MAT 526: Proofs for Seminar Presentation

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Episode 1

Theorem 1 (Whitney)**.** *Let M be a smooth manifold of dimension $\leq n$. Then, a "generic"* ***smooth*** *map from $M \to \mathbb{R}^{2n+1}$ is injective.*

Theorem 2 (Menger–Nöbeling)**.** *Let X be a topological space of Lebesgue covering dimension $\leq n$. Then, a "generic"* ***continuous*** *map from $X \to \mathbb{R}^{2n+1}$ is injective.*

Theorem 3 (Mañé)**.** *Let X be a compact Euclidean set of upper Minkowski dimension $\leq n$. Then, Lebesgue a.e.* ***linear*** *transformation from $X \to \mathbb{R}^{2n+1}$ is injective.*

Definition 4 (Dimension of a Measure)**.** *Let μ be a Borel measure on $\mathbb{R}^N$. We define the Hausdorff dimension of μ as*

$$\dim_H(\mu) := \inf\{\dim_H(X) : X \text{ is a Borel set of full } \mu\text{-measure}\}$$

Episode 2

Theorem 5 (3.1 from [1], Short). *Let μ be a σ-finite Borel measure on $\mathbb{R}^N$ of Hausdorff dimension $\dim_H(\mu) < n$. Then, for Lebesgue a.e. linear transformation $L : \mathbb{R}^N \to \mathbb{R}^n$ there exists a Borel set $X_L \subseteq \mathbb{R}^N$ such that $\mu(X_L) = 1$ and L is injective on X_L.*

Proof of Theorem. Let λ represent the Lebesgue measure on $\mathbb{R}^N$. Define

$$E := \{L : \mathbb{R}^N \to \mathbb{R}^k : \; Lx = (\langle \ell_1, x\rangle, \dots, \langle \ell_n, x\rangle) \text{ for } \|\ell_j\| \leq 1\}$$

to be the set of $N \times k$ matrices whose rows have norm ≤ 1. Define a measure η on this set to be the normalized Lebesgue measure, i.e.

$$\eta := \bigotimes_{k=1}^{n} \frac{1}{\omega_N} \lambda|_{B_1(0)}$$

such that $\eta(E) = 1$. We will first show the result for η-a.e. $L \in E$, from which the main theorem follows by rescaling.

Let $X \subseteq \mathbb{R}^n$ be a set of full μ-measure such that $\dim_H(X) < n$, and so $\mathcal{H}^n(X) = 0$. Define

$$A := \{(x, L) \in X \times E : \; Lx = Ly \text{ for some } y \in X \setminus \{x\}\}$$

and split it into $A = \bigcup_{k \in \mathbb{N}} A_k$, where

$$A_k := \left\{(x, L) \in X \times E : \; Lx = Ly \text{ for some } y \in X \text{ with } \|x - y\| \geq \frac{1}{k}\right\}$$

Lemma 6. *A_k is Borel for each $k \in \mathbb{N}$, and therefore so is A.*

Proof of Lemma. We may suppose that X is σ-compact since μ is σ-finite and a regular measure. By construction, E is σ-compact. Let $\pi : X \times X \times E$ be the projection sending $(x, y, L) \to (x, L)$. Then,

$$A_k = \pi\left(\left\{(x, y, L) \in X \times X \times E : \; Lx = Ly \text{ and } \|x - y\| \geq \frac{1}{k}\right\}\right)$$

Since $X \times X \times E$ is σ-compact, we know that $\{(x, y, L) \in X \times X \times E : Lx = Ly\}$ is also σ-compact (the preimage under ϕ will be a countable union of closed subsets of a σ-compact space). So, by continuity of π and $\|\cdot\|$, A_k is σ-compact. In particular, it is Borel. Then, so is A. ■

For $x \in X$ and $L \in E$, define the slices $A_{k,x}, A_x \subseteq E$ and $A^L \subseteq X$ via

$$A_{k,x} := \{L \in E : \; (x, L) \in A_k\}, \quad A_x := \{L \in E : \; (x, L) \in A\}, \quad A^L := \{x \in X : \; (x, L) \in A\}$$

As a slices of a Borel set, these are all Borel. We aim to show that $\eta(A_{k,x}) = 0$ for all $x \in X$ and $k \in \mathbb{N}$.

To this end, let $x \in X$ and $k \in \mathbb{N}$ be arbitrary. For notation, write $K := \{y \in X : \; \|x - y\| \geq \frac{1}{k}\}$. Let $\epsilon > 0$ be arbitrary, and so there is a countable covering of K by balls $\{B_{r_j}(y_j)\}_j$ such that $y_j \in K$ and

$$K \subseteq \bigcup_j B_{r_j}(y_j) \quad \text{and} \quad \sum_{j \in \mathbb{N}} r_j^n < \epsilon$$

Suppose that $L \in A_{k,x}$ and $y \in K$ is such that $Lx = Ly$. Then, $y \in B_{r_j}(y_j)$ for some j, and so

$$\|L(y_j - x)\|^2 = \|L(y_j - y)\|^2 \leq N\|y_j - y\|^2 \leq N r_j^2,$$

where we used Cauchy-Schwartz and our construction of E. Thus,

$$A_{k,x} \subseteq \bigcup_{j \in \mathbb{N}} \{L \in E : \; \|L(y_j - x)\| \leq \sqrt{N} r_j\}$$

We will bound the η measure of these sets with the following key geometric lemma.

Lemma 7 (2.1 in [1]). *For any $a \in \mathbb{R}^N \setminus \{0\}$ and $b \in \mathbb{R}^n$ and any $\delta > 0$, we have*

$$\eta(\{L \in E : \ \|La + b\| \leq \delta\}) \leq (CN)^{n/2} \frac{\delta^n}{\|a\|^n}$$

Proof of Lemma. Let $b = (b_1, \dots, b_n)$. Using the product structure of η and our definition of E,

$$\begin{aligned}\eta(\{L \in E : \ \|La + b\| \leq \delta\}) &\leq \prod_{i=1}^n \eta(\{L \in E : \ |L_i a + b_i| \leq \delta\}) \\ &\leq \prod_{i=1}^n \left(\frac{1}{\omega_N}\right) \lambda(\{\ell \in B_1(0) : \ |\langle \ell, a\rangle + b_i| \leq \delta\})\end{aligned}$$

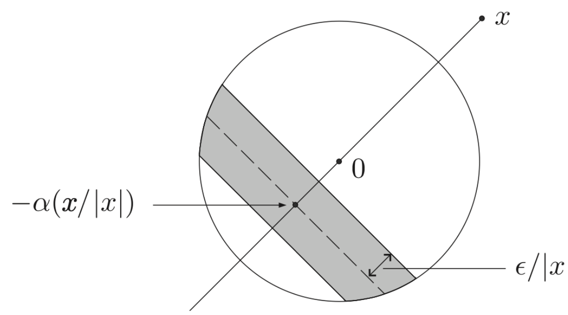

Figure 4.1 The shaded region indicates those $l \in B_N$ with $|\alpha + (l \cdot x)| \leq \epsilon$.

By the image, we compute the volume of the shaded region to be

$$\lambda(\{\ell \in B_1(0) : \ |\langle \ell, a\rangle + b_i| \leq \delta\}) = \omega_{N-1} \int_{\max\{-1, -(\delta + b_i)/\|a\|\}}^{\min\{1, (\delta - b_i)/\|a\|\}} (1 - r^2)^{(N-1)/2} dr$$

Since the integrand is ≤ 1 and the range of integration is $\leq \frac{2\delta}{\|a\|}$,

$$\lambda(\{\ell \in B_1(0) : \ |\langle \ell, a\rangle + b_i| \leq \delta\}) \leq 2\omega_{N-1} \frac{\delta}{\|a\|}$$

It can be computed through Gamma function magic that $2\frac{\omega_{N-1}}{\omega_N} \leq C'\sqrt{N}$ for some geometric constant $C' > 0$ independent of N. We may conclude that

$$\eta(\{L \in E : \ \|La + b\| \leq \delta\}) \leq \prod_{i=1}^n \left(C'\sqrt{N} \frac{\delta}{\|a\|}\right) = (CN)^{n/2} \frac{\delta^n}{\|a\|^n}$$

as desired. ■

Applying this estimate with $a = y_j - x$, $b = 0$, and $\delta = \sqrt{N} r_j$, we find

$$\eta(A_{k,x}) \leq C^{n/2} N^n \sum_{j \in \mathbb{N}} \frac{r_j^n}{\|y_j - x\|^n} \leq C^{n/2} N^n k^n \sum_{j \in \mathbb{N}} r_j^n \leq C^{n/2} N^n k^n \epsilon$$

where we used that $y_j \in K \implies \|y_j - x\| \geq \frac{1}{k}$ and the ϵ-efficiency of the cover $\{B_{r_j}(y_j)\}_j$ of K. Taking $\epsilon \to 0$, we see that $\eta(A_{k,x}) = 0$. Since $A_x = \bigcup_{k \in \mathbb{N}} A_{k,x}$, we see that $\eta(A_x) = 0$ for every $x \in X$. By Fubini's theorem, we get that $(\mu \otimes \eta)(A) = 0$, and so $\mu(A^L) = 0$ for η-a.e. $L \in E$. Thus, η-a.e. $L \in E$ is injective on the set $X_L := X \setminus A_L$ which is Borel and of full μ-measure. ■

In fact, the following stronger result holds.

Theorem 8 (3.1 from [1])**.** *Let μ be a σ-finite Borel measure on $\mathbb{R}^N$ of Hausdorff dimension $\dim_H(\mu) < \beta n$. Let $\phi : \mathbb{R}^N \to \mathbb{R}^n$ be locally β-Hölder. Then, for Lebesgue a.e. linear transformation $L : \mathbb{R}^N \to \mathbb{R}^n$ there exists a Borel set $X_L \subseteq \mathbb{R}^N$ such that $\mu(X_L) = 1$ and $\phi_L := \phi + L$ is injective on X_L.*

To prove this, one would just split X into a countable covering of open sets $\{U_j\}_j$ on which ϕ is M_j-Holder. We would then get the bound that $\|L(y_j - x)\| \leq (M_j + \sqrt{N})r_j^\beta$ and proceed to show that each $\eta(A_{k,j,x}) = 0$ in exactly the same way. There is some cleverness required to make the rescaling argument that allows us to use E work, but it's not too bad.

Episode 3

Definition 9 (s-analytic)**.** *A Borel measure μ on $\mathbb{R}^N$ is s-analytic ($s \in \{1, \ldots, m\}$) if for each Borel $U \subseteq \mathbb{R}^N$ with $\mu(U) > 0$ there exists a Borel $A \subseteq \mathbb{R}^s$ with $\lambda^s(A) > 0$ and a real analytic mapping $h : \mathbb{R}^s \to \mathbb{R}^N$ of s-dimensional Jacobian $Jh \not\equiv 0$ such that $h(A) \subseteq U$.*

Proposition 10 (Lemma IV.3(ii) in [2])**.** *If μ is s-analytic, then $\mu \ll \mathcal{H}^s$. So, if μ is nontrivial then $\dim_H(\mu) \geq s$.*

Proof of Proposition. Let $U \subseteq \mathbb{R}^N$ be Borel with $\mu(U) > 0$; we wish to show that $\mathcal{H}^s(U) > 0$. By definition, $\exists\ A \subseteq \mathbb{R}^s$ Borel with $\lambda^s(A) > 0$ and a real analytic $h : \mathbb{R}^s \to \mathbb{R}^N$ of s-dimensional Jacobian $Jh \not\equiv 0$ such that $h(A) \subseteq U$. We may assume WOLOG that $h|_A$ is an embedding since Jh vanishes on a set of measure 0 (as it is real analytic). By the area formula,

$$\int_{\mathbb{R}^N} |A \cap h^{-1}(\{y\})| d\mathcal{H}^s(y) = \int_A Jh(x) d\lambda^s(x) > 0$$

where we know that $Jh > 0$ on A and $\lambda^s(A) > 0$. Since h is injective and locally-Lipschitz (by real analyticity), we find that

$$\int_{\mathbb{R}^N} |A \cap h^{-1}(\{y\})| d\mathcal{H}^s(y) = \mathcal{H}^s(h(A))$$

So, $\mathcal{H}^s(U) \geq \mathcal{H}^s(h(A)) > 0$, as desired. Letting X be any set of full μ-measure, we therefore know that $\mathcal{H}^s(X) > 0 \implies \dim_H(X) \geq s$, and so taking an infimum we see that $\dim_H(\mu) \geq s$. ■

Theorem 11 (Theorem IV.1 in [2])**.** *Let μ be an s-analytic measure on $\mathbb{R}^N$. Let $f : \mathbb{R}^N \to \mathbb{R}^n$ be real analytic. Then, if f is injective on a set of positive μ-measure, it must be that $n \geq s$.*

Proof of Theorem. Suppose by way of contradiction that the claim is false. Then, we may find $s' \in \mathbb{N}$ such that for some $n < s'$, $f : \mathbb{R}^N \to \mathbb{R}^n$ is injective on a set U with $\mu(U) > 0$. Let s be the smallest such s' for which the above statement holds. We will show that this implies the above statement holds with $s-1$ and $n-1$, producing a contradiction. As part of the proof we will end up showing that f being injective on U is only possible if $n \geq 2$. We proceed.

By s-analyticity of μ there is a Borel $A \subseteq \mathbb{R}^s$ with $\lambda^s(A) > 0$ and a real analytic $h : \mathbb{R}^s \to \mathbb{R}^N$ with s-dimensional Jacobian $Jh \not\equiv 0$ such that $h(A) \subseteq U$, and so f is injective on $h(A)$. Write $f(x) = (f_1(x), \ldots, f_n(x))^\top$ with $f_j : \mathbb{R}^N \to \mathbb{R}$ as the coordinates, and define $\psi_j := f_j \circ h$ and $\psi = f \circ h$. Then, ψ and ψ_j are also real analytic. Define

$$A_0 := \{z \in A : \ D\psi(z) = 0 \text{ or } Jh(z) = 0\}\}$$

and

$$\begin{aligned} A_i : &= \{z \in A : \ D\psi_i(z) \neq 0 \text{ and } Jh(z) > 0\} \quad (i \in \{1, \ldots, n\}) \\ &= \{z \in A : \ J\psi_i(z) > 0 \text{ and } Jh(z) > 0\} \quad (i \in \{1, \ldots, n\}) \end{aligned}$$

So, $A = A_0 \cup \bigcup_{i=1}^n A_i$. We know that $D\psi$ is real analytic, and so it is either identically 0 or nonzero Lebesgue a.e.. Suppose BWOC that $D\psi$ is identically 0, in which case ψ is constant on A and so f is constant on $h(A)$. By injectivity of f we must have that h is constant on A, and so $Jh \equiv 0$, a contradiction. So, $D\psi$ and

Jh vanish on sets of Lebesgue measure 0, and so $\lambda^s(A_0) = 0$. Since $\lambda^s(A) > 0$, this means that $\lambda^s(A_i) > 0$ for some $i4 \leq n$. For each $y \in \mathbb{R}$ define

$$\mathcal{M}_y := \psi_i^{-1}(\{y\})$$

Since ψ_i is real analytic and so locally-Lipschitz and $n < s$, the coarea formula gives

$$\int_{\mathbb{R}} \mathcal{H}^{s-1}(A_i \cap \mathcal{M}_y) dy = \int_{A_i} J\psi_i(x) d\lambda^s(x) > 0$$

So, there must be a set $D \subseteq \mathbb{R}$ with $\lambda^1(D) > 0$ and

$$\mathcal{H}^{s-1}(A_i \cap \mathcal{M}_y) > 0 \quad (y \in D)$$

By Sard's theorem (ψ_i is C^∞) and the fact that $\lambda^1(D) > 0$, there is some $w \in D$ for which $J\psi_i(x) > 0$ for all $x \in \mathcal{M}_w$. Therefore, $\mathcal{M}_w$ is an $(s-1)$-dimensional real analytic submanifold of $\mathbb{R}^s$. By the Lindelof property and countable subadditivity, there must be some $z \in A_i \cap \mathcal{M}_w$ for which

$$\mathcal{H}^{s-1}(B_r(z) \cap A_i \cap \mathcal{M}_w) > 0 \quad (r > 0)$$

We may then find a real analytic embedding $\phi : \mathbb{R}^{s-1} \to \mathbb{R}^s$ and $\eta > 0$ such that

$$\phi(0) = z, \quad B_\eta(z) \cap \mathcal{M}_w \subseteq \phi(\mathbb{R}^{s-1})$$

and $\phi(\mathbb{R}^{s-1})$ Borel in $\mathbb{R}^s$. By monotonicity of measure,

$$\mathcal{H}^{s-1}(A_i \cap \phi(\mathbb{R}^{s-1})) > 0$$

Define the set $C := \phi^{-1}(A_i \cap \phi(\mathbb{R}^{s-1})) \subseteq \mathbb{R}^{s-1}$ and the map $\tilde{h} := h \circ \phi : \mathbb{R}^{s-1} \to \mathbb{R}^N$. We claim that C and $\tilde{h}$ produce the contradiction along with the map

$$\tilde{f} : \mathbb{R}^N \to \mathbb{R}^{n-1} \quad \text{defined by} \quad f(x) = (f_1(x), \dots, f_{i-1}(x), f_{i+1}(x), \dots, f_n(x))^\top$$

To see this, note first that C is Borel. Since ϕ is injective and locally-Lipschitz and $\mathcal{H}^{s-1}(\phi(C)) > 0$, we find that $\lambda^{s-1}(C) > 0$ as well. $\tilde{h}$ is clearly real analytic. To show that $J\tilde{h} \not\equiv 0$, we will show $J\tilde{h}(0) > 0$. This follows since ϕ is an embedding, and so $rank(D\phi(0)) = s-1$. Also, since $Jh(z) > 0$, $rank(Dh(z)) = s$, and so the chain rule tells us that $D\tilde{h}(0)$ is full rank. In particular, $J\tilde{h}(0) > 0$. It remains to show that f is injective on $\tilde{h}(C)$. Note that for all $x \in C$,

$$f_i(\tilde{h}(x)) = f_i(h(\phi(x)) = \psi_i(\phi(x)) = w$$

where we used that $\phi(x) \in \mathcal{M}_w$ by construction of ϕ. So, f_i is constant on $\tilde{h}(C)$. Since f was injective on $h(A)$ and therefore on $\tilde{h}(C)$, this means that $\tilde{f}$ must also be injective on $\tilde{h}(C)$. This reasoning also shows that $n > 1$, since if $n = 1$ then $f = f_i$ and so f is constant on a set of positive measure and cannot be injective. We arrive at a contradiction, and the theorem is proved. ■

We have shown that if $n < s$ then f cannot be injective on a set of **positive** μ-measure, let alone a set of full μ-measure. This is what makes this a strong converse; if we want to encode with a decoding error probability < 1, it requires $n \geq s$ embedding dimensions.

References

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(2). Giovanni Alberti, Helmut Bölcskei, Camillo De Lellis, Günther Koliander, and Erwin Riegler. Lossless Linear Analog Compression. arXiV preprint, 2016. https://arxiv.org/pdf/1605.00912.pdf