes in each dimension, and so we see that we may cover C_R with $N := \left\lceil \frac{R\sqrt{2}}{r} \right\rceil^n$ sets of diameter r. The result of the lemma follows. ■

Now, for each $i \in \mathbb{N}$ we may apply Lemma 1 to instantiate a cover $\{F_{i,j}\}_{j=1}^{N(i)}$ of B_R with $N(i) = \left\lceil \frac{R\sqrt{2}}{\operatorname{diam}(E_i)} \right\rceil^n$ and $\operatorname{diam}(F_{i,j}) = \operatorname{diam}(E_i)$. Therefore, since $\{E_i\}_{i\in\mathbb{N}}$ covers A, we find that

$$A \times B_R \subseteq \bigcup_{i\in\mathbb{N}} \bigcup_{j=1}^{N(i)} (E_i \times F_{i,j})$$

Furthermore, we know that

$$\operatorname{diam}(E_i \times F_{i,j})^2 = \operatorname{diam}(E_i)^2 + \operatorname{diam}(F_{i,j})^2 = 2\operatorname{diam}(E_i)^2$$

$$\implies \operatorname{diam}(E_i \times F_{i,j}) = \sqrt{2}\operatorname{diam}(E_i) < \delta\sqrt{2}$$

So, we have that $\{E_i \times F_{i,j}\}_{i\in\mathbb{N},\, j\in[N(i)]}$ is a countable cover of $A \times B_R$ by sets of diameter $< \delta\sqrt{2}$, and so

$$\begin{aligned}
\mathcal{H}^{d_A+n+\epsilon}_{\delta\sqrt{2}}(A \times B_R) &\le \frac{\omega_{d_A+n+\epsilon}}{2^{d_A+n+\epsilon}} \sum_{i\in\mathbb{N}} \sum_{j=1}^{N(i)} \operatorname{diam}(E_i \times F_{i,j})^{d_A+n+\epsilon} \\
&= \frac{\omega_{d_A+n+\epsilon}}{2^{d_A+n+\epsilon}} \sum_{i\in\mathbb{N}} N(i)(\sqrt{2})^{d_A+n+\epsilon} \cdot \operatorname{diam}(E_i)^{d_A+n+\epsilon} \\
&= \frac{\omega_{d_A+n+\epsilon}}{2^{d_A+n+\epsilon}} \sum_{i\in\mathbb{N}} \left\lceil \frac{R\sqrt{2}}{\operatorname{diam}(E_i)} \right\rceil^n (\sqrt{2})^{d_A+n+\epsilon} \cdot \operatorname{diam}(E_i)^{d_A+n+\epsilon} \\
&< \frac{\omega_{d_A+n+\epsilon}}{2^{d_A+n+\epsilon}} \sum_{i\in\mathbb{N}} \left(\frac{R\sqrt{2}+\delta}{\operatorname{diam}(E_i)} \right)^n (\sqrt{2})^{d_A+n+\epsilon} \cdot \operatorname{diam}(E_i)^{d_A+n+\epsilon} \\
&= \left(\frac{\omega_{d_A+n+\epsilon}}{\omega_{d_A+\epsilon}} (R\sqrt{2}+\delta)^n (\sqrt{2})^{d_A-n+\epsilon} \right) \cdot \left(\frac{\omega_{d_A+\epsilon}}{2^{d_A+\epsilon}} \sum_{i\in\mathbb{N}} \operatorname{diam}(E_i)^{d_A+\epsilon} \right) \\
&< \left(\frac{\omega_{d_A+n+\epsilon}}{\omega_{d_A+\epsilon}} (R\sqrt{2}+\delta)^n (\sqrt{2})^{d_A-n+\epsilon} \right) \cdot \gamma,
\end{aligned}$$

where for the first inequality we used the definition of $\mathcal{H}^\alpha_\delta$ as an infimum over such covers, for the second inequality we used that $\left\lceil \frac{R\sqrt{2}}{\operatorname{diam}(E_i)} \right\rceil \le \frac{R\sqrt{2}}{\operatorname{diam}(E_i)} + 1 = \frac{R\sqrt{2}+\operatorname{diam}(E_i)}{\operatorname{diam}(E_i)} < \frac{R\sqrt{2}+\delta}{\operatorname{diam}(E_i)}$ as $\operatorname{diam}(E_i) < \delta$, and for the last inequality we used our selection criteria for $\{E_i\}_i$. Since such a bound holds for all $\gamma > 0$, we may take $\gamma \to 0$ to find that

$$\mathcal{H}^{d_A+n+\epsilon}_{\delta\sqrt{2}}(A \times B_R) = 0$$

Since this holds for all $\delta > 0$, taking a supremum over δ reveals that

$$\mathcal{H}^{d_A+n+\epsilon}(A \times B_R) = 0$$

Since this holds for all $R > 0$, we may use the countable subadditivity of measure to see that

$$\mathcal{H}^{d_A+n+\epsilon}(A \times \mathbb{R}^n) = \mathcal{H}^{d_A+n+\epsilon}\left(\bigcup_{R\in\mathbb{N}} (A \times B_R) \right) \le \sum_{R\in\mathbb{N}} \mathcal{H}^{d_A+n+\epsilon}(A \times B_R) = 0$$

Since $A \times B \subseteq A \times \mathbb{R}^n$, monotonicity of measure grants that

$$\mathcal{H}^{d_A+n+\epsilon}(A \times B) \le \mathcal{H}^{d_A+n+\epsilon}(A \times \mathbb{R}^n) = 0,$$

and so $\dim_H(A \times B) \le d_A + n + \epsilon$. Since this holds for all $\epsilon > 0$, we find that

$$\dim_H(A \times B) \le \dim_H(A) + n$$

We may apply identical logic as above with the roles of A and B reversed to see that

$$\dim_H(A \times B) \le k + \dim_H(B)$$

Thus,

$$\dim_H(A \times B) \le \min\{\dim_H(A) + n,\ \dim_H(B) + k\}$$

as desired. ■

Problem 2

Prove that there is an uncountable closed set $E \subseteq \mathbb{R}$ which has zero Hausdorff dimension.

Solution

Proof. Let $\phi(n) := n(n+1)/2$ for all $n \in \mathbb{N}$ for notation. We will construct a Cantor ternary-type set where at step n we remove a middle portion so that what remains on the left and right sides are each a proportion $\frac{1}{3^n}$ of the original interval. Let E_n denote the set that we have after step n. So, $E_1 \equiv \left[0, \frac{1}{3}\right] \sqcup \left[\frac{2}{3}, 1\right]$. Next, for each interval of size $\frac{1}{3}$ we remove all but the left and right $\frac{1}{3^2}$ proportions of that interval; so, $E_2 \equiv \left[0, \frac{1}{27}\right] \sqcup \left[\frac{8}{27}, \frac{1}{3}\right] \sqcup \left[\frac{2}{3}, \frac{19}{27}\right] \sqcup \left[\frac{26}{27}, 1\right]$. We continue by removing from each interval of size $\frac{1}{27}$ a chunk such that the proportion of the interval remaining is $\frac{1}{3^3}$ on the left and right; so, each interval in E_3 would be of size $\frac{1}{27} \cdot \frac{1}{3^3} = \frac{1}{3^6}$. Continuing this indefinitely, we see that $E_{n+1} \subseteq E_n$ for all $n \in \mathbb{N}$ and each set E_n is a disjoint union of 2^n closed intervals, each interval having size $\frac{1}{3^{1+\dots+n}} = \frac{1}{3^{\phi(n)}}$. Define the set

$$E := \bigcap_{n \in \mathbb{N}} E_n$$

We claim this set E has the desired properties.

Firstly, each E_n is a finite union of closed intervals, and so is closed. Since E is an intersection of closed sets, E is closed. To see that E is uncountable, note that we may form a injection sending any real number in $[0, 1]$ with no 1's in its ternary expansion (an uncountable set) to elements of E. The proof of this proceeds exactly as in the proof of the uncountability of the usual ternary Cantor set. For each $x \in [0, 1]$, write its ternary expansion as $x = \sum_{n=1}^{\infty} \frac{c_n(x)}{3^n}$ with $c_n(x) \in \{0, 1, 2\}$ denoting the ternary coefficient in digit n (to make this unique, we may avoid having coefficients of 1 as much as possible, such that we select $.0\overline{22222}$ instead of $.1$ and $.2$ instead of $.\overline{11111}$). Define

$$A := \{x \in [0, 1] : \quad c_n(x) \neq 1 \quad \forall n \in \mathbb{N}\}$$

Then, A is uncountable since its cardinality is as large as $\{0, 2\}^{\mathbb{N}}$, which is itself uncountable. Define a map $f : A \to E$ as follows: for $n \in \mathbb{N}$, use the n^{th} digit to determine whether we map x to an element in a left interval (if the n^{th} digit is 0) or right interval (if it is 2) of E_n. To see that f is well defined, note that at the n^{th} digit we are restricted to an interval of width $\frac{1}{3^{\phi(n)}}$, and so traversing the ternary expansion of an element $x \in A$ provides a Cauchy sequence in $\mathbb{R}$, which will converge to $f(x)$. This will certainly be an injective map as if two ternary expansions disagree, say at position k, they must be mapped to elements that are in disjoint intervals in E_k. So, as there exists an injective map from an uncountable set to E, we see that E is also uncountable. Thus, it suffices to show that E has Hausdorff dimension 0.

To this end, let $\alpha > 0$ be arbitrary; we want to show that $\mathcal{H}^\alpha(E) = 0$. Let $\epsilon > 0$ be arbitrary. Let $\delta > 0$ be arbitrary; we want to show that $\mathcal{H}^\alpha_\delta(E) < \epsilon$. Let n be large enough that $\alpha\phi(n) > n$ and $\phi(n) > \max\left\{-\log_3(\delta), \frac{1}{\alpha}\log_{2/3}\left(\frac{2^\alpha \epsilon}{\omega_\alpha}\right)\right\}$. Then, $E \subseteq E_n$, and so $\mathcal{H}^\alpha_\delta(E) \leq \mathcal{H}^\alpha_\delta(E_n)$. Since E_n is a disjoint union of 2^n intervals, each of size $\frac{1}{3^{\phi(n)}} < \delta$ (and so it is a valid δ-cover of E_n), we find by definition of an infimum that

$$\mathcal{H}^\alpha_\delta(E_n) \leq \frac{\omega_\alpha}{2^\alpha} \sum_{k=1}^{2^n} \left(\frac{1}{3^{\phi(n)}}\right)^\alpha = \frac{\omega_\alpha}{2^\alpha} \cdot \frac{2^n}{3^{\alpha\phi(n)}} \leq \frac{\omega_\alpha}{2^\alpha} \cdot \left(\frac{2}{3}\right)^{\alpha\phi(n)} \leq \frac{\omega_\alpha}{2^\alpha} \cdot \frac{2^\alpha \epsilon}{\omega_\alpha} = \epsilon$$

So, $\mathcal{H}^\alpha_\delta(E) \leq \epsilon$. Since this holds for all $\delta > 0$, by taking a supremum we know that $\mathcal{H}^\alpha(E) \leq \epsilon$. Since this holds for all $\epsilon > 0$, we know that $\mathcal{H}^\alpha(E) = 0$. Lastly, since this holds for all $\alpha > 0$, we find that $\dim_H(E) = 0$. ■

Problem 3

Consider a Borel set $A \subseteq \mathbb{R}^n$ with $0 < H^\alpha(A) < \infty$ and a Borel set $B \subseteq \mathbb{R}^m$ with positive Lebesgue measure. Prove that

$$H^{\alpha+m}(A \times B) > 0.$$

Solution

Proof. Let $\mathcal{L}^m$ denote the Lebesgue measure on $\mathbb{R}^m$. Let $\mu := (\mathcal{H}^\alpha \llcorner A) \times (\mathcal{L}^m \llcorner B)$ denote the product measure of the measure restrictions. For all $x = (x_a, x_b) \in A \times B$ we may write

$$\Theta^{\alpha+m,*}(\mu, x) = \limsup_{r\to 0} \frac{\mu(B_r(x))}{\omega_{\alpha+m} r^{\alpha+m}}$$

Observe that for $B_r(x) \subseteq \mathbb{R}^n \times \mathbb{R}^m$, $B_r(x_a) \subseteq \mathbb{R}^n$, and $B_r(x_b) \subseteq \mathbb{R}^m$, we have

$$\begin{aligned} y = (y_a, y_b) \in B_r(x) &\implies \|y - x\|^2 = \|y_a - x_a\|^2 + \|y_b - x_b\|^2 < r^2 \\ &\implies \|y_a - x_a\|^2 < r^2 \quad \text{and} \quad \|y_b - x_b\|^2 < r^2 \\ &\implies y_a \in B_r(x_a) \quad \text{and} \quad y_b \in B_r(x_b) \\ &\implies y \in B_r(x_a) \times B_r(x_b), \end{aligned}$$

and so $B_r(x) \subseteq B_r(x_a) \times B_r(x_b)$. By monotonicity of measure and the defintion of product measures,

$$\mu(B_r(x)) \le \mu(B_r(x_a) \times B_r(x_b)) = \mathcal{H}^\alpha(A \cap B_r(x_a)) \cdot \mathcal{L}^m(B \cap B_r(x_b))$$

So, for all $x = (x_a, x_b) \in A \times B$ we have

$$\begin{aligned} \Theta^{\alpha+m,*}(\mu, x) &\le \limsup_{r\to 0} \frac{\mathcal{H}^\alpha(A \cap B_r(x_a)) \cdot \mathcal{L}^m(B \cap B_r(x_b))}{\omega_{\alpha+m} r^{\alpha+m}} \\ &= \frac{\omega_\alpha \omega_m}{\omega_{\alpha+m}} \limsup_{r\to 0} \left(\frac{\mathcal{H}^\alpha(A \cap B_r(x_a))}{\omega_\alpha r^\alpha} \cdot \frac{\mathcal{L}^m(B \cap B_r(x_b))}{\omega_m r^m} \right) \end{aligned}$$

Observe that $\mathcal{L}^m(B_r(x_b)) = \omega_m r^m$, and so by monotonicity of measure

$$\mathcal{L}^m(B \cap B_r(x_b)) \le \mathcal{L}^m(B_r(x_b)) \implies \frac{\mathcal{L}^m(B \cap B_r(x_b))}{\omega_m r^m} = \frac{\mathcal{L}^m(B \cap B_r(x_b))}{\mathcal{L}^m(B_r(x_b))} \le 1$$

Thus, for all $x = (x_a, x_b) \in A \times B$ we have

$$\Theta^{\alpha+m,*}(\mu, x) \le \frac{\omega_\alpha \omega_m}{\omega_{\alpha+m}} \limsup_{r\to 0} \frac{\mathcal{H}^\alpha(A \cap B_r(x_a))}{\omega_\alpha r^\alpha}$$

Now, we know that since $\mathcal{H}^\alpha(A) < \infty$ it holds that for $\mathcal{H}^\alpha$-a.e. $x_a \in A$,

$$\frac{1}{2^\alpha} \le \Theta^{\alpha,*}(A, x_a) = \limsup_{r\to 0} \frac{\mathcal{H}^\alpha(A \cap B_r(x_a))}{\omega_\alpha r^\alpha} \le 1$$

Therefore, for μ-a.e. $x \in A \times B$ we have

$$\Theta^{\alpha+m,*}(\mu, x) \le \frac{\omega_\alpha \omega_m}{\omega_{\alpha+m}}$$

Since the density $\Theta^{\alpha+m,*}(\mu, x)$ is bounded above for μ-a.e. $x \in A \times B$ and $A \times B$ is Borel, we find that

$$\mu(A \times B) \le \frac{\omega_\alpha \omega_m}{\omega_{\alpha+m}} \mathcal{H}^{\alpha+m}(A \times B) \implies \mathcal{H}^\alpha(A) \cdot \mathcal{L}^m(B) \le \frac{\omega_\alpha \omega_m}{\omega_{\alpha+m}} \mathcal{H}^{\alpha+m}(A \times B),$$

where we used the fact that $\mu(A \times B) = \mathcal{H}^\alpha(A) \cdot \mathcal{L}^m(B)$ by definition. Since $\mathcal{H}^\alpha(A), \mathcal{L}^m(B) > 0$ by assumption, the result follows. ■

Problem 4

Assume the validity of the following statement: every Borel $A \subseteq \mathbb{R}^k$ with $H^\alpha(A) > 0$ has a Borel subset E with $0 < H^\alpha(E) < \infty$. Use Problem 3 to prove that, if $A \subseteq \mathbb{R}^k$ is Borel with $H^\alpha(A) > 0$ and $R \subseteq \mathbb{R}^n$ is k-rectifiable with $H^k(R) > 0$, then

$$H^{\alpha+k}(A \times R) > 0$$

Solution

Proof. When $k = n$ then the result comes from a routine application of Problem 3. So, suppose without loss of generality that $k < n$. We start by applying the assumed statement to find $\tilde{A} \subseteq A$ such that $0 < \mathcal{H}^\alpha(\tilde{A}) \leq \mathcal{H}^\alpha(A) < \infty$.

Next, since R is k-rectifiable, we may write

$$R = N \cup \bigcup_{j \in \mathbb{N}} E_j,$$

where $\mathcal{H}^k(N) = 0$ and each E_j is a Borel subset of a k-dimensional Lipschitz graph. In other words,

$$\mathrm{E}_j \subseteq \Gamma_j = \left\{(x, f_j(x)) : x \in \mathbb{R}^k\right\}$$

for some Lipschitz $f_j : \mathbb{R}^k \to \mathbb{R}^{n-k}$. By countable subadditivity of the $\mathcal{H}^k$ measure, we know that there must be some $j \in \mathbb{N}$ for which $\mathcal{H}^k(\mathrm{E}_j) > 0$, since otherwise we would have that $\mathcal{H}^k(R) = 0$, a contradiction. For notation purposes, write $E := E_j$ and $f := f_j$ for this choice of j. Note that we may lift f to a map $F : \mathbb{R}^k \to \mathbb{R}^n$ via $x \mapsto (x, f(x))$. It follows that F is Lipschitz via

$$\|F(x) - F(y)\|^2 = \|x - y\|^2 + \|f(x) - f(y)\|^2 \leq \left(1 + \mathrm{Lip}(f)^2\right)\|x - y\|^2 \implies \mathrm{Lip}(F) \leq \sqrt{1 + \mathrm{Lip}(f)^2}$$

Recall the following lemma from class.

Lemma 2. *If $F : \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz and $E \subseteq \mathbb{R}^k$ has Lebesgue measure 0, then $\mathcal{H}^k(F(E)) = 0$.*

By the contrapositive of Lemma 2, we see that $F^{-1}(E)$ must have positive Lebesgue measure in $\mathbb{R}^k$ since $\mathcal{H}^k(E) > 0$. So, by Problem 3 (which we may apply since F Lipschitz $\implies$ F continuous $\implies$ $F^{-1}(E)$ Borel) it holds that

$$\mathcal{H}^{\alpha+k}\left(\tilde{A} \times F^{-1}(E)\right) > 0,$$

where $\tilde{A} \times F^{-1}(E) \subseteq \mathbb{R}^k \times \mathbb{R}^k$. Now, note by definition of F (as a lifted version of f) that $F^{-1}(E)$ is simply the projection of E onto the first k coordinates. More precisely, consider the map $P : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^k$ sending $((x_1, \ldots, x_k), (y_1, \ldots, y_k, y_{k+1}, \ldots, y_n)) \mapsto ((x_1, \ldots, x_k), (y_1, \ldots, y_k))$; then, we have that

$$\tilde{A} \times F^{-1}(E) = P(\tilde{A} \times E)$$

Furthermore, since the projection map P is clearly 1-Lipschitz, we get

$$0 < \mathcal{H}^{\alpha+k}\left(\tilde{A} \times F^{-1}(E)\right) = \mathcal{H}^{\alpha+k}\left(P(\tilde{A} \times E)\right) \leq \mathcal{H}^{\alpha+k}(\tilde{A} \times E)$$

By construction, we know that $\tilde{A} \subseteq A$, and also that $E = E_j \subseteq R$. Thus, $\tilde{A} \times E \subseteq A \times R$, and so by monotonicity of measure we get

$$0 < \mathcal{H}^{\alpha+k}(\tilde{A} \times E) \leq \mathcal{H}^{\alpha+k}(A \times R)$$

■

Problem 5

Consider the function $f_m : [0,1] \to \mathbb{R}$ which takes the value $m2^{-m^2}$ on every interval $[2k2^{-m^2}, (2k+1)2^{-m^2}] \subseteq [0,1]$ and the value $-m2^{-m^2}$ on the remaining portion of the domain. Define

$$f = \sum_{m=1}^{\infty} f_m$$

Show that the graph $\Gamma(f)$ of f has positive and finite $\mathcal{H}^1$ measure and that it is purely unrectifiable.

Solution

Proof. We will prove the following claims:

1. The infinite sum defining f converges everywhere on $[0,1]$, and so f is well-defined.
2. $\mathcal{H}^1(\Gamma(f)) > 0$
3. $\mathcal{H}^1(\Gamma(f)) < \infty$
4. For all Lipschitz functions $g : [0,1] \to \mathbb{R}$, the graph $\Gamma(g)$ has that $\mathcal{H}^1(\Gamma(f) \cap \Gamma(g)) = 0$.

All these claims taken together certainly prove the desired results, and so we proceed in order.

(1) We show that the sum is absolutely convergent. For any $x \in [0,1]$, we know that $|f_m(x)| = m2^{-m^2}$ for all $m \in \mathbb{N}$. As such,

$$\sum_{m \in \mathbb{N}} |f_m(x)| = \sum_{m \in \mathbb{N}} m2^{-m^2}$$

Let $N \in \mathbb{N}$ be large enough that for all $m > N$ we know $m > \sqrt{3\log_2(m)}$ (we can do this since the LHS grows faster asymptotically than the RHS). Note that for $m > N$, we have

$$m > \sqrt{3\log_2(m)} \implies -m^2 < -3\log_2(m) \implies 2^{-m^2} < \frac{1}{m^3}$$

Then,

$$\sum_{m \in \mathbb{N}} |f_m(x)| = \sum_{m=1}^{N} |f_m(x)| + \sum_{\substack{m \in \mathbb{N} \\ m > N}} m2^{-m^2} < \sum_{m=1}^{N} |f_m(x)| + \sum_{\substack{m \in \mathbb{N} \\ m > N}} \frac{1}{m^2} < \infty,$$

which clearly converges. So, the infinite series defining $f(x)$ converges absolutely, which means that it converges. Since this holds for all x, we see that f is well-defined.

(2) We show that the projection of $\Gamma(f)$ to the x-axis has positive $\mathcal{H}^1$ measure. In particular, let $\mathbb{P}_x : [0,1] \times \mathbb{R} \to [0,1]$ be the projection map sending $(x,y) \mapsto x$; certainly, it holds that $\mathbb{P}_x$ is 1-Lipschitz. So,

$$\mathcal{H}^1(\mathbb{P}_x(\Gamma(f))) \le \mathcal{H}^1(\Gamma(f))$$

However, for each $x \in [0,1]$ we know that $(x, f(x)) \in \Gamma(f) \implies x \in \mathbb{P}_x(\Gamma(f))$, and so $\mathbb{P}_x(\Gamma(f)) = [0,1]$. Thus,

$$\mathcal{H}^1([0,1]) \le \mathcal{H}^1(\Gamma(f)) \implies \mathcal{H}^1(\Gamma(f)) \ge 1 > 0$$

(3) We continue with the following lemma.

Lemma 3. *Let f be defined as above. Let $m \in \mathbb{N}$, $j \in \{0, \dots, 2^{m^2} - 1\}$, and $x, y \in (j2^{-m^2},\ (j+1)2^{-m^2})$. Then, we know that*

$$|f(x) - f(y)| \leq \frac{2^{-m^2}}{\ln(2)}$$

Proof of Lemma 3. By selection of x and y, we know that $f_n(x) = f_n(y)$ for all $n \leq m$ (this is because an interval on which f_{k+1} is constant is contained in an interval on which f_k is constant). As such, we have that

$$|f(x) - f(y)| = \left| \sum_{k=m+1}^{\infty} f_k(x) - f_k(y) \right| \leq \sum_{k=m+1}^{\infty} |f_k(x) - f_k(y)|$$

We know by definition of f_k that $|f_k(x) - f_k(y)| \leq 2k2^{-k^2}$ always. So, $|f(x) - f(y)| \leq 2\sum_{k=m+1}^{\infty} k2^{-k^2}$. Since the function $k2^{-k^2}$ is decreasing in k for all $k \geq 1$, we may upper bound this sum by the integral that it is the right Riemann sum of. Explicitly,

$$|f(x) - f(y)| \leq 2 \sum_{k=m+1}^{\infty} k2^{-k^2} \leq 2 \int_m^{\infty} x2^{-x^2} dx$$

Computing this integral with a substitution $u = x^2$,

$$|f(x) - f(y)| \leq \int_{m^2}^{\infty} 2^{-u} du = \frac{2^{-m^2}}{\ln(2)}$$

as desired. ∎

Lemma 3 is the tool that we need to succeed. We will show that $\mathcal{H}^1(\Gamma(f)) \leq \sqrt{1 + \frac{4}{\ln(2)^2}}$. To this end, let $\delta_n := 2^{-n^2}$ for all $n \in \mathbb{N}$; this sequence clearly approaches 0. Let $n \in \mathbb{N}$ be arbitrary. Then, $\frac{1}{\delta_n} \in \mathbb{N}$ is even. For each $k = 0, ..., \frac{1}{2\delta_n} - 1$, define the rectangles in the plane

$$I_k = [2k\delta_n,\ (2k+1)\delta_n] \times \left[f\left(2k\delta_n + \frac{1}{2}\right) - \frac{\delta_2}{\ln(2)},\ f\left(2k\delta_n + \frac{1}{2}\right) + \frac{\delta_n}{\ln(2)} \right]$$

and

$$J_k = [(2k+1)\delta_n,\ (2k+2)\delta_n] \times \left[f\left(2k\delta_n + \frac{3}{2}\right) - \frac{\delta_2}{\ln(2)},\ f\left(2k\delta_n + \frac{3}{2}\right) + \frac{\delta_n}{\ln(2)} \right]$$

We know by Lemma 3 that the rectangle I_k contains the graph of $f_{[2k\delta_n, (2k+1)\delta_n]}$ because over this interval, the function can only vary by a maximum of $\frac{\delta_n}{\ln(2)}$; the analogous result holds for each J_k. As such, we see that

$$\Gamma(f) \subseteq \bigcup_{k=0}^{\frac{1}{2\delta_n} - 1} I_k \cup J_k$$

Furthermore, we see that each I_k and J_k is a rectangle of width δ_n and height $\frac{2\delta_n}{\ln(2)}$, and so it is of diameter $\delta_n \sqrt{1 + \frac{4}{\ln(2)^2}}$. Thus, by definition of the Hausdorff premeasure as an infimum,

$$\mathcal{H}^1_{\delta_n \sqrt{1 + \frac{4}{\ln(2)^2}}} \left(\bigcup_{k=0}^{\frac{1}{2\delta_n} - 1} I_k \cup J_k \right) \leq \frac{\omega_1}{2} \sum_{k=0}^{\frac{1}{2\delta_n} - 1} (\operatorname{diam}(I_k) + \operatorname{diam}(J_k)) = \frac{1}{2\delta_n} \cdot 2\delta_n \sqrt{1 + \frac{4}{\ln(2)^2}} = \sqrt{1 + \frac{4}{\ln(2)^2}}$$

By monotonicity, for all $n \in \mathbb{N}$ we have

$$\mathcal{H}^1_{\delta_n \sqrt{1 + \frac{4}{\ln(2)^2}}} (\Gamma(f)) \leq \sqrt{1 + \frac{4}{\ln(2)^2}}$$

Since the sequence $\left\{\delta_n\sqrt{1+\frac{4}{\ln(2)^2}}\right\}_n$ approaches 0 and this bound holds for all n, it also holds in the limit. So,

$$\mathcal{H}^1(\Gamma(f)) = \lim_{\delta\to 0}\mathcal{H}^1_\delta(\Gamma(f)) \le \sqrt{1+\frac{4}{\ln(2)^2}}$$

as desired.

(4) Consider the lower density

$$\Theta^1_*(\Gamma(f),(x,f(x))) = \liminf_{\delta\to 0}\frac{\mathcal{H}^1(\Gamma(f)\cap B_\delta((x,f(x))))}{2\delta}$$

Let $\epsilon > 0$ be arbitrary. Let $\eta > 0$ be arbitrary. We will show that for almost every $x \in [0,1]$, there is a $\delta < \eta$ for which

$$\mathcal{H}^1(\Gamma(f)\cap B_\delta((x,f(x)))) \le \delta(1+\epsilon)$$

To this end, write $c := 2 - \frac{2}{3\ln(2)} \approx 1.04 > 0$ and select a $m \in \mathbb{N}$ large enough that

$$m > \max\left\{3,\ \frac{1}{c\epsilon},\ 2^{\left(\sqrt{\frac{1}{4}-\log_2(\eta/c)}+\frac{1}{2}\right)}\right\}$$

Then, we have the following properties:

1.
$$m > 3 \implies cm2^{-m^2} < \left(2-\frac{2}{m\ln(2)}\right)m2^{-m^2} = 2m2^{-m^2} - \frac{2}{\ln(2)}2^{-m^2}$$

2.
$$m > \frac{1}{c\epsilon} \implies \frac{1}{cm} < \epsilon$$

3.
$$\begin{aligned} m > 2^{\left(\sqrt{\frac{1}{4}-\log_2(\eta/c)}+\frac{1}{2}\right)} &\implies \left(\log_2(m)-\frac{1}{2}\right)^2 > \frac{1}{4}-\log_2(\eta/c) \\ &\implies -\log_2(m)^2 + \log_2(m) < \log_2(\eta/c) \\ &\implies -m^2 + \log_2(m) < \log_2(\eta/c) \\ &\implies m2^{-m^2} < \frac{\eta}{c} \implies cm2^{-m^2} < \eta, \end{aligned}$$

where to get from the second to third line we used that $m > \log_2(m)$ for $m > 0$.

Let $x \in [0,1]\setminus\{k2^{-j^2}:\quad j\in\mathbb{N},\ k\in\{0,\dots,2^{j^2}-1\}\}$ (i.e. x is any point that is not a jump), which is a set of full measure. Define $\delta := cm2^{-m^2} < \eta$. For all $k \in \{0,\dots,2^{m^2}-1\}$, define $I_k := (k2^{-m^2},\ (k+1)2^{-m^2})$. Let n be the unique integer such that $x \in I_n$ (the intervals are disjoint and we know x is not on their boundaries). Then,

$$I_k \cap (x-\delta, x+\delta)\begin{cases} = \emptyset & |k-n| > cm \\ \neq \emptyset & \text{else}\end{cases}$$

By construction of δ, there are at most $2cm+1$ distinct intervals I_k with nonempty intersection with $(x-\delta,x+\delta)$. Furthermore, we know by definition of f_m that for all $y \in I_k$,

$$|f_m(x)-f_m(y)| = \begin{cases} 0 & |k-n| \text{ even} \\ 2m2^{-m^2} & |k-n| \text{ odd}\end{cases}$$

By Lemma 3 and the reverse triangle inequality, when $|k-n|$ is odd we see that for all $y \in I_k$,

$$|f(x)-f(y)| \geq \left| 2m2^{-m^2} + \sum_{j>m} (f_j(x) - f_j(y)) \right| \geq 2m2^{-m^2} - \frac{2}{\ln(2)} 2^{-m^2} > \delta,$$

where the last inequality holds by property 1 from earlier and our selection of delta. So, we see that if $B_\delta((x, f(x)))$ is the ball of radius δ around $(x, f(x)) \in \mathbb{R}^2$ and $\Gamma(f_{I_k})$ is the graph of the restriction of f to I_k, then

$$\Gamma(f_{I_k}) \cap B_\delta(x, f(x)) \begin{cases} \neq \emptyset & |k-n| \text{ even and } \leq cm \\ = \emptyset & \text{else} \end{cases}$$

There are at most $cm + 1$ different intervals I_k satisfying the first possibility. So, since the endpoints of the I_k's contribute no mass and each I_k is of length 2^{-m^2}, we may apply similar logic to that of (3) (i.e. covering $\Gamma(f) \cap B_\delta(x, f(x))$ with rectangles of heights given by Lemma 3) to see that

$$\mathcal{H}^1(\Gamma(f) \cap B_\delta(x, f(x))) \leq (cm+1)2^{-m^2} = cm2^{-m^2}\left(1 + \frac{1}{cm}\right) < \delta(1+\epsilon),$$

where for the final inequality we used property 2 from earlier as well as our definition of δ. To recap, we have shown that for all $\epsilon > 0$, all $\eta > 0$, and almost every $x \in [0,1]$, there exists a $\delta < \eta$ such that $\mathcal{H}^1(\Gamma(f) \cap B_\delta(x, f(x))) < \delta(1+\epsilon)$. Thus, for all $\epsilon > 0$ and almost every $x \in [0,1]$, we find that

$$\Theta^1_*(\Gamma(f), (x, f(x))) = \liminf_{\delta \to 0} \frac{\mathcal{H}^1(\Gamma(f) \cap B_\delta((x, f(x))))}{2\delta} \leq \liminf_{\delta \to 0} \frac{\delta(1+\epsilon)}{2\delta} = \frac{1}{2} + \frac{\epsilon}{2}$$

Since this holds for all $\epsilon > 0$, we may take $\epsilon \to 0$ to see that for a.e. $x \in [0,1]$, $\Theta^1_*(\Gamma(f), (x, f(x))) \leq \frac{1}{2}$. This immediately guarantees that $\Gamma(f)$ is purely unrectifiable. To see this, suppose for contradiction that $\Gamma(f)$ has positive measure intersection with some Lipschitz graph. (i.e. $\exists$ Lipschitz g s.t. $\mathcal{H}^1(\Gamma(f) \cap \Gamma(g)) > 0$). Clearly, it also holds by monotonicity of measure that for $\mathcal{H}^1$-a.e. $z \in \Gamma(f) \cap \Gamma(g)$,

$$\Theta^1_*(\Gamma(f) \cap \Gamma(g), z) = \liminf_{\delta \to 0} \frac{\mathcal{H}^1(\Gamma(f) \cap \Gamma(g) \cap B_\delta(z))}{2\delta} \leq \liminf_{\delta \to 0} \frac{\mathcal{H}^1(\Gamma(f) \cap B_\delta(z))}{2\delta} \leq \frac{1}{2}$$

Since g is Lipschitz, $\Gamma(f) \cap \Gamma(g)$ should be rectifiable, contradicting the fact that the lower density is $\leq \frac{1}{2}$ almost everywhere. So, $\Gamma(f)$ cannot have positive measure intersection with any Lipschitz graph, and therefore $\Gamma(f)$ is purely unrectifiable. ∎