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Recall the following

Def: (Hausdorff measure & dimension)

Let $E \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $\delta \in (0, \infty]$. Define the Hausdorff premeasure

$$\mathcal{H}^\alpha_\delta(E) = \frac{\omega_\alpha}{2^\alpha} \inf \left\{ \sum_{i \in I} \operatorname{diam}(E_i)^\alpha : \{E_i\} \text{ is a cover of } E \text{ with sets of diam} \leq \delta \right\}$$

We define

$$\mathcal{H}^\alpha(E) := \lim_{\delta \downarrow 0} \mathcal{H}^\alpha_\delta(E) = \sup_{\delta > 0} \mathcal{H}^\alpha_\delta(E)$$

Remark:
- $\mathcal{H}^\alpha_\delta(E) \geq \mathcal{H}^\alpha_{\delta'}(E)$ if $\delta \leq \delta'$, so the limit is well-defined $\forall E \subseteq \mathbb{R}^n$.
- $\mathcal{H}^\alpha(\cdot)$ is an outer or exterior measure
- If $\alpha = n$, then $\mathcal{H}^\alpha(\cdot) = \mathcal{L}^n(\cdot)$ ← lebesgue measure
- If $\alpha = 0$ then $\mathcal{H}^\alpha(\cdot) = \#(\cdot)$ ← counting measure

Defn (Exterior measure)

An exterior measure is a set fn $\mu : \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}_+$ if

$$\mu(\emptyset) = 0 \quad \text{and} \quad \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) \leq \sum_{i \in \mathbb{N}} \mu(A_i)$$

countably subadditive

Prop: $\mathcal{H}^\alpha(A \cup B) = \mathcal{H}^\alpha(A) + \mathcal{H}^\alpha(B)$ if $\inf_{x \in A, y \in B} |x - y| = d(A, B) > 0$

Proof: do this

Defn (Caratheodory's Construction)

Let $\mathcal{M} := \{ E \subseteq \mathbb{R}^n \text{ s.t. } \mu(A) = \mu(E \cap A) + \mu(A \setminus E) \ \forall A \}$

Then $\mathcal{M}$ is a σ-algebra, containing Borel sets & sets of measure 0 as desired.

Defn: (Outer regularity)

An (outer) measure is regular if

$$\forall A \subseteq \mathbb{R}^n, \ \exists E \text{ Hausdorff-}\alpha\text{-measurable s.t. } A \subseteq E \text{ and } \mathcal{H}^\alpha(A) = \mathcal{H}^\alpha(E)$$

Replacing "Hausdorff-α-measurable" with "Borel", we get a Borel (outer) measure.

If E is $\mathcal{H}^\alpha$-measurable and $\mathcal{H}^\alpha(E) < \infty$ then $\mu := \mathcal{H}^\alpha \llcorner E$ is a Radon measure.

Remark: $\mathcal{H}^\alpha$ is a Borel, regular outer measure!

Defn (Restriction of measures) $\quad (\mathcal{H}^\alpha \llcorner E)(A) := \mathcal{H}^\alpha(A \cap E)$

> **Things to know:**
> - weak* topo on space of Radon measures
> - metrizability of bounded subsets on the space of Radon measures

Lemma:

Let ν_i be a sequence of Radon measures s.t.

$$\nu_i \rightharpoonup^* \nu \quad \left(\text{i.e. } \int f \, d\nu_i \to \int f \, d\nu \quad \forall f \in C_c(\mathbb{R}^n)\right)$$

Then,

$$\liminf_{i \to \infty} \nu_i(U) \geq \nu(U) \quad \forall U \text{ open}$$

$$\limsup_{i \to \infty} \nu_i(K) \leq \nu(K) \quad \forall K \text{ closed}$$

Thus, $\lim_{i \to \infty} \nu_i(U) \to \nu(U)$ if $\nu(\partial(U)) = 0$ for Borel U.

Remark:

$\mathcal{H}^\alpha(E) < \infty \implies \mathcal{H}^\beta(E) = 0 \quad \forall \beta > \alpha.$ & $\mathcal{H}^\alpha(E) > 0 \Rightarrow \mathcal{H}^\beta(E) = \infty \quad \forall \beta < \alpha$

So, there is a unique $\alpha \in \mathbb{R}_+$ s.t. $\mathcal{H}^\alpha(E) \notin \{0, \infty\}$.

We call this unique α to be the Hausdorff dimension $\dim_H(E)$

The w_α in Hausdorff measure:

Recall
$$\mathcal{H}^\alpha_\delta(E) = w_\alpha \inf\left\{ \sum_{i\in I}\left(\frac{\operatorname{diam}(E_i)}{2}\right)^\alpha : \begin{array}{l}\{E_i\} \text{ is a cover of } E \\ \text{with sets of diam} \le \delta\end{array}\right\}$$

If $A_i = B_{r_i}(x_i)$ are balls then $\frac{\operatorname{diam}(A_i)}{2} = r_i$.

So, we select $\quad w_k := \lambda^k(B_1(0))$ where $B_1(0)$ is a unit ball in $\mathbb{R}^k$ for $k \in \mathbb{N}$

We may extend $\quad w_\alpha := \pi^{\alpha/2}\,\Gamma(1+\frac{\alpha}{2})$ where $\Gamma(t) = \int_0^\infty s^{t-1}e^{-s}\,ds$

We select this so that

- $w_\alpha = w_k$ s.t. $\mathcal{H}^k = \lambda^k$ for integer k
- w_α is holomorphic w.r.t. α

Prop:

If $f:\mathbb{R}^m \to \mathbb{R}^n$ is L-Lipschitz, then $\mathcal{H}^\alpha(f(E)) \le L^\alpha \mathcal{H}^\alpha(E)$

Moreover, $\quad \mathcal{H}^\alpha(\lambda E) = |\lambda|^\alpha \mathcal{H}^\alpha(E)$ for $\lambda \neq 0$, $\lambda E := \{\lambda x : x \in E\}$

Remark:

our measure space

Recall that μ is σ-finite if $X = \bigcup_{i=1}^\infty E_i$ for $\mu(E_i) < \infty$

We note that $\mathcal{H}^\alpha(\cdot)$ need not be σ-finite. Prove this!

Guiding Question:	If $\dim_{\mathcal{H}}(E) = k$ and $0 < \mathcal{H}^k(E) < \infty$, how far is E from a C^1 k-dim submanifold of $\mathbb{R}^n$?

Rectifiability

Defn: (Rectifiability)

We say $E \subseteq \mathbb{R}^n$ (should be $\mathcal{H}^k$-measurable) is (countably) k-rectifiable if E can be covered $\mathcal{H}^k$-a.e. by countably many C^1 k-dim submanifolds.

I.e. $E = E_0 \cup \bigcup_{i=1}^{\infty} E_i$, where $\mathcal{H}^k(E_0) = 0$ and $E_i = E \cap \Gamma_i$ (C^1 k-dim submanifold)

Such sets are close enough to C^1 submanifolds!

Remarks

1) Rectifiable sets are approximable efficiently by affine subspaces.

2) The area formula holds! So, $\mathcal{H}^k(E)$ is computable using diff geo defn of volume.

3) If $k=n-1$, we treat "sets of finite perimeter" as those with rectifiable (almost C^1 submanifold) boundary, and then we can do Green's Thm and such.

4) Rectifiable sets play well with product structure & Fubini slices.

Prop

An $\mathcal{H}^k$-measurable $E \subseteq \mathbb{R}^n$ is k-rectifiable $\iff$ $\exists \{\Gamma_i\}_{i\in\mathbb{N}}$ of Lipschitz k-dim graphs s.t. $\mathcal{H}^k(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$

Note that these are Lipschitz graphs, not just C^1 graphs!

Theorem (Rademacher)

If $f: U \to \mathbb{R}^k$ ($U \subseteq \mathbb{R}^n$ open) is Lipschitz, then f is diff. $\mathcal{L}^n$-a.e.
I.e. $\exists$ linear map $D|_x : \mathbb{R}^n \to \mathbb{R}^k$ s.t. $f(y) - (f(x) + D(y-x)) = o(|y-x|)$

Theorem (Whitney)

If $f: U \to \mathbb{R}^k$ ($U \subseteq \mathbb{R}^n$ open) is Lipschitz, then $\forall \varepsilon > 0$ $\exists \tilde{f}: U \to \mathbb{R}^k$ C^1 s.t. $\mathcal{L}^n(\{f \neq \tilde{f}\}) < \varepsilon$

So, C^1 functions approximate Lipschitz fns up to sets of arbitrarily small measure.

Theorem (Extension)

If $f: K \to \mathbb{R}^\ell$ ($K \subset \mathbb{R}^n$; $K \subseteq \mathbb{R}^n$) Lipschitz, $\exists$ an extension $\tilde{f}: \mathbb{R}^n \to \mathbb{R}^\ell$ which is Lipschitz.

Remark: $\ell = 1$, it's easy to show $\exists \tilde{f}$ with $\mathrm{lip}(\tilde{f}) = \mathrm{lip}(f)$.
It's true, but hard to show that it holds for $\ell > 1$ (Kirszbraun)

Prop:
If E is $\mathcal{H}^k$-measurable and $E \subseteq \Gamma$ C^1 submanifold, then E is rectifiable!

Proof: Duh $\square$

Corollary:
Any σ-finite $\mathcal{H}^k$-measurable $E \subseteq \mathbb{R}^n$ can be decomposed as

$$E = \underbrace{R}_{k\text{-rect}} \cup \underbrace{P}_{\text{"purely } k\text{-unrectifiable"}}, \quad \text{where} \quad \mathcal{H}^k(P \cap \Gamma) = 0 \quad \forall \Gamma \ C^1 \ k\text{-submanifold}$$

Proof: Iteratively remove the intersection with C^1 submanifolds. $\square$

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Example: purely unrectifiable sets!

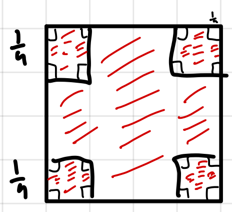
$\exists$ $\mathcal{H}^k$-measurable $E \subseteq \mathbb{R}^n$ with $0 < \mathcal{H}^k(E) < \infty$, $1 \le k \le n-1$ s.t. E is unrectifiable (in fact, E will be compact).

We focus on $n=2$, $k=1$. So, $\exists E \subseteq \mathbb{R}^2$ s.t. $\mathcal{H}^1(E) \in (0, \infty)$.

Method 1:

Define F via the 1D "ternary" Cantor-type set by starting with $[0,1]$, dropping each connected piece into $[0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{3}{4})$, $[\frac{3}{4}, 1]$ and iterating. We know $\mathcal{H}^1(F) = \frac{1}{2}$. Set $E = F \times F$.

Alternatively,

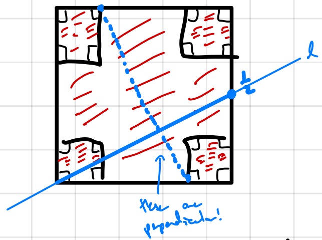


We can cover E_k with 4^k cubes Q_j^k of size $\frac{1}{4^k}$, and $\mathrm{diam}\, Q_j^k = \frac{\sqrt{2}}{4^k}$. Then,

$$\mathcal{H}^1_{\frac{\sqrt{2}}{4^k}}(E) \le \omega_1 \sum_{j=1}^{4^k} \frac{\mathrm{diam}(Q_j^k)}{2} = 4^k \frac{\sqrt{2}}{4^k} = \sqrt{2}$$

$$\Rightarrow \mathcal{H}^1(E) \le \sqrt{2}$$

We can show that this is the best we can do in the following way:



The orthogonal projection $\mathbb{P}_\ell(E_k) = \sigma = \sqrt{1+\left(\frac{1}{2}\right)^2} = \sqrt{3}/2$

$\Rightarrow \mathbb{P}_\ell(E) = \sigma$

However, $\mathbb{P}_{x_1}(E)$ and $\mathbb{P}_{x_2}(E)$ have Lebesgue 0.

Since $\mathbb{P}_\ell$ is 1-Lipschitz, $\mathcal{H}^1(\sigma) \le Lip(\mathbb{P}_\ell)\, \mathcal{H}^1(E) = \mathcal{H}^1(E)$

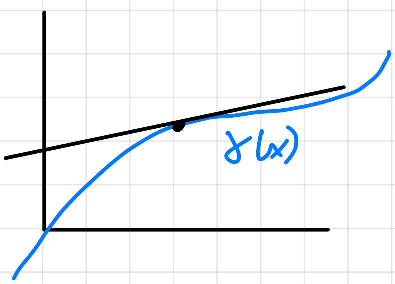
Since $\mathcal{H}^1$ agrees with the Lebesgue $\mathcal{L}^1$, we see $\mathcal{H}^1(E) \ge \sqrt{3}/2$.

Now, let Γ be a C^1 curve with a param $\gamma : \mathbb{R} \to \mathbb{R}^2$ s.t.

$$\mathcal{H}^1(\gamma(F)) = \int_F |\dot{\gamma}(t)|\, dt$$

If $\mathcal{H}^1(\Gamma \cap E) > 0$, then $\exists F \subseteq \mathbb{R}$ measurable with $\mathcal{L}^1(F) > 0$ s.t. $\gamma(F) \subseteq E$

Pick a x s.t. $\mathcal{L}(F \cap B_\delta(x)) > 0 \quad \forall \delta > 0$.



Note that one of the following always holds:

$\mathcal{L}^1(\mathbb{P}_{x_1}\gamma(F \cap B_\delta(x))) > 0$ or $\mathcal{L}^1(\mathbb{P}_{x_2}(\gamma(F \cap B_\delta(x)))) > 0$ TF?

However, $\mathcal{L}^1(\mathbb{P}_{x_1}(E)) = \mathcal{L}^1(\mathbb{P}_{x_2}(E)) = 0$. So, E hides from the graphs of all C^1 curves, and is unrectifiable!

$\square$

Covering Lemmas:

$(5x)$-Covering Theorem:

Let X be a separable metric space and $\{B_{\Omega_i}(x_i)\}_{i\in I}$ be a collection of open balls and $\sup_{i \in I}\{\Omega_i\}$ is finite.

Then, $\exists \mathcal{F} \subset I$ s.t. $\{B_{\Omega_j}(x_j)\}_{j \in \mathcal{F}}$ consists of pairwise disjoint balls and $\bigcup_{i\in I} B_{\Omega_i}(x_i) \subseteq \bigcup_{\mathcal{F}\subset I} B_{5\Omega_j}(x_j)$

Besicovitch Covering Theorem:

Let $A \subseteq \mathbb{R}^n$ be a Borel bounded set. Let $\mathcal{F} = \{\overline{B_\Omega}(x)\}$ be a Vitali cover of A (i.e. $\forall x \in A\ \forall \varepsilon > 0$, $\exists B_\delta(x) \in \mathcal{F}$ s.t. $\delta < \varepsilon$).

Let μ be a Radon measure.

Then, $\exists \mathcal{F}' \subseteq \mathcal{F}$ consisting of pairwise disjoint balls s.t.

$$\mu\Big(A \setminus \bigcup_{B_\Omega(x)\in\mathcal{F}'} B_\Omega(x)\Big) = 0$$

($\mathcal{F}'$ is pairwise disjoint and covers A μ-a.e.)

Theorem (Radon-Nikodym)

If μ and ν are Radon measures on $\mathbb{R}^n$, then $\exists \mu_s$ s.t.
$$\mu = f\nu + \mu_s \quad \text{s.t.} \quad f \in L^1(\mathbb{R}^n, \nu)$$
and $\exists A$ s.t. $\nu(A) = 0$ and $\mu_s(\mathbb{R}^n \setminus A) = 0$ (i.e. $\mu_s \perp \nu$).

In fact, $\mu_s = \mu \llcorner E$ where $E := \left\{ \lim_{\delta \to 0} \frac{\mu(B_\delta(x))}{\nu(B_\delta(x))} = \infty \right\}$.

restriction to E

Also,
$$f(x) := \begin{cases} \lim_{\delta \to 0} \frac{\mu(B_\delta(x))}{\nu(B_\delta(x))} & \forall x \text{ where the limit exists and is finite} \\ 0 & \text{else} \end{cases}$$

Besicovitch Diff. Theorem

relates to the derivative of a certain function, that is BV

(holds in more general spaces)

Density Talk:

Defn:

We define the upper density of a set E at x by
$$\Theta^{\alpha *}(E, x) := \limsup_{\delta \to 0} \frac{\mathcal{H}^\alpha(E \cap B_\delta(x))}{\omega_\alpha \delta^\alpha}$$

Similarly, the lower density is the liminf.

is this the same E?

For any $\mu = \mathcal{H}^\alpha \llcorner E$, we can define the upper/lower densities w.r.t. μ.

Theorem (Besicovitch - Preiss)

Let $0 < \mathcal{H}^k(E) < \infty$ for $k \in \mathbb{N}$, E $\mathcal{H}^k$-measurable.

Then,

$\mathcal{H}^k$-a.e. or 1-a.e.?

$$E \text{ is rectifiable} \iff \Theta^{k*}(E, x) = \Theta^k_*(E, x) = 1 \quad \text{for a.e. } x \in E$$

There are analogs of this for Radon measures μ and when they are rectifiable

Prop (Marstrand)

$\forall \alpha \notin \mathbb{N}$, $\quad \Theta^{\alpha *}(E, x) > \Theta^\alpha_*(E, x)$ for $\mathcal{H}^\alpha$-a.e. x

So, no E can be $\mathcal{H}^\alpha$-rectifiable.

We mentioned that $\mathcal{H}^\alpha(E) = \sup\{\mathcal{H}^\alpha(K) : K \subseteq E \text{ closed}\}$

Prop:

Let μ be a Radon measure, E be Borel-measurable.

(a) If $\Theta^{\alpha *}(\mu, x) \geq \gamma > 0 \quad \forall x \in E$, and E closed! then $\mathcal{H}^\alpha(E) \leq \frac{5^\alpha}{\gamma} \mu(E)$

(b) If $\Theta^{\alpha *}(\mu, x) \leq \gamma < \infty \quad \forall x \in E$, then $\mu(E) \leq \gamma \mathcal{H}^\alpha(E)$

So, these densities allow us to compare μ with $\mathcal{H}^\alpha$. Compare this with Lebesgue density stuff.

Proof: (a) Fix $\delta > 0$. $\forall x \in E, \exists r_j \downarrow 0$ st. $\mu(B_{r_j}(x)) \geq (\gamma - \delta) \omega_\alpha r_j^\alpha$

$\forall x$ pick $r(x)$ st. $r(x) < \delta/10$ and $\mu(B_{r(x)}(x)) \geq (\gamma - \delta) \omega_\alpha r(x)^\alpha$

By the 5r-covering theorem, $\exists \{B_{r_j}(x_j)\}$ pairwise disjoint st. $\{B_{5r_j}(x_j)\}$ covers E. So,

$$\mathcal{H}^\alpha_\delta(E) \leq \omega_\alpha \sum_{j=1}^\infty (5r_j)^\alpha \leq 5^\alpha \sum_{j=1}^\infty r_j^\alpha \omega_\alpha \leq \frac{5^\alpha}{\gamma - \delta} \sum_{j=1}^\infty \mu(B_{r_j}(x_j)) \underset{\text{by pairwise disjoint}}{\leq} \frac{5^\alpha}{\gamma - \delta} \mu\left(\bigcup_{j=1}^\infty B_{r_j}(x_j)\right)$$

(b) We know $\Theta^{\alpha}_{*}(\mu, x) \leq \gamma$. Suppose WLOG that $\mathcal{H}^\alpha(E) < \infty$. Take $\nu := \mathcal{H}^\alpha \llcorner E$ and apply Besicovitch diff'n. Then,

$$\mu \llcorner E = f \mathcal{H}^\alpha \llcorner E + \mu_s \quad \text{with} \quad f(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\mathcal{H}^\alpha(E \cap B_r(x))} \cdots$$

something's wrong. Lets do (b) next time

Lemma:

If $\mathcal{H}^\alpha(E) < \infty$, then

$$\frac{1}{2^\alpha} \leq \Theta^{\alpha *}(E, x) \leq 1 \quad \text{for } \mathcal{H}^\alpha\text{-a.e. } x$$

Proof: (≤ 1) Assume WLOG $\Theta^{\alpha *}(E, x) \geq 1 + \delta \quad \forall x \in E'$ with E' measurable and $\mathcal{H}^\alpha(E') > 0$.

(For $\mathcal{H}^\alpha$-a.e. $x \in E'$, we know by Bes. Diff'n that
$$\lim_{R \to 0} \frac{\mathcal{H}^\alpha(E' \cap B_R(x))}{\mathcal{H}^\alpha(E \cap B_R(x))} = 1, \quad \text{so WLOG}$$
)

By Besicovitch covering $\exists$ a pairwise disjoint covering of E' of balls of diam $\leq \eta$ st.

$$\mathcal{H}^\alpha(E' \cap B_{r_i}(x_i)) \geq \omega_\alpha(1 + \delta - \eta) r_i^\alpha \quad \text{and} \quad \mathcal{H}^\alpha(\underbrace{E' \setminus \bigcup B_{r_i}(x_i)}_{\text{call this } E''}) = 0.$$

$$\Rightarrow \sum_{i=1}^\infty \omega_\alpha r_i^\alpha \leq \frac{1}{1 + \delta - \eta} \mathcal{H}^\alpha(E')$$

We can show that $\mathcal{H}^\alpha(E'') = 0 \iff \mathcal{H}^\alpha_\infty(E'') = 0$ (good exercise).

$\forall \eta > 0$, we may therefore cover E' with $\{A_i\}$ st $\operatorname{diam}(A_i) < \eta$ and $\omega_\alpha \sum_i \frac{\operatorname{diam}(A_i)^\alpha}{2^\alpha} < \eta$

Therefore, we may estimate

$$\mathcal{H}^\alpha_\eta(E') \le w_\alpha \sum_i \frac{\operatorname{diam}(A_i)^\alpha}{2^\alpha} + w_\alpha \sum r_i^\alpha \le \eta + \frac{\mathcal{H}^\alpha(E')}{1+\delta-\eta}$$

Letting $\eta \to 0$, $\mathcal{H}^\alpha(E') \le \frac{\mathcal{H}^\alpha(E')}{1+\delta} \Rightarrow \mathcal{H}^\alpha(E') = 0$.

So, $\Theta^{\alpha *}(E,x) \le 1+\delta$ $\mathcal{H}^\alpha$-a.e. x. Taking $\delta \to 0$, we are done.

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We turn now to Besicovitch's theory of 1D sets ($\subseteq \mathbb{R}^2, \subseteq \mathbb{R}^n$), and we will work our way up to the $\frac{1}{2}$ conjecture.

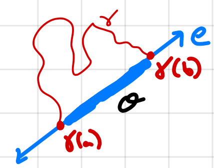
Defn:

A rectifiable curve is the image of a continuous, injective map $\gamma : [0,1] \to \mathbb{R}^n$ with finite $\mathcal{H}^1$-measure.

or S^1, if you leave out a point and cover w/ closed intervals

Lemma:
A rectifiable curve is a 1-rectifiable set.

Proof: Certainly $\mathcal{H}^1(\gamma([a,b])) \geq |\gamma(b) - \gamma(a)|$ since projection to the line e is a 1-Lipschitz map, and so



$$|\gamma(b) - \gamma(a)| = \mathcal{H}^1(\sigma) = \mathcal{H}^1(\pi_e(\gamma([a,b]))) \leq \mathcal{H}^1(\gamma([a,b]))$$

Next, we WTS the map $t \mapsto \mathcal{H}^1(\gamma([0,t]))$ is continuous. Define $\mu := \mathcal{H}^1 \llcorner \gamma([0,1])$. Then,

$$\mathcal{H}^1(\gamma([s,t])) \leq \mu(\overline{B}_r(\gamma(s))), \text{ where } r := \max_{\tau \in [s,t]} |\gamma(\tau) - \gamma(s)|$$

So,

$$\lim_{t \to s} \mu(\overline{B}_r(\gamma(s))) = \mu(\{\gamma(s)\}) = 0 \Rightarrow \lim_{t \to s} \mathcal{H}^1(\gamma([s,t])) = 0$$

Next, we will reparametrize via arc length. Define

$$\tilde{\gamma}(\tau) := \{\gamma(s) : \mathcal{H}^1(\gamma([0,s])) = \tau\} \quad \text{for} \quad \tau \in [0, \mathcal{H}^1(\gamma([0,1]))]$$

By injectivity of γ, this is well-defined (?). Then, $\tilde{\gamma}$ is 1-Lipschitz, and $\operatorname{im} \tilde{\gamma} = \operatorname{im} \gamma$. Via Whitney's Thm and implicit fn theorem (?), covering by Lipschitz graphs (as in the defn of "rectifiable") $\Leftrightarrow$ covering by images of Lipschitz fns. (is this only in 1D?)

$\square$

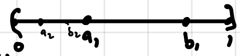
Lemma:

If $\gamma : [0,1] \to \mathbb{R}^n$ is continuous and $\gamma(0) \neq \gamma(1)$, then $\exists \tilde{\gamma} : [0,1] \to \mathbb{R}^n$ continuous s.t.

i) $\gamma(0) = \tilde{\gamma}(0)$ ii) $\gamma(1) = \tilde{\gamma}(1)$ iii) $\tilde{\gamma}$ injective iv) $\tilde{\gamma}([0,1]) \subseteq \gamma([0,1])$

Proof: Let a_1, b_1 be s.t. $\gamma(a_1) = \gamma(b_1)$ and $|b_1 - a_1|$ is maximal. Then,

$$\forall t \notin [0,1] \setminus [a_1, b_1], \quad \gamma(t) \neq \gamma(a_1) = \gamma(b_1).$$

Also, if $\gamma(a_2)=\gamma(b_2)$, then 

Keep picking maximal non-injective pairs; there may be countably many.
Let $I_j := [a_j, b_j]$ and consider removing $\bigcup_j I_j$ and squishing the domain together.
Then, we get $\gamma_N : [0, 1-\sum_{j=1}^{N}(b_j-a_j)] \to \mathbb{R}^n$ and $\bar{\gamma} : [0, 1-\sum_{j=1}^{\infty}(b_j-a_j)] \to \mathbb{R}^n$

with $\gamma_N \to \bar{\gamma}$ pointwise (by continuity of γ). So, since each γ_N is continuous, so is $\bar{\gamma}$.
Furthermore, $\bar{\gamma}$ injective by our algorithm. Define $\tilde{\gamma} := \begin{cases} \bar{\gamma} & \in [0, 1-\sum_{j=1}^{\infty}(b_j-a_j)) \\ \gamma(1) & \text{else} \end{cases}$

∎

Defn:

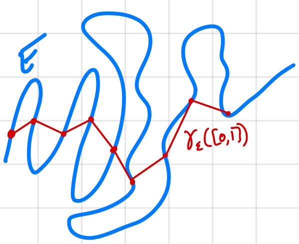
A continuum is a closed, connected set.

Theorem:

A continuum E with finite $\mathcal{H}^1$ measure is rectifiable.

Proof: The idea of the proof is to cover E with countably many continuous curves w/ finite $\mathcal{H}^1$ measure + a set of measure 0. First a lemma:

Lemma: Continuum of finite $\mathcal{H}^1$ measure is arcwise? (check) connected.
Proof: Fix $x_0, y_0 \in E$ arbitrary. Find a chain $x_0 = x_1, \dots x_N = y_0$ s.t. $x_i \in E$ and $|x_i - x_{i+1}| \le \varepsilon$ and $B_{\frac{\varepsilon}{2}}(x_{2j+1}) \cap B_{\frac{\varepsilon}{2}}(x_{2k+1}) = \emptyset \quad \forall j \ne k$



The piecewise-linear fn going through this chain (call it $\gamma_\varepsilon : [0,1] \to \mathbb{R}^n$) has that $\gamma_\varepsilon(0) = x_0$ and $\gamma_\varepsilon(1) = y_0$. Also, γ_ε will be Lipschitz and so will $\lim_{\varepsilon \downarrow 0} \gamma_\varepsilon$. ?why So, all need to show is finite $\mathcal{H}^1$ measure.

For each j, define $f_j : \mathbb{R}^n \to \mathbb{R}$ s.t. $f_j(x) := |x - x_{2j+1}|$
So, f_j is 1-Lipschitz and $f_j(E) \supseteq [0, \frac{\varepsilon}{2}]$. Furthermore, $f_j(E)$ is connected and

$$\mathcal{H}^1(B_{\frac{\varepsilon}{2}}(x_{2j+1}) \cap E) \ge \frac{\varepsilon}{2}$$

Accumulating this, $\left(\frac{N-1}{2}\right)\frac{\varepsilon}{2} \le \mathcal{H}^1(E)$. So,

$$\mathcal{H}^1(\gamma_\varepsilon([0,1])) = \sum_{i=1}^{N} |x_i - x_{i-1}| \le 4\mathcal{H}^1(E) + \varepsilon$$

∎

Back to the theorem. Take γ_1 to be the geodesic connecting the two most distant points.
w.r.t. implicit metric

Take $\gamma_2 :=$ geodesic connecting most distant points in $E \setminus \gamma_1$ to γ_1 ($E\setminus\gamma_1$ and γ_1 are connected since E is)
Take $\gamma_3 :=$ " $E \setminus (\gamma_1 \cup \gamma_2)$ to $\gamma_1 \cup \gamma_2$

If this ends finitely, then we have fully covered E and are done.
If not, we have $\mathcal{H}^1(E) \ge \sum_{i=1}^{\infty} \mathcal{H}^1(\gamma_i)$

We must show that the points left over are $\mathcal{H}^1$-null.

Claim: $E \setminus \bigcup_{i=1}^{\infty} \gamma_i$ has $\mathcal{H}^1$ measure 0.

Let $\varepsilon > 0$.

Define $\mathcal{B}_k := \{ \overline{B_r(x)} \subseteq \mathbb{R}^n \setminus \bigcup_{i=1}^{k} \gamma_i$ s.t. $r > 0$, $x \in E \setminus \bigcup_{i=1}^{k} \gamma_i \}$

We also impose on $B_r(x)$ that $\mathcal{H}^1(B_r(x) \cap E) \leq (1+\varepsilon)\, \mathrm{diam}(B_r(x))$

Since $\Theta^{1*}(E,x) \leq 1$ for $\mathcal{H}^1$-a.e. x, this doesn't change that $\mathcal{B}_k$ is a fine cover of $\underbrace{E \setminus (E' \cup \bigcup_{j=1}^{k} \gamma_j)}_{=:F}$, where $\mathcal{H}^1(E') = 0$.

Note that $\forall B \in \mathcal{B}_k$, $\mathcal{H}^1(B \cap \underbrace{\bigcup_{i=k+1}^{\infty} \gamma_i}) \geq \frac{\mathrm{diam}(B)}{2}$ $\Big\}$ we threw away a set E' of measure 0 to get this condition

note that $B \cap \bigcup_{i=1}^{k} \gamma_i = \emptyset$ by construction

Since $\mathcal{B}_k$ is fine cover, by Besicovitch Covering Theorem, let $\{B_j\}_j \subseteq \mathcal{B}_k$ be disjoint balls covering F $\mathcal{H}^1$-a.e.

Then, $\mathcal{H}^1(E \setminus \bigcup_{i=1}^{k} \gamma_i) \leq \sum_j \mathcal{H}^1(E \cap B_j) \leq (2+2\varepsilon) \sum_j \mathcal{H}^1\Big(\big(\bigcup_{i=k+1}^{\infty} \gamma_i\big) \cap B_j\Big)$

Taking $k \to \infty$, $\mathcal{H}^1(E \setminus \bigcup_{i=1}^{\infty} \gamma_i) = 0$.

$\square$

Remark: we actually only need $5r$ covering to do things like this $\mathcal{H}^k$-a.e. If we wanted μ-a.e., we need Besicovitch covering.

Defn:

Let $E \subseteq \mathbb{R}^n$ be Borel with $0 < \mathcal{H}^1(E) < \infty$. We say $x \in E$ is a regular point if $\Theta^1(E,x) = 1$ $\left(\text{i.e. iff } \Theta^1_*(E,x) = \liminf_{R \downarrow 0} \frac{\mathcal{H}^1(E \cap B_R(x))}{2R} = 1\right)$

Let $E^R := \{x \in E \text{ regular}\}$ be the regular points. Then, $E = E^R \cup \underbrace{E^I}_{E \setminus E^R}$. In fact, E^R is the "rectifiable part" of E.

Theorem:

If for $\mathcal{H}^1$-a.e. $x \in E$, $\Theta^1_*(E,x) > \frac{3}{4}$, then E is 1-rectifiable.

Remarks:

- This has been generalized to $\mathbb{R}^n$ and even to any metric space (Preiss-Tišer) with $\Theta^1_*(E,x) > \alpha$, $(\alpha = 0.7314..)$ — this paper is a good read

- Eventually, we will prove that if density exists ($\Theta_* = \Theta^*$), then E rectifiable.

★ Besicovitch conjectured that $\Theta^1_*(E,x) > \frac{1}{2}$ for $\mathcal{H}^1$-a.e. $x \Rightarrow E$ 1-rectifiable. He constructed an example of a purely unrectifiable set E s.t. $\Theta^1_*(E,x) = \frac{1}{2}$ a.e.

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Defn:

We define the convex upper density of $E \subseteq \mathbb{R}^n$ w.r.t. $\alpha \geq 0$ via

$$D_c^{\alpha *}(E,x) = \lim_{R \to 0} \sup_{\substack{\operatorname{diam}(U) < R \\ U \text{ convex} \\ U \ni x}} \left\{ \frac{\mathcal{H}^\alpha(E \cap U)}{w_\alpha \left(\frac{\operatorname{diam}(U)}{2}\right)^\alpha} \right\}$$

convex

Remark: $\forall F \subseteq \mathbb{R}^n$, $\operatorname{diam}(F) = \operatorname{diam}(\text{convex hull of } F)$. So, in our definition of the Hausdorff measure, we could have used convex sets in our covers without changing diameter. So,

$$\mathcal{H}_\delta^\alpha(E) = \inf \left\{ \sum_{i=1}^\infty w_\alpha \left(\frac{\operatorname{diam}(U_i)}{2}\right)^\alpha : \{U_i\}_i \text{ covers } E \text{ w/ closed, } \underline{\text{convex}} \ U_i \text{ s.t. } \operatorname{diam}(U_i) < \delta \right\}$$

Prop:

Let $E, E' \subseteq \mathbb{R}^n$ Borel with $0 < \mathcal{H}^\alpha(E) < \infty$. Then,

① $D_c^{\alpha *}(E,x) = 0$ for $\mathcal{H}^\alpha$-a.e. $x \notin E$
② $D_c^{\alpha *}(E,x) = D_c^{\alpha *}(E',x)$ for $\mathcal{H}^\alpha$-a.e. $x \in E \cap E'$
③ $D_c^{\alpha *}(E,x) = 1$ for $\mathcal{H}^\alpha$-a.e. $x \in E$

(We know $D_c^{\alpha *} \leq 1$. To see ≥ 1, look at the proof that $\Theta^{\alpha *} \geq 2^{-\alpha}$. We don't need to pay the 2x price)

Proof: ① Clearly, $D_c^{\alpha *}(E,x) \leq 2^\alpha \Theta^{\alpha *}(E,x)$, and so since $\Theta = 0$ for a.e. $x \in E$, we get ①.

② Follows from ①.

③ $D_c^{\alpha *}(E,x) \geq \Theta^{\alpha *}(E,x) = 1$ $\mathcal{H}^\alpha$-a.e. on E. So, we must prove the upper bound.

So, suppose BWOC that $D_c^{\alpha *}(F,x) \geq 1 + \bar{\varepsilon}$ for all $x \in F$ for some $F \subseteq E$ of positive measure and $\bar{\varepsilon} > 0$.

We will use covering arguments to show that $\mathcal{H}^\alpha(F) = 0$.

Fix $\rho > 0$ s.t. $\mathcal{H}^\alpha(F) < \mathcal{H}_{6\rho}^\alpha(F) + \varepsilon$ for some $\varepsilon > 0$.

$$\mathcal{V} := \left\{ U : U \text{ closed, convex, and } \mathcal{H}^\alpha(F \cap U) \geq \left(1 + \frac{\bar{\varepsilon}}{2}\right) w_\alpha \left(\frac{\operatorname{diam}(U)}{2}\right)^\alpha \text{ and } \operatorname{diam}(U) \leq \rho \right\}$$

← fine cover

Choose U_1 s.t. $\operatorname{diam}(U_1) > \frac{1}{2} \sup\{\operatorname{diam}(U) : U \in \mathcal{V}\} \leq \frac{\rho}{2}$

Choose U_2 s.t. $\operatorname{diam}(U_2) > \frac{1}{2} \sup\{\operatorname{diam}(U) : U \in \mathcal{V} \text{ and } U \cap U_1 = \emptyset\}$

$\vdots$

Note that $\mathcal{V}$ covers F by construction. Also,

$$\sum_{i=1}^\infty w_\alpha \left(\frac{\operatorname{diam}(U_i)}{2}\right)^\alpha \leq \frac{\mathcal{H}^\alpha(F)}{1 + \frac{\bar{\varepsilon}}{2}} < \infty, \text{ and so } \operatorname{diam}(U_j) \downarrow 0.$$

We claim that

$$\mathcal{H}_{6\rho}^\alpha(F) \leq \sum_{i=1}^\infty w_\alpha \left(\frac{\operatorname{diam}(U_i)}{2}\right)^\alpha$$

If we truncate $U_1, \ldots, U_m$ and take $B_i := B_{3\operatorname{diam}(U_i)}(x_i)$ for $x_i \in U_i$, $i > m$, then $\{U_1, \ldots, U_m, B_{m+1}, \ldots\}$ covers F.

So, $\mathcal{H}^\alpha_{6\delta}(F) \le \omega_\alpha \sum_{i=1}^{m} \left(\frac{\mathrm{diam}(U_i)}{2}\right)^2 + 6^\alpha \omega_\alpha \sum_{i=m+1}^{\infty} \left(\frac{\mathrm{diam}(U_i)}{2}\right)^\alpha$

If $x \notin F \setminus \bigcup_{i=1}^{m} U_i$, we find $x \in U \in \mathcal{V}$ for which $U \cap \left(\bigcup_{i \le m} U_i\right) = \emptyset$
Pick m_0 to be the one integer s.t. $\mathrm{diam}(U_{m_0}) < \frac{\mathrm{diam}(U)}{2}$
and $\mathrm{diam}(U_j) > \frac{\mathrm{diam}(U_j)}{2}$ $\forall j < m_0$, which exists since $\mathrm{diam}(U_j) \downarrow 0$.
We claim $U \cap U_j \neq \emptyset$ for some $j < m_0$. Since

$$\mathrm{diam}(U_{m_0}) > \frac{1}{2} \sup\left\{\mathrm{diam}(U) : U \in \mathcal{V} \text{ s.t. } \left(\bigcup_{j=1}^{m_0-1} U_j\right) \cap U = \emptyset\right\}$$

we must have $U \cap U_j \neq \emptyset$ for some $m < j < m_0$.
So, $U \subset B_{\mathrm{diam}(U) + \mathrm{diam}(U_j)} \subseteq B_{3\,\mathrm{diam}(U_j)}(x_j)$.

$\square$

Lemma:

If $E \subseteq \mathbb{R}^2$ Borel and $0 < \mathcal{H}^1(E) < \infty$, then

$$\lim_{R \downarrow 0} \underset{\substack{B_\rho(y) \ni x \\ \rho < R}}{\overset{\text{monotone in } R}{\sup}} \left\{\frac{\mathcal{H}^1(E \cap B_\rho(y))}{2\rho}\right\} = 1 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in E.$$

Proof: From prev. proposition.

Theorem:

Let $E \subseteq \mathbb{R}^2$ Borel with $0 < \mathcal{H}^1(E) < \infty$. If
$$\Theta^{1*}(E, x) > \frac{3}{4} \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in E,$$
then $\exists$ a continuum G with $\mathcal{H}^1(G) < \infty$ s.t. $\mathcal{H}^1(G \cap E) > 0$.

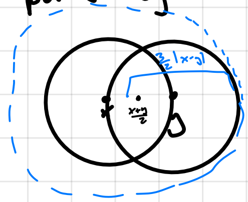
Proof: By regularity of measures, $\forall \beta > 0$ $\forall \gamma > 0$ there exist $E_0 \subseteq E$, $\alpha > 0$, $\Delta > 0$, $\bar{\delta} < \frac{\Delta}{10}$ s.t.

i) E_0 closed
ii) $\mathcal{H}^1(E_0) > 0$
iii) $\forall x \in E_0$, $\forall R < \Delta$. $\mathcal{H}^1(E \cap B_R(x)) \ge \left(\frac{3}{4} + \alpha\right) 2R$
iv) $\mathcal{H}^1(E \cap B_s(y)) \le (1+\beta) 2s$ $\forall y$ and $s < \frac{2\Delta}{3}$ s.t. $B_s(y) \cap E = \emptyset$.
v) $\mathcal{H}^1((E \setminus E_0) \cap B_r(0)) \le \gamma r$ $\forall r < 5\bar{\delta}$
vi) $B_{2\bar{\delta}} \cap E_0 \neq \emptyset$ (use an upper density estimate!)
vii) $\mathcal{H}^1(E_0 \cap B_r) \ge \frac{3}{4} \cdot 2r$ $\forall r < 3\bar{\delta}$

Define the Besicovitch circle pair of two points by

$$R(x,y) := B_{|x-y|}(x) \cap B_{|x-y|}(y)$$

$$\Rightarrow B_{\frac{3}{2}|x-y|}\left(\tfrac{x+y}{2}\right) \supseteq B_{|x-y|}(x) \cup B_{|x-y|}(y)$$



So, if $x,y \in E_0$ s.t. $|x-y| =: R < \delta$. Then,

$$\mathcal{H}^1(R(x,y)\cap E) \ge \mathcal{H}^1(B_R(x)\cap E) + \mathcal{H}^1(B_{|x-y|}(y)\cap E) - \mathcal{H}^1((B_R(x)\cup B_R(y))\cap E)$$

$$\underset{\substack{(iii)\\(iv)}}{\ge} \left(\tfrac{3}{4}+\alpha\right)4R - (1+\beta)2\cdot\tfrac{3}{2}R = (4\alpha-3\beta)R \ge \alpha R > 0$$

Define $\mathcal{G} := \{ \overline{B_R}(x) : x\in E_0\cap \overline{B_{\bar\delta}}(0)$ s.t. $R\le\delta$ and $\mathcal{H}^1((E\setminus E_0)\cap \overline{B_R}(x)) \ge \alpha R\}$ via the circle pairs. By the $5r$-covering theorem, $\exists$ a disjoint subcollection $\{B_i\}_i$ s.t.

$$\bigcup_i 5B_i \supseteq \bigcup_{B\in\mathcal{G}} B$$

Define $H := (E_0\cap B_{\bar\delta}) \cup \partial B_{\bar\delta} \cup \left(\bigcup_i 5B_i\right)$

$$G := \left(\left((E_0\cap B_{\bar\delta})\cup \partial B_{\bar\delta}\right) \setminus \left(\bigcup_i 5B_i\right)\right) \cup \left(\bigcup_i \partial(5B_i)\right)$$

Step 1: H is closed

Let $\{x_n\}_n \subseteq H$ s.t. $x_i \to x_0$. If the sequence accumulates in $(E\cap B_{\bar\delta})\cup\partial B_{\bar\delta}$ we are ok. So, suppose BWOC $x_i \in 5\overline{B_{j(i)}}$ where $j(i)\to\infty$.
Let $y_{j(i)}$ be the center of each $\overline{B_{j(i)}}$; then $|y_{j(i)} - x_i| \to 0$. Since each $y_{j(i)}\in E_0$ which is closed, $x_0\in E_0$. Thus, $x_0\in H$.

Clearly, this means G is closed.

Step 2: H is connected

Suppose BWOC H_1, H_2 closed and disjoint s.t. $H = H_1\cup H_2$.
Suppose WLOG that $\overline{B_{\bar\delta}} \subseteq H_1$. Also, each $5B_i$ is either in H_1 or H_2. So, $H_1\cap E_0 \neq\emptyset$. If one disk is in H_2, then $H_2\cap E_0\neq\emptyset$ by (iii) and (v).
Take $x_1\in H_1\cap E_0$, $x_2\in H_2\cap E_0$ at minimal distance. Draw the circle pair, and so

$R(x_1,x_2)\cap E_0 = \emptyset$ because anything else would contradict minimal distance.

So, by the circle pair bound,

$$\mathcal{H}^1(E\cap R(x_1,x_2)) \ge \alpha|x_1-x_2|$$ ← enough mass in both

But $\mathcal{H}^1(R(x_1,x_2)\cap(E\setminus E_0)) \le \mathcal{H}^1(B_{|x_1-x_2|}(x_1)\cap(E\setminus E_0)) \le \frac{\alpha}{2}|x_1-x_2|$ ← not enough mass in remainder

Since $R(x_1,x_2)\cap E_0=\emptyset$, we see that H is connected.

So, G is connected, and is thus a continuum.

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Theorem: (Besicovitch)

Let $E \subseteq \mathbb{R}^2$ be Borel with $0 < \mathcal{H}^1(E) < \infty$ s.t. $\Theta^1_*(E,x) > \frac{3}{4}$ for $\mathcal{H}^1$-a.e. $x \in E$,

then E is rectifiable.

Proof: Suppose BWOC it is not, then use measure theory to find a closed, purely unrect. $E' \subseteq E$ with $\Theta^1_*(E',x) \geq \frac{3}{4} + \alpha$ at $\mathcal{H}^1$-a.e. $x \in E'$. Find a continuum G with $0 < \mathcal{H}^1(G) < \infty$ s.t. $\mathcal{H}^1(G \cap E') > 0$. we now repeat the proof of finding such a G.

Out of measure theory and standard considerations, we may find $F \subseteq E$ s.t. $0 \in F$ and

- $\mathcal{H}^1(E' \cap B_R(x)) \geq (\frac{3}{4} + \alpha) 2R \quad \forall x \in F,\ \forall R < \Delta$
- $\mathcal{H}^1(E' \cap U) \leq \operatorname{diam}(U)(1+\beta) \quad \forall U \ni x \in F$ with $\operatorname{diam}(U) < \Delta$ (choose β s.t. circle pair property)
- $\mathcal{H}^1((E' \setminus F) \cap B_R(0)) < \gamma R \quad \forall R < \Delta$ (we can pick any γ!)
- $\partial B_{\bar\Delta} \cap F$ for all $\bar\Delta < \frac{\Delta}{10}$

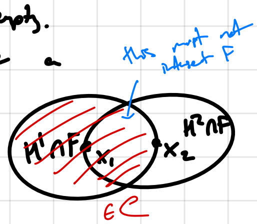
Define $\mathcal{C} := \{ \overline{B}_R(x) : x \in F \cap \overline{B}_{\bar\Delta},\ R < \Delta, \text{ and } \mathcal{H}^1(\overline{B}_R(x) \cap (E' \setminus F)) \geq \alpha R \}$

This is a Vitali cover, and so $\exists \{\overline{B}_i\}_i \subseteq \mathcal{C}$ s.t.

$\bigcup_i 5\overline{B}_i \supseteq \bigcup_{B \in \mathcal{C}} B$ and $\overline{B}_i \cap \overline{B}_j = \emptyset$ if $i \neq j$.

We define
$$H := \partial B_{\bar\Delta} \cup \left(F \setminus \left(\bigcup_i 5\overline{B}_i\right)\right) \cup \left(\bigcup_i 5\overline{B}_i\right)$$
$$G := \partial B_{\bar\Delta} \cup \left(F \setminus \left(\bigcup_i 5\overline{B}_i\right)\right) \cup \left(\bigcup_i \partial(5\overline{B}_i)\right)$$

We wish to show connectedness of H, which will imply G connected.
Suppose BWOC $H = H_1 \cup H_2$ with H_i disjoint, closed, and nonempty.
We know $H_1 \cap F \neq \emptyset$ and $H_2 \cap F \neq \emptyset$; by closure we may take a pair $x_1 \in H_1 \cap F$, $x_2 \in H_2 \cap F$ of minimal distance. So,



We know that $B_{|x_1 - x_2|}(x_1) \in \mathcal{C}$, and so there is a path connecting x_1 and x_2. So, H is connected $\Rightarrow$ G connected.

Next, note that
$$\mathcal{H}^1(G) \leq \mathcal{H}^1(F) + 2\bar\Delta\pi + 10\pi \sum_{i=1}^{\infty} R_i \leq \mathcal{H}^1(F) + 2\bar\Delta\pi + 10\pi \sum_{i=1}^{\infty} \frac{\mathcal{H}^1((E' \setminus F) \cap \overline{B}_i)}{\alpha} \leq \mathcal{H}^1(F) + 2\bar\Delta\pi + \frac{\mathcal{H}^1(E' \setminus F)}{\alpha}$$

So,
$$\mathcal{H}^1(E' \cap G) \geq \mathcal{H}^1(F \cap G) \geq \mathcal{H}^1(F \cap \overline{B}_{\bar\Delta}) - \sum_{i=1}^{\infty} \mathcal{H}^1(5\overline{B}_i \cap E') \geq \mathcal{H}^1(F \cap B_{\bar\Delta}) - (1+\beta)\, 10\pi \sum_i R_i$$
$$\geq \mathcal{H}^1(F \cap B_{\bar\Delta}) - (1+\beta) \frac{\mathcal{H}^1((E' \setminus F) \cap B_{5\bar\Delta})}{\alpha} \geq \mathcal{H}^1(E \cap B_{\bar\Delta}) - \frac{2(1+\beta)}{\alpha} \gamma \cdot 5\bar\Delta$$
$$\underset{\text{by density}}{\geq} \frac{3}{4}(2\bar\Delta) - \frac{2(1+\beta)\gamma}{\alpha} 5\bar\Delta \geq \frac{3}{8}(2\bar\Delta) > 0.$$

This reproves the earlier theory and so G is a continuum.

why does this imply rectifiable?

$\square$

Besicovitch-Federer

We now turn to proving the Besicovitch-Federer theorem.

First, we must handle some ugliness.

1. We want to put a measure on $O(n)$, the orthogonal group $\{A \in \mathbb{R}^{n\times n} : A^T A = I_n\}$ (i.e. space of all linear isometries)

2. we want to put a measure on $G(n,m)$, the Grassmannian $\left\{V \subseteq \mathbb{R}^n : \begin{array}{l} V \text{ is an } m\text{-dim} \\ \text{linear subspace} \end{array}\right\}$

Remarks

1. $O(n) \subseteq \mathbb{R}^{n\times n}$ is a compact submanifold of dimension $\frac{n(n-1)}{2}$
So, take $\mu := \mathcal{H}^{\frac{n(n-1)}{2}} \llcorner O(n)$. We know that for all $A \in O(n)$,
since A is a linear isometry, $\mu(AU) = \mu(U) = \mu(A^{-1}(U))$ $\forall U$.
Define $\theta_n := \frac{1}{\mathcal{H}^{\frac{n(n-1)}{2}}(O(n))} \mu$. It turns out this is the Haar measure, which is how Mattila defines it.

2. We may identify $G(n,m) \cong P(n,m) \subset \mathbb{R}^{n\times n}$ for some P under the map $V \mapsto \mathbb{P}_V$ (orthogonal proj. onto V)
We know
$$\mathbb{P}_V^2 = \mathbb{P}_V, \qquad \mathbb{P}_V^T = \mathbb{P}_V, \qquad \dim \operatorname{range} \mathbb{P}_V = m$$
Incidentally, any matrix with these properties is a projection. So,
$P(n,m) \subseteq \mathbb{R}^{n\times n}$ is a compact $m(n-m)$-dim submanifold. Let us place the measure
$$\mathcal{H}^{m(n-m)} \llcorner P(n,m) =: \gamma_{n,m}$$
Another way to define $\gamma_{n,m}$ is to require that $\gamma_{n,m}(U) = \gamma_{n,m}(O\,U)$ $\forall O \in O(n)$
So, we may define $\gamma(U) := \theta_n(\{O \in O(n) : O(\mathbb{R}^m \times \{0\}) \in U\})$,

which will have the same invariants of $\gamma_{n,m}$. In fact, $\gamma = \frac{1}{\mathcal{H}^{m(n-m)}(P(n,m))} \gamma_{n,m}$
which is how Mattila defines it.

Note that $G(n,m) \cong G(n,n-m)$ via $V \mapsto V^\perp$ and $\mathbb{P}_V \mapsto I_n - \mathbb{P}_{V^\perp}$

Now, on to the theorem!

Theorem: (Besicovitch–Federer)

Let $0<\mathcal{H}^m(E)<\infty$ for some $E\subseteq\mathbb{R}^n$ $\mathcal{H}^k$-measurable. Then,

$$E \text{ is purely unrectifiable} \iff \mathcal{H}^k(\mathbb{P}_V(E))=0 \text{ for } \gamma_{n,k}\text{-a.e. } V\in G(n,k)$$

(equivalently, $\mathcal{H}^m(\mathbb{P}_{V^\perp}E)=0$ for $\gamma_{n,n-m}$-a.e. $V\in G(n,n-m)$)

Remarks:
- we cannot get, in general, have that the projection to *every* subspace is 0, though this does sometimes happen
- We will prove with Federer's method. - WLOG assume E Borel & compact

Firstly some lemmas.

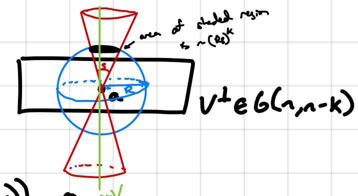
Lemma 1: Let $A\subseteq\mathbb{R}^n$ closed, purely k-unrectifiable w/ $0<\mathcal{H}^k(A)<\infty$, $\delta>0$, and $V\in G(n,n-k)$

only place we need unrectifiability!

Set

$$A_{1,\delta}(V) := \left\{ a\in A : \limsup_{s\downarrow 0} \sup_{0<R<\delta} \left\{(R\delta)^{-k}\mathcal{H}^k(A\cap B_R(a)\cap C(a,V,\delta))=0\right\}\right\}$$

cone centred with axis a with axis V and opening δ



"take skinny cones, take ball maximizing ratio of intersection to cross-section"

Then, $\mathcal{H}^k(A_{1,\delta}(V))=0$.

Remark: If A were Lipschitz graph, then its intersection with the cone is always tiny!

Lemma 2: Let $A\subseteq\mathbb{R}^n$ compact w/ $0<\mathcal{H}^k(A)<\infty$, $\delta>0$, and $V\in G(n,n-k)$. Set

$$A_{2,\delta}(V) := \left\{ a\in A : \limsup_{s\downarrow 0} \sup_{0<R<\delta} \left[(R\delta)^{-k}\mathcal{H}^k(A\cap B_R(a)\cap C(a,V,\delta))\right]=+\infty\right\}$$

Then, $\mathcal{H}^k(\mathbb{P}_{V^\perp}(A_{2,\delta}))=0$.

Lemma 3: Let $A\subseteq\mathbb{R}^n$ compact w/ $0<\mathcal{H}^k(A)<\infty$ and $V\in G(n,n-k)$. Set

$$A_3(V) := \left\{ a\in A : \#(A\cap(a+V))=\infty\right\}$$

Then, $\mathcal{H}^k(\mathbb{P}_{V^\perp}(A_3(V)))=0$.

Lemma 4: Let $\delta>0$. For $\gamma_{n,n-k}$-a.e. $V\in G(n,n-k)$, $\mathcal{H}^k$-a.e. $a\in A$ satisfies one of

① $a\notin A_{1,\delta}(V)$ ② $a\in A_{2,\delta}(V)$ ③ $(A\setminus\{a\})\cap(a+V)\cap B_\delta(a)\neq\emptyset$

Proof of Theorem: From the lemmas, we see that for $\gamma_{n,n-k}$-a.e. $V \in G(n,n-k)$ and $\mathcal{H}^k$-a.e. $a \in A$,

① $a \in \bigcup_n A_{1,\frac{1}{n}}(V) \Rightarrow \mathcal{H}^k($

② $a \in \bigcup_n A_{2,\frac{1}{n}}(V) \rightarrow \mathcal{H}^k($

③ We are in the fiber $a+V$ infinitely many times

So, our projection is in the countable intersection of A_{1s}'s, A_{2s}'s, and A_3. □

Proof of Lemma 3: Apply the Coarea inequality w/ $s=n-k$, and $f = \mathbb{P}_{V\perp}$,

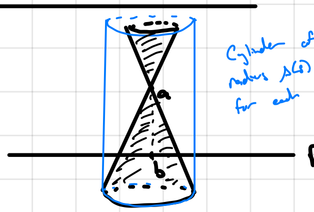
counting measure $\quad f^{-1}(\{a\})$

$$\int^*_{\mathbb{R}^k} \mathcal{H}^0(A \cap (a+V))\, da \le C\,\mathcal{H}^k(A)$$

$$\Rightarrow \int_{\mathbb{P}_{V\perp}(A_3)} +\infty \le C\,\mathcal{H}^k(A) \Rightarrow \mathcal{H}^k(\mathbb{P}_{V\perp}(A_3)) = 0.$$ □

Proof of Lemma 2: Let $B := \mathbb{P}_{V\perp}(A_{2,\delta}(V))$. Fix $M>0$ large. By defn of limsup, for all $b \in B$, $\forall s$, $\exists \Delta(s) \le c_0 s \delta$ s.t.

handle sloppiness



Cylinder of radius $\Delta(s) := c_0 \cdot R \cdot s$ for each $R \in B$.

$$\mathcal{H}^k\left(\mathbb{P}^{-1}_{V\perp}\left(\overline{B^k_{\Delta(s)}}(b)\right) \cap A\right) \ge M \Delta(s)^k$$

Note that $\mathcal{C} := \{\overline{B^k_{\Delta(s)}}(b) : b \in B\}$ is a fine cover of B.

So, $\exists$ pairwise disjoint balls covering $\mathcal{H}^k$-a.e. $x \in B$. Thus,

$$\mathcal{H}^k(B) \le \sum_i \lambda\left(\overline{B_{\Delta_i}}(b_i)\right) \le \omega_k \sum_i \Delta_i^k \le \omega_k \sum_i \frac{1}{M} \mathcal{H}^k\Big(\underbrace{\mathbb{P}_{V\perp}^{-1}\left(\overline{B_{\Delta_i}}(b_i)\right)}_{\text{the cylinder!}} \cap A\Big)$$

$$\le \frac{\omega_k}{M} \mathcal{H}^k(A)$$

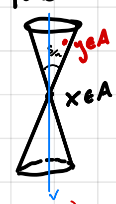
Take $M \to \infty$, and we are done. □

Proof of Lemma 1: Let $\varepsilon > 0$. Then, $\exists s > 0$ st. "stuff after limsup" is bounded uniformly. Apply proposition from below to cover most of $A_{1s}(V)$, and cover the small rest.

Prop:

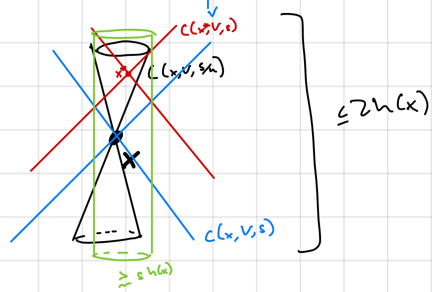
Let A be purely k-unrectifiable. Let $s \in (0,1)$, $\lambda, \delta \in (0,\infty)$.
If
$$\sup_{0<R<\delta} \mathcal{H}^k(A \cap B_R(a) \cap C(a,V,s)) \leq \lambda(\delta s)^k \quad \forall a \in A,$$
then
$$\mathcal{H}^k(A \cap B_{\delta/6}(a)) \leq \underset{2\cdot 20^k}{\text{const}}\ \lambda \delta^k \quad \forall a \in A.$$

Proof: Fix $0 \in A$, and suppose WLOG that $A \subseteq B_{\delta/6}(0)$. Define a function
$$h(x) := \sup\{|y-x| : y \in A \cap C(x, V, \tfrac{s}{4})\}$$



By pure unrectifiability, $h(x) > 0$ for $\mathcal{H}^k$-a.e. x.

Let $x^* \in A$ be s.t. $|x - x^*| \geq \frac{3}{4} h(x)$
We claim that the green cylinder
$$\mathbb{P}_{V^\perp}^{-1}(B_{\frac{s h(x)}{4}}(x) \cap A) \subseteq (A \cap B_{2h(x)} \cap C(x,V,s)) \cup (A \cap B_{2h(x)} \cap C(x^*,V,s))$$



Applying our assumed estimate, $\mathcal{H}^k(\mathbb{P}_{V^\perp}^{-1}(B_{\frac{s h(x)}{4}}(x) \cap A)) \leq C\lambda(h(x)s)^k$

So, for $\mathcal{H}^k$-a.e. $z \in \mathbb{P}_{V^\perp}(A \cap B_{s/6})$, $\exists R(z)$ s.t.
$$\mathcal{H}^k(\mathbb{P}_{V^\perp}^{-1}(B_{R(z)}(z) \cap A)) \leq C\lambda R(z)^k$$
$$\Rightarrow \mathcal{H}^k(\mathbb{P}_V(A \cap B_{s/6}(0)) \cap B_{R(z)}(z)) \leq C\lambda R(z)^k \quad ?$$
$$\mathcal{H}^k(A \cap B_{\frac{s}{6}}(0) \cap \mathbb{P}_{V^\perp}^{-1}(B^k_{R(z)}(\mathbb{P}_{V^\perp}(z))) \leq C\lambda R(z)^k$$

Let $\rho(a) = R(a)/5$ and $\mathcal{G} := \{B^k_{\rho(a)}(\mathbb{P}_{V^\perp}(a)) : a \in A \cap B_{s/6}\}$

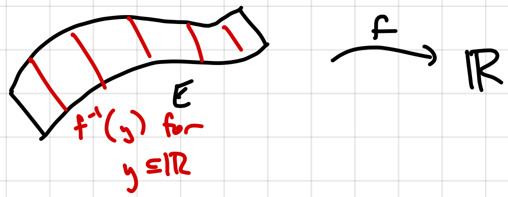
$\mathcal{G}$ covers $\mathbb{P}_{V^\perp}(A \cap B_{s/6})$ and so there is a pairwise disjoint collection of balls s.t.
$$\bigcup_i 5B_i \supseteq \bigcup_{B \in \mathcal{G}} B \supseteq \mathbb{P}_{V^\perp}(A \cap B_{s/6})$$

So,
$$\mathcal{H}^k(A \cap B_{s/6}) \leq \sum_i \mathcal{H}^k(A \cap B_{s/6} \cap \mathbb{P}_{V^\perp}^{-1}(5B_i))$$
$$\leq \sum_i C\lambda(5\rho(a_i))^k \leq \frac{C\lambda s^k 5^k}{\omega_k} \sum_i \omega_k \rho(a_i)^k$$
by pairwise disjointness
$$\leq C\lambda s^k$$

$\square$

10/5-

Let E rectifiable and $f: E \to \mathbb{R}^j$ be Lipschitz. Then, the coarea formula applies.
If E 2-rect (i.e. a surface) and $j=1$, then:



The <u>coarea formula</u> allows one to find measure of a set by integrating level sets of the function, Fubini-style, using $J_f(x)$ to account for distortion. We knew this for smooth E and differentiable f, but the coarea formula holds for rectifiable E and Lipschitz f. However, a general inequality does hold.

Recall the upper integral $\int^* f = \inf_{\substack{\psi \ge f \\ \psi \text{ meas.}}} \int \psi$. Fatou's lemma holds!

<u>Prop:</u> (Coarea inequality)

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz, $s \ge m$, $A \subseteq \mathbb{R}^n$. Then,

need not even be measurable!

$$\int^*_{\mathbb{R}^m} \mathcal{H}^{s-m}(A \cap f^{-1}(\{y\}))\,dy \le C(s,m)\,\mathrm{Lip}(f)^m\,\mathcal{H}^s(A)$$

dimensional constant $= \frac{\omega_{s-m}\,\omega_m}{\omega_s} \cdot 2^m$

This is true from a metric space to a metric space! See Federer.

<u>Proof:</u> Cover A with $\{E_{k,i}\}_{i \in \mathbb{N}}$ with $\mathrm{diam}(E_{k,i}) \le \frac{1}{k}$ s.t.

$$\frac{\omega_s}{2^s} \sum \mathrm{diam}(E_{k,i})^s \le \mathcal{H}^s(A) + \frac{1}{k} + \delta_k$$

Next, let $F_{k,i} = f(E_{k,i}) = \{y \in \mathbb{R}^m : E_{k,i} \cap f^{-1}\{y\} \ne \emptyset\}$

Then,

$\forall y, z \in F_{k,i}$: $|y - z| \le \mathrm{Lip}(f)\,\mathrm{diam}(E_{k,i}) \Rightarrow \mathrm{diam}(F_{k,i}) \le \mathrm{Lip}(f)\,\mathrm{diam}(E_{k,i})$

So,

$$\int^* \mathcal{H}^{s-m}(A \cap f^{-1}(\{y\}))\,dy = \int^* \lim_{k\to\infty} \mathcal{H}^{s-m}_{\frac{1}{k}}(A \cap f^{-1}(\{y\}))\,dy$$

$$\le \int^* \liminf_{k \to \infty} \underbrace{\frac{\omega_{s-m}}{2^{s-m}} \sum_i \mathrm{diam}\big(\overbrace{E_{k,i} \cap f^{-1}\{y\}}^{\in F_{k,i}}\big)^{s-m}}_{\ge \mathcal{H}^{s-m}_{\frac{\mathrm{Lip}(f)}{k}}(A \cap f^{-1}\{y\})}\,dy$$

Fatou

$$\le \liminf_{k\to\infty} \sum_i \int^*_{F_{k,i}} \frac{\omega_{s-m}}{2^{s-m}} \sum_i \mathrm{diam}(E_{k,i} \cap f^{-1}\{y\})^{s-m}\,dy$$

diam is 0 outside of $F_{k,i}$

$$\leq \liminf_{k\to\infty} \Big(\sum_i \frac{w_{s-m}}{2^{s-m}} \operatorname{diam}(E_{k,i})^{s-m} \Big) \underbrace{\lambda(F_{k,i}^{\subseteq \mathbb{R}^m})}_{\leq w_m \operatorname{diam}(F_{k,i})^m}$$

$$\leq \liminf_{k\to\infty} \Big(\frac{w_{s-m}\cdot w_m}{2^{s-m}} \operatorname{Lip}(f)^m \sum_i \operatorname{diam}(E_{k,i})^s \Big)$$

$$= \frac{w_{s-m}\, w_m\, 2^m \operatorname{Lip}(f)^m}{w_s} \mathcal{H}^s(A).$$

$\square$

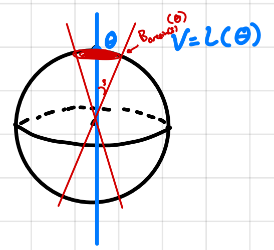
<u>Remark:</u> If f Hölder, you could still do this with the estimate $\operatorname{diam}(F_{k,i}) \lesssim \operatorname{diam}(E_{k,i})^{\alpha}$.

10/10 -

<u>Proof of Lemma 4 -</u> There are two parts. We will prove it first for $k=n-1$ (codimension 1), after which we will do it in generality.

① Let $k=n-1$. We may think of $G(n,1)$ as the sphere/$\mathbb{RP}^n$ (up to double counting), and $\gamma_{n,1}$ as the Haar measure/uniform prob. on the sphere.

Let $L(\theta)$ be the line parametrized by a point θ on the n-sphere.
Let $L(E) = \bigcup_{\theta\in E} L(\theta)$ $\forall E \subseteq S^{n-1}$. Then, $\forall$ openings s,



$$C(0,V,s) = \bigcup_{\theta' \in B_{\arcsin(s)}(\theta)} L(\theta') = L(B_{\arcsin(s)}(\theta)).$$

Define the set function $\Psi(E) := \sup_{0<R\leq\delta} R^{1-n} \mathcal{H}^{n-1}(A \cap B_R(0) \cap L(E))$ for $E \subseteq S^{n-1}$.

Then, $$\underbrace{\limsup_{s\to 0} s^{1-n} \Psi(B_s(\theta))}_{\text{basically } \Theta^{n-1,*}(\Psi,\theta)} = \limsup_{s\to 0} \sup_{0<R\leq\delta} (Rs)^{1-n} \mathcal{H}^{n-1}(A\cap B_R \cap C(0,\overset{L(\theta)}{V},s))$$

So, we WTS that for $\gamma_{n,1}$-a.e. θ, either

(*)
① $\Theta^{n-1,*}(\Psi,\theta) = \infty$
② $\Theta^{n-1,*}(\Psi,\theta) = 0$
③ $\underbrace{L(\theta)\cap A\setminus\{0\}\cap B_\delta(0)}_{\text{this is Borel and so Lebesgue measurable}} \neq \emptyset$

Here, we are proving it for all points (by shifting the origin) and a.e. V.
This would imply Lemma 4 via Fubini applied to the product measure $\gamma_{n,1}\times(\mathcal{H}^k \llcorner A)$, i.e. we can swap the almost everys for points and lines.

We want to show Ψ is an outer measure (homework), since then we would be able to apply the following:

<u>Lemma (Mickle-Radó):</u>

Let Ψ be an outer measure on $\mathbb{R}^m$ and E a $\mathcal{L}^m$-meas. set s.t. $\Psi(E)=0$. Then, for $\mathcal{L}^m$-a.e. $x\in E$, $$\Theta^{m,*}(\Psi,x) = \limsup_{R\downarrow 0} R^{-m}\Psi(B_R(x)) \in \{0,\infty\}$$

Proof of Mattila-Radó: Suppose WLOG that E is closed. Define

$$F_j := \{x \in E : \psi(B_R(x)) \leq j R^m \ \ \forall\, 0 < R < \tfrac{1}{j}\}$$

It isn't hard to show F_j is closed. It's harder to see that

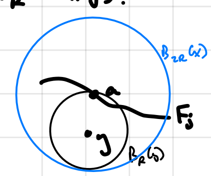
$$\bigcup_{j=1}^{\infty} F_j = \{x \in E : \limsup_{R \downarrow 0} R^{-m} \psi(B_R(x)) < \infty\}$$

for the limsup

Since $F_j \subseteq E$, we know $\psi(F_j) = 0$. So, $\psi(B_R(x)) = \psi(B_R(x) \setminus F_j)$.

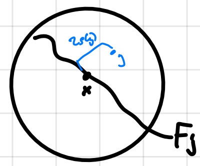
For any y, if $B_R(y) \cap F_j \neq \emptyset$ and $R < \delta < \frac{1}{2j}$, then

$$\psi(B_R(y)) \leq \psi(B_{2R}(w)) \leq (2R)^k j$$



Fix $x \in F_j$ and $0 < R < \delta/5$. Pick a point $y \in B_R(x) \setminus F_j$
(if it doesn't exist then $\psi(B_R(x) \setminus F_j) = 0$ $\forall R < R$ and the limsup is 0).

Set $s_y := \dfrac{d(y, F_j)}{2}$



We know $\{B_{s(y)}(y) : y \in B_R(x) \setminus F_j\}$ covers $B_R(x) \setminus F_j$.
By the $5r$-covering theorem we grab pairwise disjoint $5r$-cover.
By countable subadditivity of ψ,

$$\psi(B_R(x) \setminus F_j) \leq \sum_i \psi(B_{5s(y_i)}(y_i)) \leq \sum_i \frac{j}{\omega_m} 10^m s(y_i)^m \cdot \omega_m$$

$$\leq c j\, \mathcal{L}^m(B_{5R}(x) \setminus F_j).$$

So, $\forall x \in F_j$,

$$\Theta^{k,*}(\psi, x) := \limsup_{R \to 0} \frac{\psi(B_R(x))}{R^k} = \limsup_{R \to 0} \frac{\psi(B_R(x) \setminus F_j)}{R^k}$$

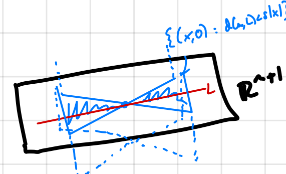
$$\leq c j \limsup_{R \to 0} \frac{\mathcal{L}^m(B_{5R}(x) \setminus F_j)}{(5R)^m} = c_j \left(\begin{matrix}\text{Lebesgue density} \\ \text{of } x \text{ in } F_j^c\end{matrix}\right) = 0 \quad \mathcal{L}^m\text{-a.e.}$$

In a sense, we took a density of an outer measure and dropped the set in a good way to get a Lebesgue density.

So, when the density is finite, $x \in F_j$ for some j, and the density is 0. □

② We now want to get to higher dimensions. Consider a k-dim plane $\mathbb{R}^k \times \{0\}$. Then, $\mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^{k+1} \times \{0\} =: W$. For a.e. line $L \in G_{k+1,1}$, we have

$$X^{k+1}(0, L, s) = \{x \in W : d(x, L) \geq s|x|\} \oplus W^{\perp}$$



For $\gamma_{k+1,1}$-a.e. $\ell \in G_1(\mathbb{R}^{k+1} \times \{0\})$ and $\mathcal{H}^k$-a.e. $a \in A$, either

① $\displaystyle \limsup_{s \to 0} \sup_{0 < R < s} (Rs)^{-k} \mathcal{H}^k(A \cap B_R(a) \cap (X^{k+1}(0, L, s) + a)) = 0$

② $\displaystyle \limsup_{s \to 0} \sup_{0 < R < s} (Rs)^{-k} \mathcal{H}^k(A \cap B_R(a) \cap (X^{k+1}(0, L, s) + a)) = \infty$

③ $(A\setminus\{a\}) \cap ((\ell \oplus V^\perp)+a) \cap B_\delta(a) \neq \emptyset$

via an application of the above kernel logic. We have the right alternative, but for a cone of the wrong shape. Here is how we will fix it.

Let $V_0 := \{x_1 = \dots = x_k = 0\}$, and so

$$C(0,V_0,s) = \left\{x \in \mathbb{R}^n : \sum_{i=1}^{k} x_i^2 < s^2 \sum_{i=1}^{\hat{}} x_i^2\right\} = \left\{x \in \mathbb{R}^n : \sum_{i=1}^{k} x_i^2 < \frac{s^2}{1-s^2} \sum_{i=k+1}^{\hat{}} x_i^2\right\}$$

and $W_j := V_0^\perp \oplus \mathbb{R}e_j$, $j \in \{k+1, \dots, n\}$ (first k dims plus another). Then,

$$X_j(0,V_0,\sigma) = \left\{x \in \mathbb{R}^n : \sum_{i=1}^{k} x_i^2 < \frac{\sigma^2}{1-\sigma^2} x_j^2\right\}$$

Note that if $s=\sigma$, then $X_j(0,V_0,\sigma) \subseteq C(0,V_0,s) \Rightarrow \hat{\bigcup_{j=k+1}} X_j(0,V_0,s) \subseteq C(0,V_0,s)$

However, $C(0,V_0,s) \subseteq \hat{\bigcup_{j=k+1}} X_j(0,V_0,s^*)$ for $\frac{s^{*2}}{1-s^{*2}} = (n-m)\frac{s^2}{1-s^2}$.

We may map V_0 to other subspaces via orthogonal transformations. So, we will reason about a.e. orthogonal transformation instead of a.e. subspace.

<u>Lemma:</u>

Let $\delta > 0$, $j \in \{k+1, \dots, n\}$. For θ_n-a.e. $g \in O(n)$, one of the following alternatives holds:

① $\limsup_{\delta \downarrow 0} \sup_{0<R<\delta} (R\delta)^{-k} \mathcal{H}^k(A \cap B_\delta(a) \cap (a + gX_j(0,V_0,\delta))) = 0$

② $\limsup_{\delta \downarrow 0} \sup_{0<R<\delta} (R\delta)^{-k} \mathcal{H}^k(A \cap B_\delta(a) \cap (a + gX_j(0,V_0,\delta))) = \infty$

③ $(A\setminus\{a\}) \cap B_\delta(a) \cap (a + gV_0)$

<u>Proof:</u> Let $j = k+1$ WOLOG. Let $W := W_{k+1} = \{x_{k+2} = \dots = x_n = 0\} = V^\perp + \mathbb{R}e_{k+1}$

Let $\mathcal{X}(g) = \begin{cases} 1 & \text{if none of the 3 properties holds} \\ 0 & \text{otherwise} \end{cases}$

We can confirm that $\mathcal{X}$ is Borel and so measurable (A compact will help).

We have $O(n) = \{\text{orthogonal transformations of } \mathbb{R}^n\}$

$O(k+1) = \{g \in O(n) : g|_{W^\perp} = \text{identity}\}$

Then, $\int_{O(k+1)} \mathcal{X}\, d\theta_{k+1} = 0$ since

and $\int_{O(n)} \mathcal{X}(h)\, d\theta_n(h) \overset{\text{invariance of Haar measure}}{=} \int_{O(n)} \mathcal{X}(hg)\, d\theta_n(h) \quad \forall g \in O(n)$.

So, since θ_{k+1} is a probability measure,

$$\int_{O(n)} \mathcal{X}(h)\, d\theta_n(h) = \int_{O(k+1)} \int_{O(n)} \mathcal{X}(h)\, d\theta_n(h)\, d\theta_{k+1}(g) \overset{\text{invariant}}{=} \int_{O(k+1)} \int_{O(n)} \mathcal{X}(hg)\, d\theta_n(h)\, d\theta_{k+1}(g)$$

$$\overset{\text{Fubini}}{=} \int_{O(n)} \underbrace{\int_{O(k+1)} \mathcal{X}(hg)\, d\theta_{k+1}(g)}_{=0}\, d\theta_n(h) = 0.$$

□

With this lemma, our proof of Lemma 4 is complete since the density is 0 for a subspace iff ① holds for a.e. $g \in O(n)$.

$\square$

With Lemma 4, we know we always have the alternative and each one happens on a set of measure 0, and so we have done it!

$\square$

Besicovitch – Preiss

Theorem: (Besicovitch-Preiss)

Let $E \subseteq \mathbb{R}^n$ Borel s.t. $0 < \mathcal{H}^k(E) < \infty$. If $0 < \Theta^{k*}(E,x) = \Theta^k_*(E,x) < \infty$ exists for $\mathcal{H}^k$-a.e. $x \in E$, then E is rectifiable.

Equivalently:

Let μ be a Radon measure and assume $\Theta^{k*}(\mu,x) = \Theta^k_*(\mu,x)$ exists and is positive and finite for μ-a.e. x. Then, $\exists E$ rectifiable of $\dim_k$ and $f: E \to \mathbb{R}^+$ Borel s.t.

$$\mu = f\,\mathcal{H}^k \llcorner E$$

call this "μ is k-rectifiable"

Remark: To show equivalence, use density of μ to show it's abs. cont. wrt $\mathcal{H}^k$.

Theorem: (Marstrand)

Suppose μ satisfies the requirements of BP, but $k \notin \mathbb{N}$. Then, $\mu = 0$.

Remark: So, non-integer-dimension sets must have holes of some sort.

Theorem: (Marstrand-Mattila)

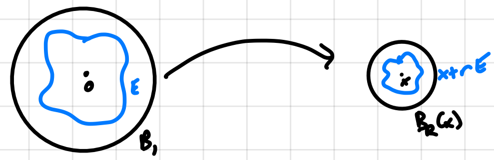
If E satisfies BP conditions and $\Theta^{k*}(E,x) = 1$ $\mathcal{H}^k$ a.e. $x \in E$, then E is rectifiable.

Remark: This is weaker than BP, and so we won't prove it. $\square$

Tangent measures

Fix a Radon measure μ and a point x at which $\Theta^{\alpha *}(\mu, x) < \infty$.

Let $R>0$ and define $\mu_{x,R}(E) := \dfrac{\mu(x+RE)}{R^\alpha}$



Then, $\forall \lambda > 0$,

$$\mu_{x,R}(B_\lambda) = \left(\frac{\mu(B_{\lambda R}(x))}{\omega_\alpha \lambda^\alpha R^\alpha}\right) \omega_\alpha \lambda^\alpha \Rightarrow \limsup_{R \downarrow 0} \mu_{x,R}(B_\lambda) \leq \Theta^{\alpha *}(\mu,x)\, \omega_\alpha \lambda^\alpha$$

← uniformly bounded! so, has weak-* subsequence

For any sequence $\{R_k\}_k \to 0$, $\exists R_{k_j}$ subsequence s.t.

$$\mu_{x,R_{k_j}} \overset{*}{\rightharpoonup} \nu$$

Def:

We call a Radon measure ν a tangent measure to μ at x if it comes about in the above way. Let $\mathrm{Tan}_\alpha(\mu,x)$ denote the set of tangent measures.

Def:

An α-uniform measure is a (Radon) measure μ for which $\exists C>0$ s.t.

$$\mu(B_R(x)) = CR^\alpha \qquad \forall R>0, \quad \forall x \in \mathrm{spt}(\mu)$$

← complement of this is largest open set of measure 0

(i.e. for μ-a.e. x)

Exercise

Let μ be a measure and $f \in L^1(\mu)$. Then,

$$\mathrm{Tan}_\alpha(f\mu, x) = f(x)\, \mathrm{Tan}_\alpha(\mu,x) \quad \text{for } \mu\text{-a.e. } x$$

Lemma: (Marstrand)

Assume μ satisfies $0 < \Theta^\alpha_*(\mu,x) = \Theta^{\alpha *}(\mu,x) < \infty$ exists for μ-a.e. x.
Then, for μ-a.e. x, all tangent measures ν at x satisfy:

① $0 \in \mathrm{spt}(\nu)$ (obvious!)

② $\forall a \in \mathrm{spt}(\nu)$ and $\forall R>0$, $\nu(B_R(a)) = \omega_\alpha \Theta^\alpha(\mu,x) R^\alpha$ (so, ν is an α-uniform measure).

Proof: ① $\forall \nu \in \mathrm{Tan}_\alpha(\mu,x)$, $\nu(B_R(0)) = \lim_{r\to 0} \dfrac{\mu(B_{Rr}(x))}{(Rr)^\alpha} R^\alpha = \omega_\alpha \Theta^\alpha(\mu,x) R^\alpha$. So, $0 \in \mathrm{spt}(\nu)$.

what is going on

② Let

$$E^{ijk} := \left\{ x : \frac{j-1}{i} \leq \frac{\mu(B_R(x))}{\omega_\alpha R^\alpha} \leq \frac{j+1}{i} \quad \forall R \leq \frac{1}{k} \right\}$$

Then, $\forall i$ we know $\mu\Big(\mathbb{R}^n \setminus \underbrace{\bigcup_{j,k} E^{ijk}}_{E_i}\Big) = 0$.

We claim that for μ-a.e. $x \in E^{ijk}$ and $\forall \nu \in \mathrm{Tan}_\alpha(\mu \llcorner E^{ijk}, x)$

$$\left| \nu(B_R(y)) - \theta^\alpha(\mu, x)\, \omega_\alpha R^\alpha \right| \le \frac{2\omega_\alpha}{i} \qquad \forall y \in \mathrm{spt}(\nu)$$

Fix $y \in \mathrm{spt}(\nu)$. Then, $\forall r > 0$,

$$\nu(B_r(y)) = \lim_{r_k \to 0} \frac{\mu(B_{rr_k}(r_k y + x) \cap E^{ijk})}{r_k^\alpha} \qquad \text{if } \nu(\partial B_r(y)) = 0$$

by measure stuff.
this holds for all but possibly countably many y

Pick x s.t. $\displaystyle \lim_{s \to 0} \frac{\mu(B_s(x) \setminus E^{ijk})}{s^\alpha} = 0$

Then,

$$B_{rr_k}(yr_k + x) \setminus E^{ijk} \subseteq B_{(|y|+r)r_k}(x) \setminus E^{ijk}$$

$$\Rightarrow \frac{\mu(B_{rr_k}(yr_k + x) \setminus E^{ijk})}{r_k^\alpha} \le \frac{\mu(B_{(|y|+r)r_k}(x) \setminus E^{ijk})}{r_k^\alpha}$$

why?
So, we may remove the $\cap E^{ijk}$ business!

Note that $(\mu \llcorner E)_{x, r_k} = \mu_{x, r_k} \llcorner \frac{E - x}{r_k}$. So, it must be that $\mathrm{dist}(y, \frac{E-x}{r_k}) \to 0$ since otherwise $y \notin \mathrm{spt}(\mu)$. Thus, $\exists \{y_k\}_k$ s.t. $y_k \in \frac{E-x}{r_k}$ and $|y_k - y| \to 0$.
Let $\varepsilon_k := |y - y_k|$

$$B_{(r-\varepsilon_k)r_k}(r_k y_k + x) \subseteq B_{rr_k}(r_k y + x) \subseteq B_{(r+\varepsilon_k)r_k}(r_k y_k + x)$$

$$\Rightarrow \frac{\mu(B_{(r-\varepsilon_k)r_k}(r_k y_k + x))}{r^\alpha r_k^\alpha} \le \frac{\mu(B_{rr_k}(r_k y + x))}{r^\alpha r_k^\alpha} \le \frac{\mu(B_{(r+\varepsilon_k)r_k}(r_k y_k + x))}{r^\alpha r_k^\alpha}$$

Since $E^{ijk} \Rightarrow \frac{\omega_\alpha r^\alpha (j-1)}{i} \le \mu(B_r(r_k y_k + x)) \le \frac{\omega_\alpha r^\alpha (j+1)}{i} \qquad \forall r \le \frac{1}{k}$

$$\Rightarrow \frac{j-1}{i} \omega_\alpha r^\alpha \le \nu(B_r(y)) \le \frac{j+1}{i} \omega_\alpha r^\alpha \qquad \forall y \in \mathrm{spt}(\nu), \quad \forall r > 0$$

However, when $y = 0$, $\nu(B_r(0)) = \omega_\alpha \theta^\alpha(\mu, x) r^\alpha$. Thus,

$$\left| \nu(B_r(y)) - \theta^\alpha(\mu, x)\, \omega_\alpha r^\alpha \right| \le \frac{2 r^\alpha \omega_\alpha}{i}$$

$\square$

<u>Remark:</u> We might expect all α-uniform measures to be Hausdorff measure on a plane. In general, this <u>isn't</u> true (what is true is that $\alpha \in \mathbb{N}$ and ν is $\mathcal{H}^\alpha$ restricted to an analytic subvariety of $\mathbb{R}^n$).

light cone
A counterexample is $C := \{x_4^2 = x_1^2 + x_2^2 + x_3^2\} \subseteq \mathbb{R}^4$. Then, $\mathcal{H}^3 \llcorner C$ is 3-uniform measure.

<u>Exercise:</u>
A tangent measure to a uniform measure is a uniform measure.

Proposition: (Marstrand)

If $\alpha = k$, at least one tangent measure at μ-a.e. x is $\Theta^k(\mu,x)\mathcal{H}^k \llcorner V$ for a k-dim subspace $V \subseteq \mathbb{R}^n$.

← which always holds for tangent measures

Remark: This is far from allowing us to apply the tangent measure criterion from week 2 to prove rectifiability, since that required unique, Hausdorff-on-a-plane tangent measures. However, it turns out we don't need uniqueness.

Theorem: (Marstrand-Mattila Rectifiability Criterion)

Let μ be a Radon measure with positive lower density and finite upper density, $k \in \mathbb{N}$, and assume that for μ-a.e. x, EVERY tangent measure at μ is of the form (*) $c\mathcal{H}^k \llcorner V$ for some $c > 0$ and k-dim subspace V, then μ is rectifiable.

Theorem: (Preiss)

Under the assumption $\Theta^k(\mu,x)$ exists μ-a.e., then for μ-a.e. x every tangent measure at x has the form (*).

Together, Preiss + Marstrand-Mattila Rectifiability $\Longrightarrow$ BP.

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Proposition: (Marstrand)

Let μ be an α-uniform measure and assume $\alpha < n$. Then, $\exists x \in \operatorname{spt}(\mu)$ and $\nu \in \operatorname{Tan}_\alpha(\mu,x)$ which is supported in a hyperplane.

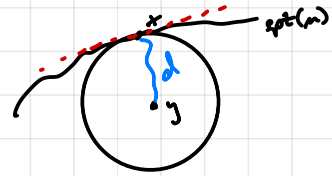
Corollary:

If $\alpha \notin \mathbb{N}$, there is no α-uniform measure.

Proof: repeat above dimensionality reduction until $n - 1 < \alpha < n$. ◻

Lemma:

If μ is nonflat and α-uniform, $\alpha < n$, then $\exists$ an α-uniform measure $\nu \in \operatorname{Tan}_\alpha(\mu,x)$ at some $x \in \operatorname{spt}(\nu)$ that is nonflat and supported in the half-space $\{x_1 \geq 0\}$.



Proof of Lemma: Let $x \in spt(\mu)$, $y \notin spt(\mu)$. Then,

$$spt(\mu_{x,r_k}) = \frac{spt(\mu) - x}{r_k} \Rightarrow \mu_{r,r_k}\left(B_{\frac{d}{r_k}}\left(\frac{y-x}{r_k}\right)\right) = 0.$$

Letting $\mu_{x,r_k} \xrightarrow{*} \nu$ completes the proof. $\square$

Lemma:

Let $\nu \in Tan_\alpha(\mu, x)$ be as given by the previous lemma.
Then, there is a hyperplane H s.t. $\forall \beta \in Tan(\nu, x)$, for some $x \in spt(\nu)$.

① $\beta \in Tan_\alpha(\mu, x)$ ② β is supported in H

Proof of Lemma: ① Exercise

② Define the barycenter $b(r) := r^{-\alpha}\int_{B_r(0)} z \, d\nu(z) \in \mathbb{R}^n$. By construction, $spt(\nu) \subseteq \{x_1 \geq 0\}$.

If $b(r) = 0$ $\forall r$, then ν itself is supported in a hyperplane already, and we are done. So, suppose $b(r) \neq 0$ for some $r > 0$. We will show

$$|\langle b(r), y\rangle| \leq C\|y\|^2 \qquad \forall y \in spt(\nu) \cap B_{2r}(0)$$

To see this, note that $2\langle b(r), y\rangle = \|y\|^2 + (r^2 - \|x-y\|^2) + (r^2 - \|x\|^2)$

$$\Rightarrow 2|\langle b(r), y\rangle| = \left| r^{-\alpha}\int_{B_r(0)} 2\langle z, y\rangle \, d\nu(z)\right|$$

$$\leq \|y\|^2 + \left| r^{-\alpha}\int_{B_r(0)} (r^2 - \|z\|^2)\, d\nu(z) - r^{-\alpha}\int_{B_r(y)} (r^2 - \|z-y\|^2)\, d\nu(z)\right|$$

If these integrals were over the whole space, translation invariance would save us. Hence, we have

$$2|\langle b(r), y\rangle| \leq \|y\|^2 + r^{-\alpha}\int_{\underbrace{(B_r(0)\setminus B_r(y)) \cup (B_r(y)\setminus B_r(0))}_{\text{symmetric difference}}} |r^2 - \|z-y\|^2| \, d\nu(z)$$

For $z \in B_r(0)\setminus B_r(y)$, $0 \leq \|z-y\|^2 - r^2 \leq \|z-y\|^2 - \|z\|^2 \leq 2\|z\|\|y\| + \|y\|^2 \leq 3r\|y\|$

For $z \in B_r(y)\setminus B_r(0)$, $0 \leq r^2 - \|z-y\|^2 \leq \|z\|^2 - \|z-y\|^2 \leq 3r\|y\|$

So,

$$2|\langle b(r), y\rangle| \leq \|y\|^2 + r^{-\alpha}3r\|y\|\mu(B_r(0) \Delta B_r(y))$$

We know

$$B_r(0) \Delta B_r(y) \subseteq B_{r+\|y\|}(z) \setminus B_{r-\|y\|}(z)$$

$$\Rightarrow 2|\langle b(r), y\rangle| \leq \|y\|^2 + r^{-\alpha}3r\|y\|\left((r+\|y\|)^\alpha - (r-\|y\|)^\alpha\right)$$

$$\leq \|y\|^2 + 3\|y\| r^{-\alpha}\left(C(\alpha)\|y\| r^{\alpha-1}\right) \leq (C(\alpha)+1)\|y\|^2$$

Now, let $\beta := \text{weak* }\lim_{k\to\infty} \nu_{0,r_k}$. Let $z \in spt(\beta) \Rightarrow z = \lim_k z_k$ for some $z_k \in spt(\nu_{0,r_k})$. Thus, $r_k z_k \in spt(\nu)$

$$\Rightarrow |b(r)\cdot z| = \lim_{k\to\infty} |b(r)\cdot z_k| = \frac{1}{r_k}\lim_{k\to\infty} |b(r)\cdot(r_k z_k)| \le C\|z_k\|^2$$

$$\Rightarrow |b(r)\cdot z| \le \lim_{k\to\infty} r_k C\|z_k\|^2 = 0$$

Since this inner product is 0 $\forall z\in spt(\beta)$, it's supported on the hyperplane.

$\square$

Exercise:

Let υ be an α-uniform measure. Then,

$$\int f(\|z\|)\, d\upsilon(z) = \int f(\|z-y\|)\, d\upsilon(z) \qquad \forall y\in spt(\upsilon)$$

By the "Bath-tub formula", $\int f(\|z\|)\, d\upsilon(z) = \int_0^\infty \upsilon(\{z : f(\|z\|) > t\})\, dt$

For monotone f, $= \int_0^\infty \upsilon(B_{f^{-1}(t)}(0))\, dt = \int_0^\infty C\|f^{-1}(t)\|^\alpha\, dt$

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Recall that we are trying to build to Besicovitch Preiss.

Theorem: (BMP)

Let $\mu\neq 0$ be Radon on $\mathbb{R}^n$ with α density existing, positive, and finite μ-a.e. Then,

① $\alpha\in\mathbb{N}$ ② μ is α-rectifiable

Let's continue our journey!

Prop:

Let μ be as in BMP. Then, for μ-a.e. $x\in\mathbb{R}^n$,

$$Tan(\mu,x) \subseteq \Theta^\alpha(\mu,x)\,\mathcal{U}^\alpha(\mathbb{R}^n)$$

uniform measures supported at 0

Defn:

$\mathcal{U}^\alpha(\mathbb{R}^n)$ is the set of all α-uniform measures μ s.t. $0\in spt(\mu)$.

Prop:

If $\upsilon\in\mathcal{U}^\alpha(\mathbb{R}^n)$, then $\exists x\in spt(\upsilon)$ and $\beta\in Tan(\upsilon,x)$ s.t. $spt(\beta)\subseteq\{\vec{e}\cdot\vec{x}=0\}$ for some $e\in\mathbb{R}^n$.

Remark: We may iterate this to find a $\beta_k = \theta^k(\upsilon,x)\,\mathcal{H}^k \llcorner V$ for some $V\in G(n,k)$.

Lemma:

ⓐ $Tan(\mu,x)$ is weak-* compact.

ⓑ If $\upsilon \in Tan(\mu,x)$, then $\upsilon_{0,R} \in Tan(\mu,x)$ $\forall R>0$.

Proof:

a) Suppose that $\{\upsilon_k\}_k \subseteq Tan(\mu,x)$ with $\upsilon_k = \lim_{j\to\infty}^{*} \mu_{x,\rho_j^k}$ s.t. $\upsilon_k \overset{*}{\rightharpoonup} \upsilon$.
By Cantor diagonalization, $\exists$ subsequence $j(k)$ s.t.

$$\mu_{x,\rho_{j(k)}^k} \overset{*}{\rightharpoonup} \upsilon.$$

So, closed. Compactness?

b) $\upsilon = \lim_{k\to\infty}^{*} \mu_{x,\rho_k}$ with $\rho_k \to 0$ $\Rightarrow$ $\upsilon_{0,R} = \lim_{k\to\infty}^{*} \mu_{x,R\rho_k}$. $\square$

Lemma:

If $\upsilon \in \mathcal{U}^d(\mathbb{R}^n)$, then $\exists$ a sequence $\{a_k\}_k \subseteq spt(\upsilon)$ and a sequence of radii $\rho_k > 0$ s.t. $\upsilon_{a_k,\rho_k} \overset{*}{\rightharpoonup} \mathcal{H}^d \llcorner V$ for some $V \in G(n,k)$.

Proof: diagonalization again. $\square$

We have that scaling preserves tangency, but it would be nice for shifts to do so as well.
It does!

Proposition:

Let μ satisfy BMP. Then, for μ-a.e. x,

- if $\upsilon \in Tan(\mu,x)$ and $a \in spt(\mu)$ and $R>0$, then $\upsilon_{a,R} \in Tan(\mu,x)$.

Proof: We know $a/R \in spt(\upsilon_{0,1/R})$ $\Rightarrow$ $(\upsilon_{a,R})_{0,1/R} = \upsilon_{a/R,1}$. So, WLOG we wish to show that $\upsilon_{a,1} \in Tan(\mu,x)$.

Introduce a distance d that metrizes weak-* convergence on Radon measures μ s.t. $\exists\, C_N$ s.t. $\mu(B_N) \leq C_N$ $\forall N$. Check Camillo's notes to see this, looks cool!

Define $A_{kj} := \left\{ x \in \mathbb{R}^n \text{ s.t. } \exists\, \upsilon \in Tan(\mu,x) \text{ and } a \in spt(\upsilon) \text{ s.t. } d(\mu_{x,R}, \upsilon_{a,1}) \geq \frac{1}{k} \ \forall R \leq \frac{1}{j} \right\}$

Note that $\bigcup_{kj} A_{kj} = \{\text{points where the claim of the prop is false}\}$.

We WTS $\mu(A_{kj}) = 0$!

However, first we must show measurability of A_{kj}. First, we must pull out some heavy stuff.

look up the proof of this!

> **Theorem:** (Universal measurability theorem)
>
> Let $E \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be Borel. Then, for every μ Radon, $P_{\mathbb{R}^n}(E)$ is μ-measurable.

countable index based on C_N's used to metrize d

Define $B_{kjR} := \{ x \in A_{kj} \text{ s.t. } R^{-1} \leq \Theta^\alpha(\mu, x) \leq R \}$; need that $\mu(B) = 0$.

drop indices cause lazy and fix j, k, R

Define $S := \{ \upsilon_{a,1} \text{ s.t. } \upsilon \in \mathrm{Tan}(\mu, x) \text{ for some } x \in B \text{ and } a \in \mathrm{spt}(\upsilon) \}$

S is a bounded set in the weak-* topology by d (why?). So, its closure is compact. Thus, we can cover S by finitely many d-balls

$$G_i := \{ \zeta : d(\zeta, \zeta_i) < \tfrac{1}{4k} \}$$

center of ball

Define

$$D_i := \left\{ x \in B \text{ s.t. } \exists \upsilon^x \in \mathrm{Tan}(\mu, x) \text{ and } a_x \in \mathrm{spt}(\upsilon^x) \text{ s.t. } d(\mu_{x,R}, \upsilon^x_{a_x,1}) \geq \tfrac{1}{k} \ \forall R \leq \tfrac{1}{j} \text{ and } \upsilon^x_{a_x,1} \in G_i \right\}$$

In words, since $x \in B \subseteq A_{jk}$, there must be some contradicting tangent measure. Since S is covered by the G_i's, we know the contradicting tangent measure must be in one of the G_i's. Measurability of D_i is a bitch, check Camillo's notes.

Note that $\forall x, y \in D_i$, $d(\upsilon^x_{a_x,1}, \upsilon^y_{a_y,1}) < \frac{1}{2k}$ by definition of G_i.

Next, choose

(1) $x \in D_i$ s.t. $\displaystyle \lim_{R \downarrow 0} \frac{\mu(D \cap B_R(x))}{\mu(B_R(x))} = 1$ by measurability of D

drop the subscript

(2) $r_\ell \downarrow 0$ s.t. $\mu_{x, r_\ell} \xrightarrow{*} \upsilon^x$ defn of tangent measure

(3) $x_\ell \in D$ s.t.

$$\| x_\ell - (x + r_\ell a_x) \| \leq \underbrace{\mathrm{dist}(x + r_\ell a_x, D)}_{= o(r_\ell) \text{ since } a_x \in \mathrm{spt}(\upsilon^x), \text{ so rescaling shrinks by } r_\ell} + \frac{r_\ell}{\ell}$$

Then, (1) $\Rightarrow \mathrm{Tan}(\mu, x) = \mathrm{Tan}(\mu \llcorner D, x)$ (since points not in D become smaller and smaller in measure)

By (3),

$$\upsilon^x_{a_x,1} = \lim_{\ell \to \infty}{}^{*} \mu_{(x + r_\ell a_x), r_\ell} = \lim_{\ell \to \infty}{}^{*} (\mu \llcorner D)_{(x + r_\ell a_x), 1}$$

We know $\mu_{x_\ell, r_\ell} \xrightarrow{*} \upsilon^x_{a_x,1}$ by (2). So, eventually $d(\mu_{x_\ell, r_\ell}, \upsilon^x_{a_x,1}) < \frac{1}{2k}$ by earlier remark.

However, since $x_\ell \in D$, we know $\frac{1}{k} \leq d(\mu_{x_\ell, r_\ell}, \upsilon^{x_\ell}_{a_{x_\ell},1})$

By construction of G_i, $d(v^{x}_{a_{x},1}, v^{x_2}_{a_{x_2},1}) < \frac{1}{2k}$. The triangle inequality yields $d(\mu_{x_2,r_2}, v^{x_2}_{a_{x_2},1}) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$, a contradiction!

So, D is μ-null $\Rightarrow \ldots \Rightarrow \bigcup_{kj} A_{kj}$ is μ-null $\Rightarrow$ prop holds μ-a.e. $\square$

We are getting there! Let's outline a plan!

Plan:

(Part I) Marstrand-Mattila Rectifiability Criterion!

Theorem: (MMRC)

Let μ be Radon s.t. for μ-a.e.

- $0 < \Theta^k_*(\mu,x) \le \Theta^{k*}(\mu,x) < \infty$
- every $v \in \mathrm{Tan}(\mu,x)$ has $v = c_v \mathcal{H}^k \llcorner V$ for some $c_v > 0$, $V \in G(n,k)$

Then, μ is rectifiable.

(Part II) Let μ be as in BMP. Let x be s.t.

- $\mathrm{Tan}(\mu,x) \subseteq \Theta^{\alpha}(\mu,x)\, \mathcal{U}^{\alpha}(\mathbb{R}^n)$ (which holds for μ-a.e. x)
- $\Theta^{\alpha}(\mu,x)\, G^{\alpha}(\mathbb{R}^n) \cap \mathrm{Tan}(\mu,x) \neq \emptyset$ (WTS this holds a.e.)

set of $\mathcal{H}^{\alpha} \llcorner V$ for $V \in G(n,\alpha)$

To do so, we will need $d(G^{\alpha}(\mathbb{R}^n), \mathcal{U}^{\alpha}(\mathbb{R}^n) \setminus G^{\alpha}(\mathbb{R}^n)) > 0$.

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Prop: (MM Rectifiability Criterion)

Let μ be Radon and $k \in \mathbb{N}$ s.t.

(a) $0 < \Theta^k_*(\mu,x) \le \Theta^{k*}(\mu,x) < \infty$ for μ-a.e. x.

(b) $\mathrm{Tan}(\mu,x) \subseteq \{c\mathcal{H}^k \llcorner V : c \in \mathbb{R}^+, V \in G(n,k)\}$

since $\mathrm{Tan}(\mu,x)$ is weak-* closed, $0 < c_\ell(x) \le c \le c_u(x) < \infty$

Then, μ is rectifiable.

Remark:

there's a proposition about this that I missed in

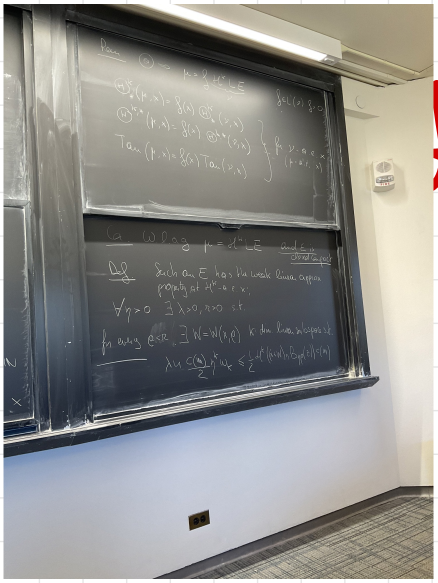
$0 < \Theta^k_*(\mu,x) \le \Theta^{k*}(\mu,x) < \infty \implies \mu = f\,\underbrace{\mathcal{H}^k \llcorner E}_{\nu}$, $f \in L^1(\nu)$ nonnegative

So, for ν-a.e. x (and so μ-a.e. x),

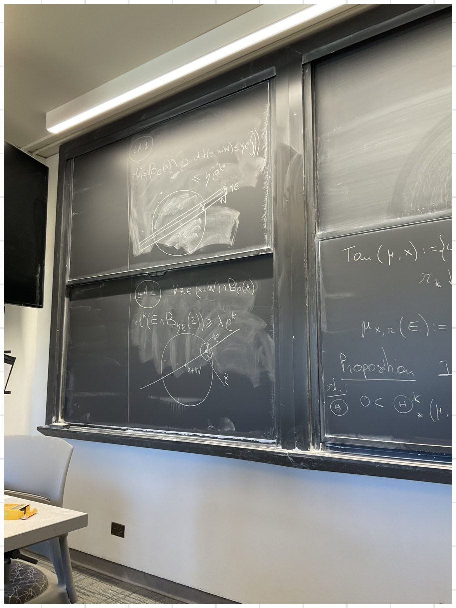
- $\Theta^k_*(\mu,x) = f(x)\,\Theta^k_*(\nu,x)$
- $\Theta^{k*}(\mu,x) = f(x)\,\Theta^{k*}(\nu,x)$
- $\mathrm{Tan}(\mu,x) = f(x)\,\mathrm{Tan}(\nu,x)$

Without loss of generality we may suppose $\mu = \mathcal{H}^k \llcorner E$ and E is compact!

Linear approximations



Live and draw pictures



(Somehow $\mathrm{Tan} \subseteq G(\mathbb{R}^n)$ implies WLAP?)

Prop: (Marstrand, then Mattila)

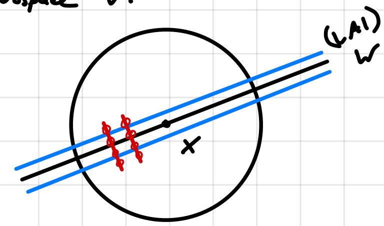
Let $k < n$, $k \in \mathbb{N}$, $E \subseteq \mathbb{R}^n$ compact with $\mathcal{H}^k(E) < \infty$. If E has the weak linear approximation property at $\mathcal{H}^k$-a.e. x, then E is rectifiable.

Proof: By the above remark, if $\nu = \mathcal{H}^k \llcorner E$ has the WLAP, then so does μ. So, we may prove WLOG that if E is purely unrectifiable, compact, and has the WLAP, then E has measure 0. finish up proof from here

perhaps we could have used BF instead?

Lemma: If E purely unrectifiable w/ WLAP at $\mathcal{H}^k$-a.e. x, then $\mathcal{H}^k(\mathbb{P}_V(E)) = 0$ for EVERY k-dim linear subspace V.

can't use BF, which holds a.e. on $G(n,k)$



Proof of lemma: Fix $\varepsilon > 0$ and $V \in G(n,k)$.

Step 1: $\exists$ a compact $C \subseteq E$ and positive r_0, η, δ s.t.

Motivation: each tangent measure has to be sort of vertical, since otherwise A projects with too much mass!

① $\mathcal{H}^k(E \setminus C) < \varepsilon$

② $\mathcal{H}^k(E \cap B_r(a)) \geq \delta r^k \quad \forall a \in C,\ r < r_0$ } lower density bounded a.e.

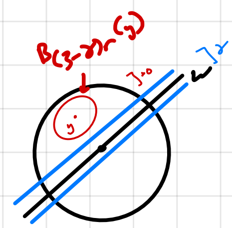
③ $\forall a \in C$, $\forall r < r_0$, $\exists$ a plane $W \in G(n,k)$ s.t.

$$C \cap B_r(a) \subseteq \{z : \operatorname{dist}(z, a+W) \leq \eta r\}$$

④ $\eta < \delta_\varepsilon < \varepsilon$

To do this, find C' s.t. ② holds and $\mathcal{H}^k(E \setminus C') < \frac{\varepsilon}{2}$, which can be done since the lower density is bounded below; this gives δ. By (LAI), for a fixed $\gamma < \delta_\varepsilon$, find $C'' \subseteq C'$ with $\mathcal{H}^k(E \setminus C'') < \varepsilon$. Then,

$$\mathcal{H}^k(E \cap B_r(x) \setminus \{z : \operatorname{dist}(z, x+W) \leq \gamma r\}) \leq \gamma r^k \quad \forall r < 2r_0.$$



Suppose BWOC that $\operatorname{dist}(y, x+W) \geq \eta$ and $y \in B_r(x) \cap C''$. Then, $\mathcal{H}^k(E \cap B_{r(\eta-\gamma)}(y)) \leq \delta((\eta-\gamma)r)^k$ by ② since $C'' \subseteq C'$.

$$\delta((\eta-\gamma)r)^k \leq \gamma((1+\eta)r)^k \quad \text{since} \quad \delta(\eta-\gamma)^k \leq \gamma$$

Imposing . . .

we get ③.

Step 2: Define $C_i := \{a \in C : C \cap B_{1/i}(a) \subseteq C(a, V, \tfrac{1}{3})\}$. Recall $C(a,V,\tfrac{1}{3}) = \{z : |\mathbb{P}_{V^\perp}(a-z)| \le \tfrac{1}{3} |\mathbb{P}_V(a-z)|\}$

So, $C_i \subseteq$ Lipschitz graph over V w/ Lip $\le \tfrac{1}{3}$

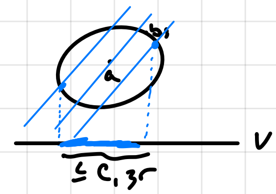
Thus, $\mathcal{H}^k(C_i) = 0$ since $C_i \subseteq E$, purely unrect. $\Rightarrow \mathcal{H}^k(\bigcup_i C_i) = 0$

So, for $\mathcal{H}^k$-a.e. $a \in C$, $\exists b_i \in C \cap B_{r_0}(a) \cap B_{1/i}(a)$ s.t.

$$|\mathbb{P}_{V^\perp}(b_i - a)| > \tfrac{1}{3} |\mathbb{P}_V(b_i - a)| \Rightarrow |\mathbb{P}_V(b_i - a)| < 3\,|b_i - a| \qquad (*)$$

Let $r := |a - b_i| < r_0$. By WLAP, $\exists W$ s.t. $C \cap B_r(a) \subseteq \boxed{\ }$.

We have:



and so

$\mathcal{H}^k(\mathbb{P}_V(C \cap B_r(a))) \le C_2 3 r^k \le C_2 \delta \varepsilon r^k$. Note that $\{B_r(a)\}$ is a fine cover of C. So, $\mathcal{H}^k$-a.e. C is covered with disjoint balls a_i, r_i

Then,

$$\mathcal{H}^k(\mathbb{P}_V(E)) \le \varepsilon + \mathcal{H}^k(\mathbb{P}_V(C)) \le \varepsilon + \mathcal{H}^k\Big(\bigcup_i \mathbb{P}_V(B_{r_i}(a_i) \cap C)\Big)$$

its a cover

$$(*) \Rightarrow \quad \le \varepsilon + C_2 3 \sum_i r_i^k \le \varepsilon + C_2 3 \sum_i \tfrac{1}{\delta} \mathcal{H}^k(B_{r_i}(a_i) \cap E) \le \varepsilon + \frac{C_2 3}{\delta} \mathcal{H}^k(E)$$

$$\le \varepsilon(1 + C\mathcal{H}^k(E)).$$

$\square$

Let $3 > 0$.

For the rest of the theorem, throw away a set of small measure and find $F \subseteq E$ s.t. $\mathcal{H}^k(F) > 0$ and $\mathcal{H}^k(E \cap B_r(a)) \ge \delta r^k \;\forall a \in F, \;\forall r < r_0$.

for δ lower bounding Θ^k_*

Find $F_1 \subseteq F$ and $r_1 > 0$ s.t. $\mathcal{H}^k(F_1) > 0$ and $\forall a \in F_1$,

comes from (LAI) → ① $F \cap B_{2r}(a) \subseteq \{z : \mathrm{dist}(z, a+W) < 3r\}$ $\quad \forall r < r_1$ and some $W \in G(n,k)$

apply 2 from Lemma first. With that radius, pick b's on that strip and use (LAI) → ② $\mathcal{H}^k(E \cap B_{3r}(b)) \ge \gamma (3r)^k \quad \forall b \in (a+W) \cap B_r(a)$ $\qquad$ (A)

density lower bound

Observe that γ and r_1 will both depend on 3. However, a better argument that target measure density is lower bounded $\Rightarrow \dots \Rightarrow$ we can take $\gamma := \tfrac{\delta}{2}\omega_n$.

Now, select $G \subseteq F_1$ s.t. $0 < \mathcal{H}^k(G) < \infty$ and $\forall a \in G$ and $\forall r < r_2$,

① $F \cap B_{2r}(a) \subseteq \{z : \mathrm{dist}(z, a+W) < 3r\}$

② $(a+W) \cap B_r(a) \subseteq \{z : \mathrm{dist}(z, F) < 3r\}$

Hausdorff distance type stuff $\qquad$ (B)

via $G := \{x \in F : \mathcal{H}^k(E \setminus F) \cap B_\rho(x) \le \chi \rho^k \;\; \forall \rho < r_2\}$ for some r_2, sufficiently small χ.

Now the clever stuff. Since $\Theta^k(E \setminus G) = 0$ a.e. and $\Theta^k(E) \le 1$ a.e., we may pick a $0 \in G$ s.t.

① $\quad \mathcal{H}^k(E \cap B_r(0)) < 2\omega_k r^k \quad \forall r < r_2$

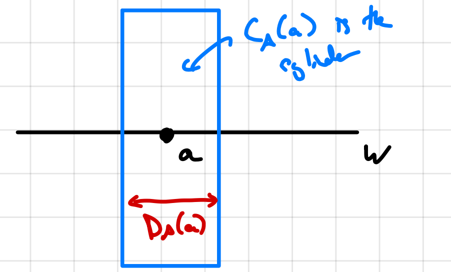
② $\quad \mathcal{H}^k((E \setminus G) \cap B_{2r}(0)) < t\, r^k \quad \forall r < r_2$

we pick later

density upper bounds $\qquad$ (C)

From (A), we get some $W\in G(n,k)$ based on a. By our lemma, $\mathcal{H}^k(\mathbb{P}_W(E))=0$.
Fix an $r=\theta$ small enough that (A), (B), (C) hold.
Define

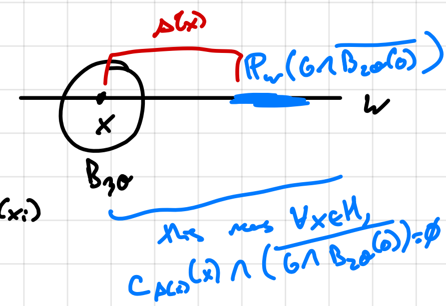
$$H := D_\theta(0)\setminus \mathbb{P}_W(G\cap B_{2\theta}(0))$$



Then, H is open and has full measure.
For $x\in H$, define $\rho(x)=\mathrm{dist}(x,\mathbb{P}_W(G\cap \overline{B_{2\theta}(0)}))$.

By (A)② and related density lower bounds,

$$\mathcal{H}^k(B_{3\theta}\cap E)\ge \gamma 3^k\theta^k \qquad \mathcal{H}^k(B_{3\theta}\cap G)\ge \frac{\gamma}{2}3^k\theta^k$$



We will run the 5r covering theorem. Cover $H\cap D_{\theta/4}(0)$ with disks $\{D_{20\rho_i}(x_i)\}_{i\in I}$ s.t. $\{D_{4\rho_i}(x_i)\}$ are pairwise disjoint and $\rho_i:=\rho(x_i)$.
Then,

$$\sum_i \omega_k(20\rho_i)^k \ge \left(\frac{\theta}{4}\right)^k\omega_k \Rightarrow \sum_i \rho_i^k\ge \frac{\theta}{80^k} \qquad (*)$$

Write $J:=\{i\in I: C_{\rho_i/2}(x_i)\cap F\cap B_\theta(0)\ne\emptyset\}$ and let $K=I\setminus J$.
Observe that $\forall j\in J$, $\exists y_j\in F\cap C_{\rho_j/2}(x_j)\cap B_\theta(0)$

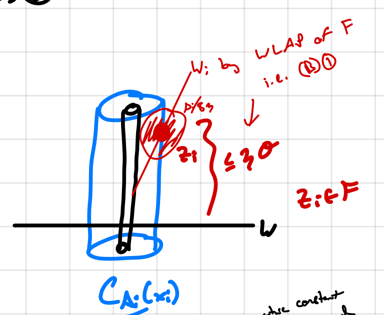
$$\mathcal{H}^k(C_{\rho_j}(x_j)\cap(E\setminus G)\cap B_{2\theta}(0))\ge \mathcal{H}^k(E\cap B_{\rho_j/2}(y_j))\ge \delta\left(\frac{\rho_j}{2}\right)^k \quad \text{by (A)②.}$$

So,

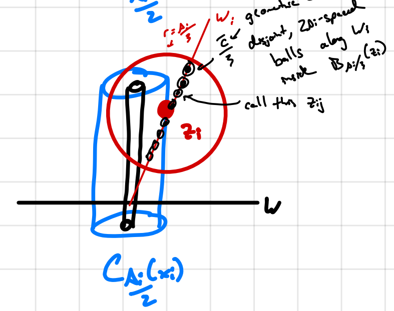
$$\sum_{j\in J}\left(\frac{\rho_j}{2}\right)^k \le \frac1\delta \mathcal{H}^k((E\setminus G)\cap B_{2\theta}(0))\le \frac1\delta t(2\theta)^k \quad \text{by (C)②}$$

For t small enough, this will surpass (*). So, $\forall i\in K$,

$$C_{\rho_i/2}(x_i)\cap F\cap B_\theta(0)=\emptyset$$



By the fact that F cannot intersect the cylinder, we know W_i must be vertical enough to avoid the cylinder.



By the second picture, $\exists\ell$ segment inside z_i+W_i and $\{z_{ij}\}_j\subseteq z_i+W_i$ s.t. $\{B_{2\rho_i}(z_{ij})\}_j$ pairwise disjoint. So, picking $B_{\rho_i}(z_{ij})\ni f_{ij}\in F$, $B_{\rho_i}(f_{ij})\subseteq C_{4\rho_i}(x_i)\cap B_{2\theta}(0)$. So,

$$\mathcal{H}^k(E\cap C_{4\rho_i}(x_i)\cap B_{2\theta}(0))\ge \frac{\bar c}{3}\delta\rho_i^k$$

By pairwise disjointness of the balls around f_{ij},

$$\mathcal{H}^k(E\cap B_{2\theta}(0))\ge \frac{\bar c}{3}\delta\sum_{i\in K}\rho_i^k$$

By (C)①,

$$\frac{\bar c}{3}\delta\sum_{i\in K}\rho_i^k \le \omega_k(2\theta)^k\zeta \overset{(*)}{\Longrightarrow} \frac{\theta^k}{(80)^k}\le \frac{\omega_k(2\theta)^k\zeta}{\bar c\,\delta}$$

$$\Rightarrow \zeta \ge \underbrace{\frac{\bar c\,\delta}{(160)^k\omega_k}}_{\text{independent of }\zeta}$$

. With ζ small enough, $\times$

$\square$

Let's zoom out. Recall we wish to prove BMP.

Theorem: (BMP)

Let $\mu \neq 0$ be Radon on $\mathbb{R}^n$ with α density existing, positive, and finite μ-a.e.
Then,

① $\alpha \in \mathbb{N}$ ② μ is α-rectifiable

Under these assumptions, we have seen the following for μ-a.e. x

① $\alpha = k$ for k integer

② $\mathrm{Tan}(\mu, x) \subseteq \mathcal{U}^k(\mathbb{R}^n)$

③ $\emptyset \neq \mathrm{Tan}(\mu,x) \cap \Theta(\mu,x)\ \mathcal{G}^k(\mathbb{R}^n)$ ← Hausdorff on a plane $\{\mathcal{H}^k \llcorner V : V \in G(n,k)\}$

With the following theorem, we would complete BMP.

Theorem (Preiss)

If all tangent measures are uniform and one tangent measure is $\mathcal{H}^k \llcorner V$ for $V \in G(n,k)$, then μ is rectifiable. (i.e. ② + ③ ⇒ rect)

Proof of Preiss

Defn:

The tangent measure at infinity of a Radon μ is

$$\mathrm{Tan}(\mu, \infty) = w^*\lim_{j \to \infty} \mu_{0, r_j}$$

Prop 1

$\forall \mu \subseteq \mathcal{U}^k(\mathbb{R}^n)$ uniform, $\mathrm{Tan}(\mu, \infty) = \{\xi\}$ is unique!

Prop 2

$\exists \varepsilon > 0$ s.t. if μ and ξ are as in Prop 1 and

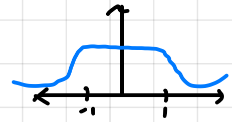
$$\min_{V \in G(n,k)} \int_{B_1} \mathrm{dist}^2(x, V)\, d\xi(x) < \varepsilon,$$

then $\xi = \mathcal{H}^k \llcorner V$ for some V.

Prop 3

If μ and ξ are as in Prop 1 and $\xi \in G^k(\mathbb{R}^n)$,
then $\mu = \xi$.

Using these, we will reason in the following way:



define

$$F(\mu) := \min_{V \in G(n,k)} \int \varphi(z)^2 \operatorname{dist}^2(z, V)\, d\mu \quad \text{for some } \varphi \geq \mathbb{1}_B,$$

if $=0$, flat for measure!

Then, F is continuous since min of Lipschitz.

Define $f(r) := F(\mu_{0,r})$. If $\mu_{0,r_j} \overset{*}{\rightharpoonup} \nu$, then $\lim_{r\to\infty} f(r) = F(\nu)$

Blow-Down Procedure

$\geq$ this why?

Let $\varepsilon > 0$ be as given by Prop 2. We know $\liminf_{r\to 0} f(r) = 0$. By Prop 2, we expect $\limsup_{r\to 0} f(r) \geq \varepsilon$. Fix $r_j \downarrow 0$ and $s_j \downarrow 0$ s.t. $f(s_j) > \varepsilon$ and $f(r_j) \to 0$. By picking subsequences, we may suppose WOLOG $s_j \leq r_j$. Then, $\exists\, \theta_j$ s.t.

$$f(\theta_j) = \varepsilon \quad \text{and} \quad f < \varepsilon \text{ on } [\theta_j, r_j]$$

Up to subsequence, $\mu_{0,\theta_j} \overset{*}{\rightharpoonup} \nu_1$, $\leftarrow$ not a tangent measure since $F(\nu_1) \geq 0$

$\mu_{0,r_j} \overset{*}{\rightharpoonup} \nu_2$ $\leftarrow$ tangent measure

So, $\frac{\theta_j}{r_j} \to 0$. Let $\xi = \operatorname{Tan}(\nu_1, \infty)$. Then, $\exists \{s_j\}_j$ s.t. $s_j \in [\theta_j, r_j]$ s.t.

$$\mu_{0,s_j} \rightharpoonup \xi, \text{ and so } F(\xi) = \lim_{j\to\infty} f(s_j) \leq \varepsilon$$

By Prop 2, ξ is flat. So, by Prop 3, since ν_1's tangent at ∞ measure is flat, then $\nu_1 = \xi \Rightarrow \nu_1$ is flat.

$\square$

So, to prove Preiss we must prove these 3 props! Next time :)

Fill in 11/7

11/9-

Theorem: (Tangent at ∞ is unique)

Let $\mu \in \mathcal{U}^m(\mathbb{R}^n)$. Then, $\exists!\ \xi \in \mathcal{U}^m(\mathbb{R}^n)$ s.t.

$$w^*\lim_{r\to\infty} \mu_{0,r} = \xi$$

prove real analyticity to ensure uniqueness at infinity

Proof: Write $\mu_r := e^{-|\cdot|^2}\mu_{0,r}$. We w.t.s. $\mu_r \overset{*}{\rightharpoonup} \_\_$ uniquely.

Def.

Define the generalized moments as

when $s \downarrow 0$, we can't see if smoothness holds

$$b_{k,s}(u_1,\dots,u_k) = \frac{(2s)^k}{k!}\left(\int e^{-s|z|^2}\,d\mu(z)\right)^{-1}\int \langle z,u_1\rangle \cdots \langle z,u_k\rangle\, e^{-s|z|^2}\,d\mu(z)$$

Remark: Theorem above $\iff \lim_{s\to 0} \frac{b_{k,s}}{s^{k/2}}$ exists and is finite

Consider the following Taylor expansions of $b_{k,s}$

(1)

a) $|b_{k,s}(u_1,\dots,u_k)| \le C\, \frac{2^k k^{k/2}}{k!}\, s^{k/2}\, |u_1|\cdots|u_k|$ (independent of k)

b) $\left|\sum_{k=1}^{2a} b_{k,s}(x^k,\dots,x^k) - \sum_{k=1}^{a}\frac{s^k|x|^{2k}}{k!}\right| \le C\,(s|x|^2)^{a+\frac12} \quad \forall x \in spt(\mu)$

(2)

c) $\forall a \in \mathbb{N}, \quad b_{k,s} = \sum_{j=1}^{a} \frac{s^j b_k^{(j)}}{k!} + o(s^a)$ for $s \downarrow 0$ (so, $b_{k,s}$ is C^∞)

$\Rightarrow \lim_{s\downarrow 0}\frac{b_{2a,s}}{s^a} = \frac{b_{2a}^{(a)}}{a!} \quad \forall x \in spt(\mu)$

d) $b_k^{(j)} = 0$ if $k > 2j$

e) $\sum_{k=1}^{2a} b_k^{(a)}(x) = |x|^{2a} \quad \forall a$ and all $x \in spt(\mu)$ (sum over taylor expansions of $b_{k,s}$ and c.w. $e^{s|x|^2}-1$'s expansion)

Proof of b_k stuff: Introduce $\odot^k \mathbb{R}^n$, the space of symmetric k-tensors, where $k = 2a$.

Let $X = X^{k,n} = \bigoplus_{j=1}^{k} \odot^j \mathbb{R}^n$ with each $\odot^j\mathbb{R}^n$ having inner product $\langle\cdot,\cdot\rangle_j$.

Then, X has an inner product $\langle u,v\rangle_X = \sum_{j=1}^{k}\frac{1}{j!}\langle P_j(u),P_j(v)\rangle$ (P_j: projection to $\odot^j\mathbb{R}^n$)

Define $b_s := b_{1,s}\oplus b_{2,s}\oplus\cdots\oplus b_{k,s} \in Hom(X,\mathbb{R}) = X^*$

We claim that $b_s \in C^a$. For any $z \in \bigoplus^i \mathbb{R}^n$, let $\underset{\swarrow\, z\otimes \cdots \otimes z}{z^j}$ be the rank-1 symmetric tensor generated by z. Then, for any $x \in \operatorname{spt}(\mu)$,

$$b_s(x + \cdots + x^{2a}) = \sum_{j=1}^{2a} b_{j,s}(x^j) \overset{(8)}{=} \sum_{j=1}^{a} \underbrace{\frac{s^j |x|^{2j}}{j!}}_{C^\infty} + o(s^a)$$

$$=: w_s(x + \cdots + x^{2a}) + o(s^a)$$

Let $V := \operatorname{Span}\{x + x^2 + \cdots + x^{2a} : x \in \operatorname{spt}(\mu)\}$. By linearity, the above holds over V. So, b_s is C^a over V.

By linearity, we can only worry about what happens in one component of b_s. So, let $\hat{A} = (0, \underset{\text{symmetric 2-tensor}}{A}, 0, \ldots)$, and then $b_s(\hat{A}) = b_{2,s}(A)$.

WLOG, $A = \frac{1}{2!}(u_1 \otimes u_2 + u_2 \otimes u_1)$ since symmetric 2-tensor. Thus,

We know

$$b_{2,s} = \frac{(2s)^2}{2!} \underbrace{\left(\int e^{-s|z|^2} d\mu(z)\right)^{-1}}_{=: I(s)} \int e^{-s|z|^2} \underbrace{\langle z, u_1 \rangle \langle z, u_2 \rangle}_{= (z\otimes z,\, u_1 \otimes u_2)_2} d\mu(z)$$

$$= I(s)^{-1} \int e^{-s|z|^2} \frac{s^2}{2} \langle A, z^2 \rangle_2 \, d\mu(z)$$

$$= I(s)^{-1} \int e^{-s|z|^2} \left\langle \sum_{k=1}^{2a} s^k P_k(A),\ z + z^2 + \cdots + z^{2a} \right\rangle d\mu(z)$$

Let $F_s := \left\{ A : \left\langle \sum_{k=1}^{2a} s^k P_k(A),\ v \right\rangle = 0 \quad \forall v \in V \right\}$ (so, $F_1 = V^\perp$).

Note that $V \oplus F_s = X^{2a,n}$. On F_s we have $b_{2,s} = 0$.

We will make a projection that does what we want. In particular,

$$Q_s : X \to X \quad \text{s.t.} \quad \begin{cases} Q_s(u) = 0 & u \in F_s \\ Q_s(v) = v & v \in V \end{cases} \quad \text{and extend linearly.}$$

So, we have $b_s = w_s \circ Q_s$ since the projection sends $\ker(b_s)$ to 0 anyway.

We claim $s \mapsto Q_s$ can be real-analytically extended to $s = 0$.

We see

$$\underbrace{\left(\sum_{k=1}^{2a} s^k P_k\right)}_{B_s} \circ \underbrace{\left(\sum_{j=1}^{2a} s^{-j} P_j\right)}_{A_s} = \sum_{k,j=1}^{2a} s^{k-j} \overbrace{P_k \circ P_j}^{\delta_{kj} P_k} = \sum_{k=1}^{2a} P_k = \mathbb{1}$$

So, B_s and A_s are inverses. So, $w \in F_s \iff B_s(w) \in V^\perp = F_1 \iff W = A_s(V^\perp)$

$\Rightarrow F_s = A_s(V^\perp)$.

Write $V^\perp := \bigoplus_{j=1}^{2a} V_j$ inductively:
$$V_1 = V^\perp \cap \bigoplus \mathbb{R}^n$$
$$V_2 = (V^\perp \cap (\bigoplus \mathbb{R}^n \oplus \bigoplus^2 \mathbb{R}^n)) \cap V_1^\perp$$
$$\vdots$$

Define $A_s' := P_k + s P_{k-1} + \cdots + s^{k-1} P_1$ on V_k, and the identity on V.

We know $X = V \oplus V_1 \oplus \cdots \oplus V_{2a}$

We know $s \mapsto A_s'$ is real-analytic since it is a power series.

As A_0' is the identity, it is invertible with real-analytic inverse.

Let $\tilde{Q}_s = P_V \circ A_s^{-1}$, and so $\tilde{Q}_s$ is real-analytic. Check $\tilde{Q}_s = Q_s$.

Since A_s is the identity on V, so too is A_s^{-1}. We WTS $\tilde{Q}_s = 0$ on F_s.
As $F_s = A_s(V^\perp)$, we want to check A_s^{-1} maps $A_s(V^\perp) \mapsto V^\perp$ $\forall s>0$.

Note that on V_j,

$$s^{-j}A_s' = s^{-j}P_j + s^{-(j-1)}P_{j-1} + \dots + s^{-1}P_1 \overset{P_\ell(V_j)=0 \ \forall \ell > j}{=} s^{-2a}P_{2a} + \dots + s^{-1}P_1 = A_s$$

So, $\tilde{Q}_0$ is C^ω, and so b_0 is as well!

□

Theorem:

Let $\mu \in \mathcal{U}^m(\mathbb{R}^n)$ and ς be the tangent at $+\infty$. Then, $\exists\, \varepsilon(m,n)$ s.t. (indep. of μ!)

$$\int_{B_1} \mathrm{dist}^2(x,V)\, d\varsigma < \varepsilon \Rightarrow \varsigma = \mathcal{H}^m \llcorner V.$$

Proof: For $m \leq 2$, we will show $\varepsilon = \infty$. We will use the fact that $\varsigma_{0,s} = \varsigma$ for all $s>0$ (conical property). To see this, note that if r_k sends $\mu_{0,r_k} \rightharpoonup \varsigma$, then $\mu_{0,sr_k} \overset{*}{\rightharpoonup} \varsigma_{0,s}$; uniqueness of tangent at ∞ means $\varsigma_{0,s} = \varsigma$. We will inspect the moments of ς.

Lemma: If $\lambda \in \mathcal{U}^m(\mathbb{R}^n)$ is conical ($\lambda_{0,r} = \lambda \ \forall r$), then

(i) $b_{2k-1,s} \equiv 0$ and $b_{2k,s} = \frac{s^k}{k!} b_{2k}^{(k)}$

(ii) $\mathrm{spt}(\lambda) \subseteq \{ b_{2a}^{(a)}(x^{2a}) = |x|^{2a} \}$.

(iii) If $u \in \mathrm{spt}(\lambda)$, then $\forall \varphi \in C_c(\mathbb{R}^2)$

$$\int \varphi(|z|, \langle z,u\rangle)\, d\lambda = \int_{\mathbb{R}^m} \varphi(|z|, |u| z_1)\, dz$$

(iv) $\mathrm{tr}(b_{2,1}) = m$.

Proof of lemma: We have seen $b_{k,s} = s^{k/2} b_{k,1}$, and so $b_{k,s} = 0$ for k odd.
There is also the Taylor expansion $b_{2k,s} = \frac{s^k}{k!} b_{2k}^{(k)}$ ← check why this is exact, seems to do with conical

From ②(e) earlier, $\sum_{k=1}^{2a} b_k^{(a)}(x^k) = |x|^{2a}$, and so $b_{2a}^{(a)}(x^{2a}) = |x|^{2a}$

For (iii), we may compute for all $x \in \mathrm{spt}(\lambda)$ the following:

$$\int e^{-s|z|^2} \langle z,x\rangle^k \, d\lambda(z) = b_{k,s}(x^k) = \begin{cases} 0 & k \text{ odd} \\ \frac{|x|^k}{(\frac{k}{2})!} & k \text{ even} \end{cases} \quad \text{by conicality}$$

Since the Lebesgue measure on $\mathbb{R}^m$ is conical, uniform, and $\mathrm{spt}(\lambda) \subseteq \mathrm{spt}(\mathcal{L}^m)$.

So, the same computation holds, i.e.

$$\int e^{-s|z|^2}\langle z,x\rangle^k\,dz = b_{k,s}(x^k) = \begin{cases} 0 & k \text{ odd} \\ \frac{|x|^k}{(\frac{k}{2})!} & k \text{ even} \end{cases}$$

Taking some derivatives, $\int e^{-s|z|^2}\langle z,x\rangle^k |z|^{2j}\,d\lambda(z) = \int_{\mathbb{R}^n} e^{-s|z|^2}|x|^k z_1{}^k |z|^{2j}\,dz$

So, the prop holds for all polynomials, taking $s \downarrow 0$. By Stone-Weierstrass and some good bounding and measure theory and compact reasoning, the result follows.

For (iv),

$$\mathrm{tr}(b_{2,1}) = \sum_{i=1}^{n} b_{2,1}(e_i^2) = 2\,I(1)^{-1}\int e^{-|z|^2}\sum_{i=1}^{n}\langle e_i,z\rangle^2\,d\lambda(z)$$

$$= 2\,I(1)^{-1}\left(\int_{\mathbb{R}^n} e^{-|z|^2}|z|^2\,dz\right) = I(1)^{-1}\int\langle(e^{-|z|^2}2z),z\rangle\,dz$$

$$\overset{\text{integration by parts}}{=} -\int -e^{-|z|^2}\underbrace{\mathrm{div}\,z}_{m}\,dz = m\,I(1).$$

□

We know that $b_{2,1}(x^2) = b_2^{(1)}(x^2)$ by definition. This is a symmetric bilinear form, and so it's positive semidefinite. Diagonalize it and write its eigenvalues

$$\alpha_1 \geq \dots \geq \alpha_m \geq \dots \geq \alpha_n \geq 0.$$

<u>Lemma:</u> Under assumptions, $\left(\begin{matrix} m\leq 2 \text{ or} \\ \int_{B_1}\mathrm{dist}^2(x,W)\,d\xi < \varepsilon\end{matrix}\right)$, then $\alpha_m \geq 1$.

<u>Proof of lemma:</u> Since ξ conical and has nonempty support, $\exists x \in \mathrm{spt}(\xi)$ with $|x|=1$.
So, $b_2^{(1)}(x^2) = b_{2,1}(x^2) = 2\,I(1)^{-1}\int_{\mathbb{R}^n} e^{-|z|^2}\langle z,x\rangle^2\,d\xi(z) \overset{(iii)}{=} 2\,I(1)^{-1}\int_{\mathbb{R}^n} e^{-|z|^2}z_1\,dz$
$= 1.$

Thus, $\alpha_1 = 1$. So, we've proven the lemma for $m=1$.

($m=2$) Now, let $x \in \mathrm{spt}(\xi)$ arbitrary. Define $S := \{y : |\langle y,x\rangle| \leq 1\}$
Then, $\xi(S) = \int \mathbb{1}_S(z)\,d\xi(z) \overset{(iii)}{=} \infty \Rightarrow \mathrm{spt}(\xi)\cap S$ is unbounded.
Taking a sequence $y_j \in \mathrm{spt}(\xi)\cap S$ that grows, we see $\frac{y_j}{|y_j|} \to y \perp x$ ← make it unit
So, $\exists y \in \mathrm{spt}(\xi)$ orthogonal to x. We have $b_2^{(1)}(y^2) = 1$
letting $V = \mathrm{span}(\{x,y\})$, $\mathrm{tr}(b_{2,1}\llcorner V) = 2$.
Since $\mathrm{tr}(b_{2,1}) = \mathrm{tr}(b_{2,1}\llcorner V) + \mathrm{tr}(b_{2,1}\llcorner V^\perp)$, we see $b_{2,1}\llcorner V^\perp = 0$.
So, $\mathrm{spt}(\xi) \subseteq V$.

($m\geq 3$) Let V be the m-dim plane spanned by the first m eigenvectors. By min/max characterization of eigenvalues,

$$\mathrm{tr}(b_2^{(1)}\llcorner V^\perp) = \min_{W\in G(n,m)} \mathrm{tr}(b_{2,1}\llcorner W^\perp)$$

Observe that $\forall W$, since $\sum_{i=1}^{n-m}\langle z,f_i\rangle^2 = \mathrm{dist}^2(z,W)$, we have ($f_i$ ONB for $W^\perp$)

$$\int e^{-|z|^2}\mathrm{dist}^2(z,W)\,d\xi(z) = \,= \mathrm{tr}(b_{2,1}\llcorner W^\perp)$$

We know a minimizer of this - it's V! In other words,

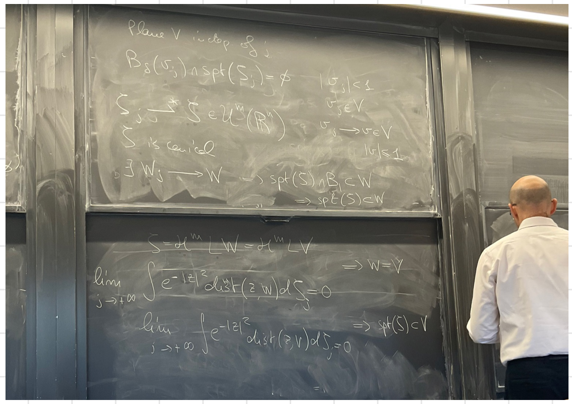
$$\min_{W} \int e^{-|z|^2} dist^2(z,W)\, d\xi(z) = tr(b_{z,1} \llcorner V^{\perp})$$

By assumption, $\min_{W} \int dist^2(z,W)\, d\xi(z) \le \varepsilon$

<u>Claim:</u> $\forall \delta>0$, $\exists \varepsilon$ small enough that $B_\delta(v)\cap spt(\xi) \neq 0$ $\forall v \in B_1 \cap V$.

$\xi_j \to \xi$

<u>Argument:</u> Fix $\delta>0$, let $\varepsilon = \frac{1}{j}$ for the ξ_j contradicting the statement.



$\Rightarrow \xi(B_{\delta/2}(v)) = 0$, contradicting $\xi = \mathcal{H}^m \llcorner V$.

Fix $\delta>0$. By the claim, $\exists x \in spt(\xi)$ s.t. $|\underset{\in B_1\cap V}{x-e_m}| < \delta$ where $e_1, \dots, e_m, \dots$ are eigenvectors of $b_{z,1}$. Then,

$$\alpha_i + (m-1)\alpha_m \le tr\, b_{z,1} = m \quad \forall i<m$$
$$\Rightarrow (\alpha_i - 1) \le (m-1)(1-\alpha_m) \quad \forall i<m$$

We also know $\alpha_i \le \alpha_m$ $\forall i \ge m$. Suppose BWOC $\alpha_m < 1$. Then,

$$b_{z,1}(x^2) = \sum_{i=1}^{n} \alpha_i \langle x, e_i\rangle^2 = |x|^2$$

$$\Rightarrow 0 = \sum_{i=1}^{n} (\alpha_i - 1)\langle x, e_i\rangle^2 \underset{\substack{\alpha_i<1\\ i\ge m}}{<} \sum_{i=1}^{m} (\alpha_i - 1)\langle x, e_i\rangle^2$$

$$\le (m-1)(1-\alpha_m)\sum_{i=1}^{m-1}\langle x, e_i\rangle^2 + (\alpha_m - 1)\langle x, e_m\rangle^2$$

$$= (m-1)(1-\alpha_m)\sum_{i=1}^{m-1}\underbrace{\langle x-e_m, e_i\rangle^2}_{\le \delta^2 \text{ C.S.}} - (1-\alpha_m)\Big(\underbrace{\langle e_m, x-e_m\rangle}_{\le \delta \text{ C.S.}} + \underbrace{\langle e_m, e_m\rangle}_{1}\Big)^2$$

$$\le (m-1)(1-\alpha_m)(m-1)\delta^2 - (1-\alpha_m)(1-\delta)^2$$

$$= \underbrace{(1-\alpha_m)}_{>0}\left((m-1)^2\delta^2 - (1-\delta)^2\right)$$

For δ small enough, this expression is negative. Contradiction. Taking the ε small enough for the claim, lemma proven. $\square$

The lemma and the fact $tr(b_{z,1}) = m$ implies $\alpha_j = \begin{cases} 1 & j \le m \\ 0 & j > m \end{cases}$. Letting V be the eigenspace spanned by the 1-eigenvectors, and so

$$b_z^{(1)}(x^2) = |P_V(x)|^2$$

By (ii) from above lemma, $spt(\xi) \subseteq \{x : |P_V(x)|^2 = |x|^2\} = V$

Since ξ is supported on an m-dimensional space, it's flat. $\square$

11/14

We needed the 3 propositions below for $\mu \in \mathcal{U}^m(\mathbb{R}^n)$

✓ **Prop A:** $\exists$ a unique tangent ξ to μ at ∞.

✓ **Prop B:** $\exists\, \varepsilon(m,n)$ s.t. ξ is flat if $m \leq 2$ or if

$$\min_{V \in G(n,m)} \int_{B_1} \operatorname{dist}^2(x,V)\, d\xi(x) < \varepsilon(m,n)$$

Prop C: ξ flat $\Rightarrow \mu$ flat. (black magic)

Fill in 11/28

Reifenberg's Topological Disc Theorem

Leon Simon *

Here $B_\rho = \{x \in \mathbf{R}^n : |x| \le \rho\}$ and $B_\rho(y) = \{x \in \mathbf{R}^n : |x - y| \le \rho\}$.

First we introduce Reifenberg's ϵ-approximation property for subsets of $\mathbf{R}^n$.

Definition: If $\epsilon > 0$ and if S is a closed subset of the ball B_2, we say that S, containing 0, has the m-dimensional ϵ-Reifenberg approximation property in B_1 if for each $y \in S \cap B_1$ and for each $\rho \in (0, 1]$, there is an m-dimensional subspace $L_{y,\rho}$ such that $d_{\mathcal{H}}(S \cap B_\rho(y),\, y + L_{y,\rho} \cap B_\rho(y)) < \epsilon$.

Here $d_{\mathcal{H}}(A_1, A_2)$ is the Hausdorff distance between A_1, A_2; thus $d_{\mathcal{H}}(A_1, A_2) = \inf\{\epsilon > 0 : A_1 \subset B_\epsilon(A_2)\ \&\ A_2 \subset B_\epsilon(A_1)\}$.

Now we can state the main theorem.

Theorem (Reifenberg's disc theorem). *There is a constant $\epsilon = \epsilon(n) > 0$ such that if S, containing 0, is a closed subset of the ball B_2 which satisfies the above ϵ-Reifenberg approximation property in B_1, then $B_1 \cap S$ is homeomorphic to the closed unit ball in $\mathbf{R}^m$.*

In fact, there is a closed subset $M \subset \mathbf{R}^n$ such that $M \cap B_1 = S \cap B_1$ and such that is homeomorphic to a subspace T_0 of $\mathbf{R}^n$ via a homeomorphism $\tau : T_0 \to M$ with $|\tau(x) - x| \le C(n)\epsilon$ for each $x \in T_0$, and $\tau(x) = x$ for each $x \in T_0 \setminus B_2$. For any given $\alpha \in (0, 1)$ we can additionally arrange that τ and τ^{-1} are Hölder continuous with exponent α provided S satisfies the ϵ-Reifenberg condition with suitable $\epsilon = \epsilon(n, \alpha)$.

We'll need the following lemma in the proof of the above theorem.

Lemma 1 (Extension Lemma). *Let $\epsilon, r > 0$, let $y_1, \ldots, y_Q$ be a finite collection of points in $\mathbf{R}^n$ with $|y_i - y_k| \ge r$ for each $i \ne k$, and assume that $f : \{y_1, \ldots, y_Q\} \to \mathbf{R}^N$ is given such that $|f(y_i) - f(y_k)| \le \epsilon$ whenever $|y_i - y_k| \le 6r$. Then there is an extension $\overline{f} : \cup_i B_{2r}(y_i) \to \mathbf{R}^N$ such that $|\nabla \overline{f}| \le C(n)\epsilon r^{-1}$ and $|\overline{f}(x) - f(y_i)| \le C(n)\epsilon$ for $x \in B_{2r}(y_i)$, $i = 1, \ldots, Q$.*

Furthermore there is $\epsilon = \epsilon(n) > 0$ such that if $N = n^2$ (where $\mathbf{R}^{n^2}$ is identified with the set of $n \times n$ matrices in the usual way) and if each $f(y_i)$ is the matrix of an orthogonal projection of $\mathbf{R}^n$ onto some m-dimensional subspace $L_i \subset \mathbf{R}^n$, then we can

*Expository lecture at Universität Tübingen, May '96; Research partially supported by NSF grant DMS-9504456 at Stanford University

choose the extension $\overline{f}$ such that each $\overline{f}(x)$ is the matrix of an orthogonal projection of $\mathbf{R}^n$ onto some m-dimensional subspace L_x.

Proof: The proof uses a partition of unity $\{\psi_j\}$ for $\cup_i B_{2r}(y_i)$ of special type. Indeed we claim that there is a partition of unity for $\cup_i B_{2r}(y_i)$ with $\psi_i \in C_c^\infty(\mathbf{R}^n)$, $\psi_i \equiv 0$ outside $B_{3r}(y_i)$, $\psi_i(y_i) = 1$, and $\sup |\nabla \psi_i| \le C(n) r^{-1}$.

We see this as follows: first let ψ^0 be a $C^\infty(\mathbf{R}^n)$ function with $\psi^0(x) \equiv 1$ for $|x| < \frac{1}{3}$, $0 < \psi^0(x) < 1$ for $< \frac{1}{3}|x| \le \frac{5}{2}$, and $\psi^0(x) \equiv 0$ for $|x| \ge \frac{5}{2}$. For each $i = 1, \ldots, Q$ let $\psi_i^0(x) = \psi^0(\frac{x-y_i}{r})$, $\widetilde{\psi}_i^0(x) = \psi_i^0 \Pi_{k\neq i}(1 - \psi_k^0(x))$, and $\psi_i(x) = \frac{\widetilde{\psi}_i^0(x)}{\sum_k \widetilde{\psi}_k^0(x)}$. This evidently gives a partition of unity with the stated properties.

It is now straightforward to check that

$$\overline{f}(x) = \sum_{i=1}^{Q} \psi_i(x) f(y_i).$$

is a suitable extension.

For the second part of the lemma we recall that the orthogonal projections onto m-dimensional subspaces of $\mathbf{R}^n$ form a smooth (in fact real-analytic) compact submanifold $\mathcal{P}$ of $\mathbf{R}^{n^2}$, and hence there is a $\delta = \delta(n) > 0$ such that there is a smooth nearest-point projection map Ψ of the δ-neighbourhood $\mathcal{N}_\delta$ of $\mathcal{S}$ onto $\mathcal{S}$.

Now by the first part of the lemma we have an extension $\overline{f}^0$ such that $|f(y_i) - \overline{f}^0(x)| \le C(n)\epsilon$ for each $x \in B_{2r}(y_i)$; but by definition $f(y_i) \in \mathcal{S}$, so this means that if ϵ is small enough (depending only on n) we have $\overline{f}^0(x) \in \mathcal{N}_{\delta/2}$ and hence we can define $\overline{f} = \Psi \circ \overline{f}^0$. Evidently then $\overline{f}$ has the correct properties.

The second lemma involves a simple observation about the subspaces $L_{y,\rho}$ appearing in the ϵ-Reifenberg condition; in particular it shows that these must vary quite slowly (up to tilts of order ϵ) as y and ρ vary.

Lemma 2. *If $\epsilon > 0$ and if S satisfies the ϵ-Reifenberg condition above, then $\|L_{y_1,\sigma} - L_{y_2,\rho}\| \le 32\epsilon$ and dist$(y_1, y_2 + L_{y_2,\rho}) \le 32\epsilon\rho$ whenever y_1, $y_2 \in S \cap B_1$ and $0 < \frac{\rho}{8} \le \sigma \le \rho \le 1$.*

The proof, which involves only the definition of the ϵ-Reifenberg condition and the triangle inequality for $d_{\mathcal{H}}$, is left as an exercise for the reader.

Finally, we need the following "squash lemma":

Lemma 3 ("Squash Lemma"). *There is a constant $\epsilon_0 = \epsilon_0(n)$ such that the following holds. If $\epsilon \in (0, \epsilon_0]$, $\rho > 0$, L is an m-dimensional subspace of $\mathbf{R}^n$,*

$$\Phi(x) = p_L(x) + e(x), \qquad x \in B_{3\rho},$$

where p_L is orthogonal projection onto L and $\rho^{-1}|e(x)| + |\nabla e(x)| \le \epsilon$ for all $x \in B_{3\rho}$, and if

$$G = \{x + g(x) \,:\, x \in B_{3\rho} \cap L\}$$

is the graph of a C^1 function $g : B_{3\rho} \cap L \to L^\perp$ with $\rho^{-1}|g(x)| + |\nabla g(x)| \le 1$ at each point x of $B_{3\rho} \cap L$, then $\Phi(G \cap B_{3\rho})$ is the graph of a C^1-function $\widetilde{g} : U \to L^\perp$ over some domain U with $B_{11\rho/4} \cap L \subset U \subset L$ and with $\rho^{-1}|\widetilde{g}| + |\nabla\widetilde{g}(x)| \le 4\epsilon$ on $B_{11\rho/4} \cap L$.

Proof of the squash lemma: All hypotheses are written in "scale invariant" form, so there is no loss of generality in taking $\rho = 1$, which we do. Now by definition

$$\Phi(x + g(x)) = x + e(x + g(x)) \tag{1}$$

for $x \in B_2 \cap L$, and, if $h(x) = e(x + g(x))$, by the chain rule we have $|d_x h| \le 2\epsilon$ at each point x of $L \cap B_2$. Now we can write $h = h^\perp + h^T$, where $h^\perp = p_L^\perp \circ h$ and $h^T = p_L \circ h$. Then (1) says

$$\Phi(x + g(x)) = x + h^T(x) + h^\perp(x), \quad x \in B_2 \cap L. \tag{2}$$

Now let

$$Q(x) = x + h^T(x), \quad x \in B_2 \cap L,$$

and observe that

$$|d\,Q - \mathrm{id}| \le 2\epsilon, \quad |Q - \mathrm{id}| \le \epsilon \quad \text{on } B_2 \cap L,$$

and hence, for small enough $\epsilon \in (0, \frac{1}{6})$, by the inverse function theorem Q is a diffeomorphism of $B_2 \cap L$ onto a subset U where $L \cap B_{11/4} \subset U \subset L$ and $|d\,Q^{-1} - \mathrm{id}| \le 2\epsilon(1 + 2\epsilon) \le 3\epsilon$. Thus (2) can be written

$$\Phi(x + g(x)) = Q(x) + \widetilde{g}(Q(x)), \qquad x \in B_{11/4} \cap L,$$

where $\widetilde{g} = p_L^\perp \circ h \circ Q^{-1}$ on U, and, since $|d\,h \circ Q^{-1}| \le 2\epsilon(1 + 3\epsilon) \le 3\epsilon$, we have $|d\,\widetilde{g}| \le 3\epsilon$ and the proof is complete.

Proof of the Reifenberg disc theorem: The proof is based on an inductive procedure, making successive approximations to $S_* = S \cap B_1$ by C^∞ embedded submanifolds.

Let $T_0 = L_{0,1}$ (which without loss of generality we could take to be $\mathbf{R}^m \times \{0\}$) be an m-dimensional subspace such that $d_{\mathcal{H}}(S \cap B_1, T_0 \cap B_1) < \epsilon$, and let $r_j = \left(\frac{1}{8}\right)^j$, $j = 0, 1, \ldots$. The quantity r_j is going the be the "scale" used at the j^{th} step of the inductive process.

We in fact define maps $\sigma_j : \mathbf{R}^n \to \mathbf{R}^n$ and subsets $M_j \subset \mathbf{R}^n$ for $j = 0, 1, \ldots,$ as follows:

For $j \ge 1$, let $B_{r_j/2}(y_{ji})$, $i = 1, \ldots, Q_j$, be a maximal pairwise disjoint collection of balls centered in $S_* = B_1 \cap S$. Then evidently $S_* \subset \cup_{i=1}^{Q_j} B_{r_j}(y_{ji})$ and also $\operatorname{dist}(S_*, \mathbf{R}^n \setminus (\cup_{i=1}^{Q_j} B_{3r_j/2}(y_{ji}))) \ge r_j/2$. When $j = 0$ we take $Q_0 = 1$, $y_{0\,1} = 0$, and $M_0 = T_0$, $\sigma_0 =$ the orthogonal projection of $\mathbf{R}^n$ onto T_0.

For $j \ge 1$ and for each $i = 1, \dots, Q_j$ let L_{ji} be one of the m-dimensional subspaces $L_{y_{ji},8r_j}$ (corresponding to $y = y_{ji}$ and $\rho = 8r_j$ in the ϵ-Reifenberg condition). Thus

$$d_{\mathcal{H}}(S \cap B_{8r_j}(y_{ji}),\, (y_{ji} + L_{ji}) \cap B_{8r_j}(y_{ji})) < 8\epsilon\, r_j, \quad i = 1, \dots, Q_j.$$

For $j \ge 1$ we have by Lemma 2 that

$$d_{\mathcal{H}}((y_{ji} + L_{ji}) \cap B_{r_j}(y_{ji}),\, (y_{\ell k} + L_{\ell k}) \cap B_{r_j}(y_{ji})) \le 264\epsilon r_j \tag{1}$$

for any pair $y_{j\,i}, y_{\ell\, k}$ with $|y_{j\,i} - y_{\ell\, k}| \le 6r_{j-1}$, where either $\ell = j - 1$ and $k \in \{1, \dots, Q_{j-1}\}$ or $\ell = j$ and $k \in \{1, \dots, Q_j\}$. Notice of course that (1) implies

$$|p_{ji} - p_{\ell k}| < 264\epsilon, \quad \operatorname{dist}(y_{ji}, y_{\ell k} + L_{\ell k}) < 264\epsilon r_j \tag{2}$$

for such j, ℓ, i, k, where p_{ji} denotes the orthogonal projection of $\mathbf{R}^n$ onto L_{ji}.

In view of the inequalities (2) (together with the fact that $|y_{ji} - y_{jk}| \ge r_j$ for each $i \ne k$), we can apply the extension lemma with $r = r_j$, with y_{ji} in place of y_i and with the orthogonal projection p_{ji} in place of $f(y_i)$, to give orthogonal projections $p_{j,x}$ of $\mathbf{R}^n$ onto m-dimensional subspaces $L_{j,x}$ such that $p_{j,x} = p_{ji}$ when $x = y_{ji}$ and

$$\begin{aligned} &\left|\frac{\partial p_{j,x}}{\partial x^\ell}\right| \le \frac{C(n)\epsilon}{r_j}, \quad x \in \cup_{i=1}^{Q_j} B_{2r_j}(y_{ji}),\ \ell = 1, \dots, n, \\ &|p_{j,x} - p_{ji}| \le C(n)\epsilon, \qquad x \in B_{2r_j}(y_{ji}),\ i = 1, \dots, Q_j. \end{aligned} \tag{3}$$

Next let ψ_{ji} be a partition of unity for $\cup_{i=1}^{Q_j} B_{3r_j/2}(y_{ji})$ such that $|\nabla\psi_{ji}| \le C(n)/r_j$ and support $\psi_{ji} \subset B_{2r_j}(y_{ji})$ for each $i = 1, \dots, Q_j$. (This is constructed in precisely the same way as our partition of unity for the extension lemma, except that we start with a smooth function φ with support in $B_2(0)$ rather than in $B_3(0)$ as before; actually the construction can be simplified here because we do not need $\psi_{ji}(y_{ji}) = 1$ and $\psi_{jk}(y_{ji}) = 0$ for $i \ne k$.)

Now we can define σ_j and M_j for $j \ge 1$. First we define [1]

$$\sigma_j(x) = x - \sum_{i=1}^{Q_j} \psi_{ji}(x) p_{j,x}^{\perp}(x - y_{ji}), \qquad x \in \mathbf{R}^n, \tag{4}$$

and then we take

$$M_j = \sigma_j(M_{j-1}). \tag{5}$$

First note that, since $\sigma_j(x) \equiv x$ for $x \in \mathbf{R}^n \setminus (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji}))$, we have

$$M_j \setminus (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji})) = M_{j-1} \setminus (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji})) \tag{6}$$

[1] of course it doesn't matter that the $p_{j,x}$ are not defined outside $\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji})$ because the ψ_{ji} vanish identically there. (If you wish to be pedantic, you can define e.g. $p_{j,x}$ to be the orthogonal projection onto T_0 for $x \in \mathbf{R}^n \setminus (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji}))$.)

for each $j \geq 1$.

We claim that each M_k is a properly embedded C^∞ m-dimensional submanifold of $\mathbf{R}^n$ and that for each $k \geq 1$ and each $i \in \{1, \dots, Q_k\}$

$$\begin{aligned} &M_k \cap B_{2r_k}(y_{ki}) = \operatorname{graph} g_{ki} \\ &\sup|\nabla g_{ki}| \leq \gamma\,\epsilon, \qquad \sup|g_{ki}| \leq \gamma\,\epsilon\,r_k. \end{aligned} \tag{7}$$

where $\gamma \geq 1$ is a constant (to be specified as a function of n alone) and where g_{ki} is a C^∞ function over a domain in the affine space $y_{ki} + L_{ki}$ with values normal to L_{ki}.

We want to inductively to check this. Observe that if $j \geq 1$ and if M_{j-1} is a smooth embedded submanifold satisfying (7) with $k = j-1$, then by the definition (4) we have

$$\begin{aligned} \sigma_j(x) - x &= -\textstyle\sum_{k=1}^{Q_j}\psi_j(x)p^\perp_{j,x}(x - y_{jk}) \\ &= -\textstyle\sum_{k=1}^{Q_j}\psi_j(x)p^\perp_{jk}(x - y_{jk}) + \sum_{k=1}^{Q_j}\psi_j(x)(p^\perp_{jk} - p^\perp_{j,x})(x - y_{jk}). \end{aligned} \tag{8}$$

Now for each $i \in \{1, \dots, Q_j\}$, we can pick an $i_0 \in \{1, \dots, Q_{j-1}\}$ such that $y_{ji} \in B_{r_{j-1}}(y_{j-1\,i_0})$. Then, assuming that (7) holds with $k = j-1$ and with some constant $\gamma = \gamma_{j-1}$, for $x \in B_{2r_j}(y_{ji}) \cap M_{j-1} (\subset B_{2r_{j-1}}(y_{j-1\,i_0}) \cap M_{j-1})$ we can write $x = \xi + g_{j-1}(\xi)$, with $g_{j-1}(\xi) \in L^\perp_{j-1\,i_0}$, $\xi \in (y_{j-1\,i_0} + L_{j-1\,i_0}) \cap B_{2r_{j-1}}(y_{j-1\,i_0})$ and with $r^{-1}_{j-1}|g_{j-1}(\xi)| + |\nabla g_{j-1}(\xi)| \leq \gamma_{j-1}\epsilon$. Then we have, for each $k \in \{1, \dots, Q_j\}$,

$$\begin{aligned} p^\perp_{jk}(x - y_{jk}) = p^\perp_{j-1\,i_0}&(\xi + g_{j-1}(\xi) - y_{j-1\,i_0}) \\ &+ p^\perp_{j-1\,i_0}(y_{jk} - y_{j-1\,i_0}) + (p^\perp_{jk} - p^\perp_{j-1\,i_0})(\xi + g_{j-1}(\xi) - y_{jk}), \end{aligned}$$

and using (2), (3) together with the fact that $p^\perp_{j-1\,i_0}(\xi - y_{j-1\,i_0}) = 0$ (because $\xi - y_{j-1\,i_0} \in L_{j-1\,i_0}$), we have clearly then that

$$|p^\perp_{jk}(x - y_{jk})| \leq C(n)\epsilon(1 + \gamma_{j-1})r_j, \quad x \in B_{2r_j}(y_{ji}) \cap M_{j-1}, \quad |y_{jk} - y_{ji}| \leq 6r_j.$$

Using this in (8), and keeping in mind that for any $i \in \{1, \dots, Q_j\}$ and for any $x \in B_{2r_j}(y_{ji})$, we have that at most $C(n)$ terms in the sums on the right of (8) can be non-zero, and that these terms correspond to the indices k such that $|y_{ji} - y_{jk}| \leq 6r_j$, hence, using also (3), we again deduce from (8) that

$$|\sigma_j(x) - x| \leq C(n)(1 + \gamma_{j-1})\epsilon r_j, \quad x \in \cup_{i=1}^{Q_j} B_{2r_j}(y_{ji}) \cap M_{j-1}. \tag{9}$$

By first differentiating in (8) and using similar considerations on the right side, we also conclude

$$\sup_{x \in M_{j-1}} |\nabla'(\sigma_j(x) - x)| \leq C(n)(1 + \gamma_{j-1})\epsilon r_j, \tag{9$'$}$$

where ∇' denotes gradient taken on the submanifold M_{j-1}.

We refer to (9) and (9)′ subsequently as "the coarse estimates" for $|\sigma_j(x)-x|$, because, although useful, they are insufficient in themselves to complete that inductive proof that there is a fixed constant $\gamma=\gamma(n)$ such that (7) holds for all k; indeed after k applications of this coarse inequality, we will only have established that (7) holds with $\gamma=C(n)^k$.

Now assume that $j\geq 2$ and that (7) holds for $k=1,\ldots,j-1$, take an arbitrary $i_0\in\{1,\ldots,Q_j\}$, and write $y_0=y_{j\,i_0}$, $p_0=p_{j\,i_0}$, and $L_0=L_{j\,i_0}$. Since $\sum_{i=1}^{Q_j}\psi_{ji}\equiv 1$ in $U_j\equiv\cup_{i=1}^{Q_j}B_{3r_j/2}(y_{ji})$ we can rearrange the defining expression for σ_j to give

$$\sigma_j(x)=y_0+p_0(x-y_0)+e(x),\quad x\in U_j, \tag{10}$$

where e is given by

$$e(x)\equiv\sum_{i=1}^{Q_j}\psi_{ji}(x)p_0^{\perp}(y_{ji}-y_0)-\sum_{i=1}^{Q_j}\psi_{ji}(x)(p_{j,x}^{\perp}-p_0^{\perp})(x-y_{ji}),\quad x\in\mathbf{R}^n. \tag{11}$$

Now observe that by (2) and (3) we have $|p_{j,x}-p_0|\leq C(n)\epsilon\, r_j$ for $x\in B_{6r_j}(y_0)$. Using additionally the first inequality in (3) and the fact that $|\nabla\psi_{ji}|\leq C(n)/r_j$, it then follows easily that

$$r_j^{-1}|e(x)|+|\nabla e(x)|\leq C(n)\epsilon,\ \text{if } x\in B_{3r_j/2}(y_0), \tag{12}$$

where $C(n)$ is a fixed constant determined by n alone (and which is independent of any properties of M_{j-1}; in particular it is independent of whatever constant γ appears in (7)).

But now we can apply the Squash Lemma with $\widetilde{\sigma}_j(x)\equiv\sigma_j(x+y_0)-y_0$ in place of Φ, $2\,r_j$ in place of ρ, and $C(n)\epsilon$ in place of ϵ. Assuming that (7) holds with γ, ϵ such that $\gamma\epsilon\leq\frac{1}{2}$, we thus conclude

$$\sigma_j(M_{j-1}\cap B_{3r_j/2}(y_0))=G_j, \tag{13}$$

where $G_j=\{x+g_j(x)\,:\,x\in\Omega_j\}$ is the graph of a C^∞ function g_j defined over a domain Ω_j contained in the affine space y_0+L_0 with $B_{11r_j/8}(y_0)\cap(y_0+L_0)\subset\Omega_j$ and with

$$r_j^{-1}|g_j|+|\nabla g_j|\leq C(n)\epsilon,\quad x\in B_{11r_j/8}(y_0)\cap(y_0+L_0), \tag{14}$$

with $C(n)$ not depending on γ. Of course since $|\sigma_j(x)-x|<C(n)\gamma\epsilon$ (by (8)), we thus have, so long as $C(n)\gamma\epsilon\leq\frac{1}{32}$ that $\sigma_j(M_{j-1}\cap B_{3r_j/2}(y_0))\supset\sigma_j(M_{j-1})\cap B_{11r_j/8}(y_0)$, and hence (13) and (14) imply

$$M_j\cap B_{11r_j/8}(y_0))=G_j, \tag{15}$$

with G_j still as in (14).

Now we actually need to establish a result like this over the ball $B_{2r_j}(y_0)$ rather than merely over $B_{11r_j/8}(y_0)$; to achieve this, we observe that each y_{ji} is contained in one of the balls $B_{r_{j-1}}(y_{j-1\,i_0})$ for some $i_0 \in \{1, \dots, Q_{j-1}\}$, and so $B_{r_{j-1}/4}(y_{ji}) \subset B_{5r_{j-1}/4}(y_{j-1\,i_0})$. Also, by using the above argument with $j-1$ in place of j and with i_0 in place of i, we deduce that

$$(15)' \qquad M_{j-1} \cap B_{11r_{j-1}/8}(y_{j-1\,i_0})) = G_{j-1},$$

where $G_{j-1} = \{x+g_{j-1}(x) : x \in \Omega_{j-1}\}$ is the graph of a C^∞ function g_{j-1} defined over a domain Ω_{j-1} contained in the affine space $y_{j-1\,i_0} + L_{j-1\,i_0}$ with $B_{11r_{j-1}/8}(y_{j-1\,i_0}) \cap (y_{j-1\,i_0} + L_{j-1\,i_0}) \subset \Omega_{j-1}$ and with

$$(14)' \qquad r_{j-1}^{-1}|g_{j-1}| + |\nabla g_{j-1}| \le C(n)\epsilon, \quad x \in B_{11r_{j-1}/8}(y_{j-1\,i_0}) \cap (y_{j-1\,i_0} + L_{j-1\,i_0}).$$

But then by using the coarse estimates (9), (9)$'$ we deduce that in fact (7) holds with $k = j$ and a fixed constant γ which depending only on n and not on γ.

Notice that since $S_* \subset \cup_{i=1}^{Q_j} B_{r_j}(y_{ji})$ it is clear from (7) and the ϵ-Reifenberg condition in the ball $B_{2r_j}(y_{ji})$, that

$$(16) \qquad S_* \subset B_{C(n)\epsilon r_j}(M_j), \quad j \ge 0.$$

Notice also that (7) tells us that for $j \ge 2$

$$M_j \cap (\cup_{i=1}^{Q_j} B_{2r_j}(y_{ji})) \subset (\cup_{i=1}^{Q_j} B_{C(n)\epsilon r_j}(y_{ji} + L_{ji})) \subset B_{C(n)\epsilon r_j}(S),$$

and hence, since $M_j \setminus (\cup_i B_{2r_j}(y_{ji})) = M_{j-1} \setminus (\cup_i B_{2r_j}(y_{ji}))$ by mathematical induction it follows that

$$(17) \qquad M_j \cap B_{1+r_j/2} \subset B_{C(n)\epsilon\, r_j}(S)$$

for each $j = 0, 1, \dots$, provided $\epsilon \le \epsilon_0$, where $\epsilon_0 = \epsilon_0(n)$.

Next we want to show that the sequence $\tau_j = \sigma_j \circ \sigma_{j-1} \circ \cdots \sigma_0 | T_0$ is a sequence of C^∞ diffeomorphisms of T_0 onto M_j which converge uniformly on T_0 to a homeomorphism τ of T_0 onto a closed set M. In fact notice that by (8) we have

$$|\tau_j(x) - \tau_{j-1}(x)| \le C(n)\epsilon \left(\frac{1}{8}\right)^j, \quad j \ge 1, \ x \in T_0,$$

and hence by iterating we get

$$(18) \qquad |\tau_{j+k}(x) - \tau_j(x)| \le C(n)\epsilon \left(\frac{1}{8}\right)^j, \quad j \ge 0, \, k \ge 1, \ x \in T_0,$$

which shows that τ_j is Cauchy with respect to the uniform norm on T_0, and hence τ_j converges uniformly to a continuous map $\tau : T_0 \to \mathbf{R}^n$. Of course τ is the identity outside B_2 because each σ_j is the identity outside B_2. We let $M = \tau(T_0)$, so that M is a closed subset of $\mathbf{R}^n$ and in fact is the Hausdorff limit (with respect to the Hausdorff metric $d_{\mathcal{H}}$) of the sequence $M_j = \tau_j(T_0)$. Notice in particular that setting $j = 0$ and taking limit as $k \to \infty$ in the above inequality, we get

$$|\tau(x) - x| \le C(n)\epsilon, \quad x \in T_0. \tag{19}$$

(Thus τ is in the distance sense quite close to the identity if ϵ is small.)

Next we want to discuss injectivity of τ_j, τ; in fact we'll show that τ_j, τ are injective and that both τ and τ^{-1} are Hölder continuous.

To establish this, we first claim

$$(1 - C(n)\epsilon)|x - y| \le |\sigma_j(x) - \sigma_j(y)| \le (1 + C(n)\epsilon)|x - y|, \quad x, y \in M_{j-1}, \tag{20}$$

or equivalently

$$|\sigma_j(x) - \sigma_j(y) - (x - y)| \le C(n)\epsilon|x - y|, \quad x, y \in M_{j-1}. \tag{20$'$}$$

To prove this, note that if $|x - y| \ge r_j$ with $x, y \in M_{j-1}$, we can write

$$\begin{aligned}|\sigma_j(x) - \sigma_j(x) - (x - y)| &= |(\sigma_j(x) - x) - (\sigma_j(y) - y)| \\ &\le |\sigma_j(x) - x| + |\sigma_j(y) - y| \\ &\le C(n)\epsilon\, r_j \le C(n)\epsilon|x - y|,\end{aligned}$$

where we used (8) in the second inequality.

Now if $|x - y| < r_j$ we use the definition (4) to write

$$\begin{aligned}(\sigma_j(x) - \sigma_j(y)) - (x - y) = \textstyle\sum_{i=1}^{Q_j}&(\psi_{ji}(x)p^{\perp}_{j,x}(x - y_{ji}) \\ &- \psi_{ji}(y)p^{\perp}_{j,y}(y - y_{ji})), \qquad x, y \in \mathbf{R}^n,\end{aligned}$$

and note that we can rearrange the sum here to give

$$\begin{aligned}(\sigma_j(x) - \sigma_j(y)) - (x - y) = \textstyle\sum_{i=1}^{Q_j}\big(&\psi_{ji}(x)(p^{\perp}_{j,x}(x - y) \\ &+ \psi_{ji}(x)(p^{\perp}_{j,x} - p^{\perp}_{j,y})(y - y_{ji}) + (\psi_{ji}(x) - \psi_{ji}(y))p^{\perp}_{j,y}(y - y_{ji})\big).\end{aligned}$$

Now the second group of terms is (by (3)) trivially $\le C(n)\epsilon\,|x - y|$ in absolute value for any $x, y \in \mathbf{R}^n$ with $|x - y| \le r_j$. Further if $x, y \in M_{j-1}$, then by virtue of (7) (used with y in place of z) we see that the first and third group of terms on the right is $\le C(n)\epsilon\,|x - y|$ in absolute value. Thus we again get (20).

Now it is easy to establish the required injectivity and continuity of τ. In fact by iterating the inequality (20) we get

$$|\tau_j(x) - \tau_j(y)| \le (1 + C\epsilon)^j|x - y|, \quad x, y \in T_0, \ j \ge 1, \tag{21}$$

and by (8) we have

$$|\tau_j(x) - \tau_{j-1}(x)| \le C\epsilon r_j, \quad x \in T_0,\ j \ge 1,$$

and so (Cf. the discussion of uniform convergence of the τ_j above)

$$|\tau_j(x) - \tau(x)| \le C\epsilon r_j. \tag{22}$$

Then by the triangle inequality, for any $j \ge 0$ we have

$$\begin{aligned}|\tau(x) - \tau(y)| &\le |\tau(x) - \tau_j(x)| + |\tau_j(x) - \tau_j(y)| + |\tau_j(y) - \tau(y)| \\ &\le 2C(n)\epsilon r_j + (1 + C(n)\epsilon)^j |x - y| \\ &\le r_j + (1 + C(n)\epsilon)^j |x - y| \text{ if } 2\epsilon C(n) \le 1.\end{aligned}$$

Now let $\alpha \in (0,\, 1)$ be arbitrary and take $x,\, y \in T_0$ with $0 < |x-y| < \frac{1}{2}$. Choose j such that $r_j \le |x - y|^\alpha$ and $(1 + C(n)\epsilon)^j \le |x - y|^{-(1-\alpha)}$; thus we need $j \ge \frac{\alpha}{\log 8} \log\left(\frac{1}{|x-y|}\right)$ and also $j \le \frac{(1-\alpha)}{\log(1+C(n)\epsilon)} \log\left(\frac{1}{|x-y|}\right)$. Since $\log(1 + C(n)\epsilon) \to 0$ as $\epsilon \downarrow 0$, we see that such a choice of $j \in \{1, 2, \ldots\}$ exists provided $\epsilon \le \epsilon_0$, where $\epsilon_0 = \epsilon_0(n, \alpha)$. Then the above inequality gives

$$|\tau(x) - \tau(y)| \le 2|x - y|^\alpha, \quad x,\, y \in T_0 \text{ with } |x - y| < \tfrac{1}{2}.$$

Thus we can arrange for Hölder continuity with any exponent $\alpha < 1$. Similarly we have from the first inequality in (20) and (22) that

$$\begin{aligned}|x - y| &\le (1 + C\epsilon)^j |\tau_j(x) - \tau_j(y)| \\ &\le (1 + C\epsilon)^j \left(|\tau_j(x) - \tau(x)| + |\tau_j(y) - \tau(y)| + |\tau(x) - \tau(y)|\right) \\ &\le (1 + C(n)\epsilon)^j \left(C(n)\epsilon r_j + |\tau(x) - \tau(y)|\right)\end{aligned}$$

and j is again at our disposal. We in fact first choose ϵ such that $C(n)\epsilon \le 1$, so that

$$|x - y| \le (1 + C(n)\epsilon)^j \left(r_j + |\tau(x) - \tau(y)|\right),$$

and then choose j such that $\alpha \in (0,\, 1)$

$$4^{-j} \le \frac{1}{2}|x - y| \text{ and } (1 + C(n)\epsilon)^j \le |x - y|^{-(\alpha/(1-\alpha)}.$$

Notice that this requires $j \ge \log(2/|x-y|)/\log\left(\frac{8}{1+C(n)\epsilon}\right)$ and $j \le \alpha^{-1}(1-\alpha)\log(1/|x-y|)/\log(1+C(n)\epsilon)$, and again certainly such a choice of j exists provided $0 < |x-y| < \frac{1}{2}$ and provided we take $\epsilon \le \epsilon_0$ for suitable $\epsilon_0 = \epsilon_0(n, \alpha)$. In this case the above inequality gives

$$\tfrac{1}{2}|x - y| \le |x - y|^{-\alpha/(1-\alpha)} |\tau(x) - \tau(y)|, \quad |x - y| < \tfrac{1}{2},$$

which of course gives

$$|x-y|^{\alpha} \le 2|\tau(x)-\tau(y)|, \quad |x-y| < \tfrac{1}{2}.$$

Thus τ is injective, and the inverse is Hölder continuous with exponent α, for any given $\alpha \in (0, 1)$, provided the ϵ-Reifenberg condition holds with $\epsilon \le \epsilon_0$, where $\epsilon_0 = \epsilon_0(n, \alpha)$.

Now the proof of the Reifenberg inequality is complete, because we have shown that τ maps T_0 Hölder continuously onto M with Hölder continuous inverse, and by (16) and (17) we have

$$M \cap B_1 = S_*,$$

because (by (19)) M_j converges to M with respect to the Hausdorff distance metric.

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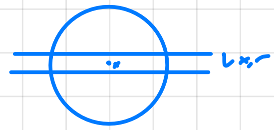
"Reifenberg" WLAP

$\forall x \in S \cap B_2,\ \forall r \leq 1,\ \exists L_{x,r} \in G(n,k)$ s.t.
$d_H\big((x+L_{x,r}) \cap B_r(x),\ S \cap B_r(x)\big) \leq \varepsilon r$



Theorem: (Reifenberg Disk)

$\exists \varepsilon = \varepsilon(n) > 0$ s.t. if S has the ε-weak k-dim linear approx. property in B_3 and $0 \in S$, then $\exists M \subseteq \mathbb{R}^n$, $T_0 \in G(n,k)$, and a map $\tau: T_0 \to M$ s.t.

i) $M \cap B_1 = S \cap B_1$ ii) $\tau: T_0 \to M$ is homeomorphic
iii) $|\tau(x) - x| \leq C(n)\varepsilon \quad \forall x$ iv) $\tau x = x \quad \forall |x| \geq B_2$
v) $\tau, \tau^{-1} \in C^\alpha$ for some $\alpha(\varepsilon,n) > 0$ vi) $\alpha \uparrow 1$ as $\varepsilon \downarrow 0$

Proof:

To prove the above, we will construct $M_0, M_1, \dots$ satisfying:

- $T_0 = L_{0,1}$
- $\sigma_j: \mathbb{R}^n \to \mathbb{R}^n$ s.t. $M_j = \sigma_j(M_{j-1})$
- $\{y_{ji}\}_{i \in [Q_j]} \subseteq S \cap B_1$ is a maximal subset s.t. $|y_{ji} - y_{jk}| \geq r_j := 8^{-j}$

covering of S

We will also use the following lemma, which is "simple to prove".

Lemma: (Squash Lemma)

$\exists \varepsilon_0 = \varepsilon_0(n)$ s.t. if

- $\varepsilon \in (0, \varepsilon_0]$
- $L \in G(n,k)$
- $\Phi(x) = P_L(x) + e(x) \quad \forall x \in B_{3R}$

C^0 norms $\Big\{$
- $R^{-1}\|e\| + \|De\| \leq \varepsilon$ in B_{3R}
- $G = \text{graph}(g) \quad g \in B_{3R} \cap L \to L^\perp$
- $R^{-1}\|g\| + \|Dg\| \leq 1$

Then, $\Phi(G \cap B_{3R})$ is the graph of a map $\tilde{g}: L \supset U \to L^\perp$ s.t. $B_{\frac{11R}{4}} \cap L \subset U$ and $R^{-1}\|\tilde{g}\|_{C^0} + \|D\tilde{g}\|_{C^0} \leq 4\varepsilon$.

From last time, we used partitions of unity to construct maps

$$\bigcup_{i=1}^{Q_j} B_{2r_j}(y_{ji}) \ni x \mapsto \mathbb{P}_{j,x} \quad \text{with} \quad \left|\frac{\partial \mathbb{P}_{j,x}}{\partial x_\gamma}\right| \leq \frac{C\varepsilon}{r_j}$$

Letting $\mathbb{P}_{y_{ji}} := \mathbb{P}_{L_{ji}}$, we also know $|\mathbb{P}_{j,x} - \mathbb{P}_{L_{ji}}| \leq C\varepsilon \quad \forall x \in B_{2r_j}(y_{ji})$

So,

$$d_H\big((y_{ji} + L_{ji}) \cap B_{r_j}(y_{ji}),\ (y_{k\ell} + L_{k\ell}) \cap B_{r_j}(y_{ji})\big) \leq C\varepsilon r_j$$

and

$$|y_{ji} - y_{k\ell}| \leq 6 r_{j-1} = 48 r_j \quad \forall i, \ell, \quad \text{when } k \in \{j, j-1\}$$

We constructed a partition of unity ψ_{ji} s.t.

$$\mathrm{spt}(\psi_{ji}) \subset B_{2r_j}(y_{ji}), \qquad \sum \psi_{ji} = 1 \text{ on } \bigcup_{i=1}^{Q_j} B_{3/2\, r_j}(y_{ji})$$

and $r_j \|D\psi_{ij}\| + \|\psi_{ij}\|_{C^0} \le C.$

We define

$$\sigma_j(x) := x - \sum_{i=1}^{Q_j} \psi_{ji}(x)\, \mathbb{P}_{j,x}^{\perp}(x - y_{ji}),$$

we will be able to apply Squash lemma here

and so

$$\sigma_j(x) - x = -\sum_{i=1}^{Q_j} \psi_{ji}(x)\, \mathbb{P}_{ji}^{\perp}(x-y_{ji}) + \sum_{i=1}^{Q_j} \psi_{ji}(x)\left(\mathbb{P}_{ji}^{\perp} - \mathbb{P}_{j,x}^{\perp}\right)(x-y_{ji})$$

$$= -\mathbb{P}_{j,i_0}^{\perp}(x-y_{j,i_0}) - \sum_{i=1}^{Q_j} \psi_{ji}(x)\left(\mathbb{P}_{ji}^{\perp} - \mathbb{P}_{ji_0}\right)^{\perp}(x-y_{ji_0}) - \underbrace{\sum_{i=1}^{Q_j} \psi_{ji}(x)\, \mathbb{P}_{ji}^{\perp}(y_{ji_0} - y_{ji})}_{\text{error term for freezing an } i_0}$$

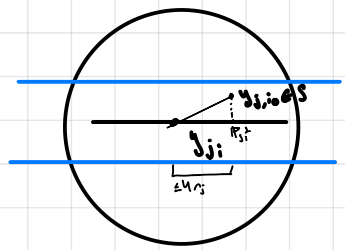
$$+ \sum_{i=1}^{Q_j} \psi_{ji}(x)\left(\mathbb{P}_{j,i}^{\perp} - \mathbb{P}_{j,x}^{\perp}\right)(x-y_{ji})$$

Call these error terms $e(x)$

Then, $\forall x \in B_{3/2\, r_j}(y_{j,i_0})$, we know

$$\sigma_j(x) = y_{ji_0} + (x - y_{ji_0}) - \mathbb{P}_{j,i_0}^{\perp}(x-y_{j,i_0}) + e(x) = y_{ji_0} + \mathbb{P}_{L_{ji_0}}(x-y_{ji_0}) + e(x)$$

Consider shifting the origin to y_{ji_0} so that we may apply the Squash lemma to σ_j.
Note that $\psi_{ji}(x) \neq 0$ only if $|y_{ji} - y_{ji_0}| \le 4r_j$



This, along with the Reifenberg WLAP, gives the estimate $r_j^{-1}\|e\|_{C^0} + \|De\|_{C^0} \le C\varepsilon$

tougher to show, but not impossible

We claim that $\exists\, g_{\kappa i}: L_{\kappa i} + y_{\kappa i} \to L_{\kappa i}^{\perp}$ and some constant γ s.t.

- $M_\kappa \cap B_{2r_\kappa}(y_{\kappa i}) = \mathrm{graph}(g_{\kappa i})$
- $\|g_{\kappa i}\|_{C^0} \le \gamma \varepsilon r_\kappa$, $\|Dg_{\kappa i}\|_{C^0} \le \gamma\varepsilon$

We will show this by induction (note that at every step we remain a submanifold). By passing through σ_j, and applying the squash lemma to σ_j, we see that the above holds for $\gamma = 4C$. Making ε small enough, we can ensure that this iteration contracts.

Define $\tau_j := \sigma_j \circ \sigma_{j-1} \circ \dots \circ \sigma_0 : T_0 \to \mathbb{R}^n$. By induction, $\tau_{j-1}(T_0) = M_{j-1}$.
We have the estimate on the error $|\tau_j(x) - \tau_{j-1}(x)| \le C\varepsilon r_j = C\varepsilon \cdot \frac{1}{8^j}$

Furthermore, by the chain rule, $\|D\tau_j\|_{C^0} \le (1+C\varepsilon)^j \Rightarrow \|D\tau_j - D\tau_{j-1}\|_{C^0} \le 2(1+C\varepsilon)^j$
So, since τ_j is uniformly close to τ_{j-1} and its C^1 norm is subexponential, we may apply an interpolation estimate on a Hölder seminorm $(\alpha < 1)$

$$[\tau_j - \tau_{j-1}]_\alpha \le C\varepsilon^\alpha \frac{1}{8^{j(1-\alpha)}}\left(2(1+C\varepsilon)^j\right)^\alpha$$

(this uses a Hölder interpolation estimate $[f]_\alpha \le C\|f\|_{C^0}^{1-\alpha}\|Df\|_{C^0}^{\alpha}$)

We choose α s.t. $\frac{(1+C\varepsilon)^\alpha}{8^{1-\alpha}} < 1$, which goes to 1 as $\varepsilon \downarrow 0$.

To get a Bi-Hölder estimate, we want to show

$$|x-y| \leq \frac{1}{1-C\varepsilon} |\sigma_j(x)-\sigma_j(y)| \quad \forall x,y \in M_{j-1}$$ ← again by induction

Iterating,

$$|x-y| \leq (1-C\varepsilon)^{-j} |\tau_j(x)-\tau_j(y)| \leq (1+C\varepsilon)^j |\tau_j(x)-\tau_j(y)| \quad \forall x,y \in T_0$$

Since the τ_j's get uniformly close, write $\tau := \lim_{j\to\infty} \tau_j$. Then,

$$|\tau(x)-\tau_j(x)| \leq C\varepsilon \frac{1}{8^j} \quad \forall j$$

So,

$$|x-y| \leq (1+C\varepsilon)^j \left(\frac{2C\varepsilon}{8^j} + |\tau(x)-\tau(y)|\right) \quad \forall j$$

Choose a j large enough that $\frac{1}{20}|x-y| \leq \frac{C\varepsilon(1+C\varepsilon)^j}{8^j} \leq \frac{1}{2}|x-y|$, and so

$$|x-y| \leq (1+C\varepsilon)^j |\tau(x)-\tau(y)|$$

We may get $(1+C\varepsilon)^j \leq c|x-y|^{-\beta}$ for small enough ε, giving the Bi-Hölder estimate

$$|x-y|^{1-\beta} \leq |\tau(x)-\tau(y)|$$

↑ certain types of decay of these ε's turn out to be enough to completely characterize rectifiability

$\square$

Consider an n-param NN. Define $\phi : \mathbb{R}^n \to \mathbb{R}$ to be the loss functional, and make whatever assumptions to ensure $\phi \in C^1$

Let $f_\Theta << \mathcal{L}^n(\mathbb{R}^n)$ be the density of some distribution over parameters. Then, this induces a distribution $f << \mathcal{L}^1(\mathbb{R})$ with density

$$f(x) = \int_{w \in \phi^{-1}(x)} \frac{f_\Theta(w)}{|\nabla \phi(w)|} \, d\mathcal{H}^{k-1}(w) \qquad \text{for a.e. } y \in \phi(\mathbb{R}^n)$$

In particular, we have expected loss

$$\mathbb{E}\, \phi(z) = \int_{x \in \phi(\mathbb{R}^n)} x \left(\int_{w \in \phi^{-1}(x)} \frac{f_\Theta(w)}{|\nabla \phi(w)|} \, d\mathcal{H}^{k-1}(w) \right) dx$$

Suppose we start at parameter z and execute one step of SGD. So, we get an estimate $\tilde{\nabla} \phi(z) \in \mathbb{R}^n$ with pdf $\tilde{f}$.

This induces a distribution over parameters in the next step via $z - 3\tilde{\nabla}\phi(z) = w$ with pdf $f_\Theta(w) = \tilde{f}\left(\frac{z-w}{3}\right)$. So, the loss has pdf

$$f(x) = \int_{w \in \phi^{-1}(x)} \frac{\tilde{f}\left(\frac{z-w}{3}\right)}{|\nabla \phi(w)|} \, d\mathcal{H}^{k-1}(w)$$

$$\mathbb{E}\, \phi(z) = \int_{x \in \phi(\mathbb{R}^n)} x \left(\int_{w \in \phi^{-1}(x)} \frac{f_t(w)}{|\nabla \phi(w)|} \, d\mathcal{H}^{k-1}(w) \right) dx$$

Dropout!

N-param nn with loss function $\phi : \mathbb{R}^N \to \mathbb{R}$. $\in C^1$

For any current param $w \in \mathbb{R}^N$, dropout induces a distribution over $\mathbb{R}^N$. The expected grad $\mathbb{E}_{z \sim \text{dropout}(w)} [|\nabla \phi(z)|]$ can be expressed as

$$\int_{\mathbb{R}^N} f_w(z) |\nabla \phi(z)| \, dz = \int_{\mathbb{R}} dy \int_{\phi^{-1}(y)} f_w \, d\mathcal{H}^{N-1}$$

In basic dropout case, we have $z_i := \begin{cases} 0 & \text{w.p. } p \\ w_i & \text{w.p. } 1-p \end{cases}$. In this case,

$$f_w(z) = \prod_{i=1}^N p^{1 - \frac{z_i}{w_i}} (1-p)^{z_i/w_i}, \text{ and so}$$

$$\int_{\phi^{-1}(y)} f_w \, d\mathcal{H}^{N-1} =$$

$$u(w) = \mathbb{1}_E \cdot \frac{1}{|\nabla\phi(w)|} \Rightarrow m(E) = \int_{\mathbb{R}} d\ell \int_{\phi^{-1}(\ell)} \mathbb{1}_E \cdot \frac{1}{|\nabla\phi(w)|} d\mathcal{H}^{N-1}(w)$$

$$\Rightarrow m(\{\phi \le \varepsilon\}) = \int_0^\varepsilon d\ell \int_{\phi^{-1}(\ell)} \frac{1}{|\nabla\phi(w)|} d\mathcal{H}^{N-1}(w)$$

We know
$$\int_{\mathbb{R}} \mathcal{H}^{N-1}(E \cap \phi^{-1}(\ell))\, d\ell = \int_E |\nabla\phi(w)|\, dw$$
for all $\mathcal{L}^N$-meas. $E \subseteq \mathbb{R}^N$.

interpret this!

Let $E := \{w \in \mathbb{R}^N : \phi(w) < \varepsilon\}$. Then,
$$\int_0^\varepsilon \mathcal{H}^{N-1}(\phi^{-1}(\ell))\, d\ell = \int_E |\nabla\phi(w)|\, dw$$

$f := \mathbb{1}_{\phi^{-1}\ell} |w - w^*| \Rightarrow \int f\, d\mathcal{H}^{N-1} = \int_0^\infty \mathcal{H}^{N-1}(\{w : f(w) \ge t\})\, dt$

$= \int_0^\infty dt\, \mathcal{H}^{N-1}\left(\left\{w : \begin{array}{l}\phi(w) = \ell \\ \text{and } |w - w^*| \ge t\end{array}\right\}\right)$

$= \int_0^\infty dt\, \mathcal{H}^{N-1}(\phi^{-1}(\ell) \cap B_{w^*}(t)^c)$

From the blue, we also know
$$\mathcal{L}^N(E) = \int_0^\varepsilon d\ell \int_{\phi^{-1}(\ell)} \frac{1}{|\nabla\phi(w)|} d\mathcal{H}^{N-1}(w)$$

For small $\ell < \varepsilon$, we expect $\nabla\phi(w) \approx \frac{\phi(w^*+h) - \phi(w^*)}{h} = \frac{\ell}{w - w^*}$
$$\Rightarrow \mathcal{L}^N(E) \approx \int_0^\varepsilon d\ell \cdot \frac{1}{\ell} \int_{\phi^{-1}(\ell)} |w - w^*|\, d\mathcal{H}^{N-1}(w) \quad (*)$$

By the red,
$$\mathcal{L}^N(E) \approx \int_0^\varepsilon d\ell \cdot \frac{1}{\ell} \int_0^\infty dt\, \mathcal{H}^{N-1}(\phi^{-1}(\ell) \cap (B_{w^*}(t))^c)$$
$$= \int_0^\varepsilon d\ell \cdot \frac{1}{\ell} \int_0^\infty dt \int_{\phi^{-1}(\ell)} \mathbb{1}_{B_{w^*}(t)^c}\, d\mathcal{H}^{N-1}$$
$$= \int_0^\infty dt \int_0^\varepsilon d\ell \int_{\phi^{-1}(\ell)} \frac{1}{\phi(w)} \cdot \mathbb{1}_{B_{w^*}(t)^c}(w)\, d\mathcal{H}^{N-1}(w)$$

$= \int_0^\varepsilon d\ell \int_0^\infty dt\, \mathcal{H}^{N-1}\left(\frac{1}{\ell^{\frac{1}{N-1}}}[\phi^{-1}(\ell) \cap \ldots]\right)$

Since $\ell^{-\frac{1}{N-1}} B_{w^*}(t)^c \underset{\text{suppose } w^*=0}{=} B_0(t\ell^{-\frac{1}{N-1}})^c$,

$= \int_0^\varepsilon d\ell \int_0^\infty dt\, \mathcal{H}^{N-1}\left(B_0(t\ell^{-\frac{1}{N-1}})^c \cap (\ell^{-\frac{1}{N-1}}\phi^{-1}(\ell))\right)$

$= \int_0^\infty dt \int_0^\varepsilon d\ell \; \ldots$

Define $\varphi : \mathbb{R}^N \to \mathbb{R}$ via $\varphi(w) = \max\{\phi(w), \varepsilon\} \Rightarrow \varphi\ C^1$ a.e.
$$= \int_0^\infty dt \int_{\mathbb{R}} d\ell \int_{\varphi^{-1}(\ell)} \frac{1}{\varphi} \mathbb{1}_{B_{w^*}(t)^c}\, d\mathcal{H}^{N-1}$$

coarea
$$= \int_0^\infty dt \int_{B_{w^*}(t)^c} \frac{1}{\varphi} |\nabla\varphi| = \int_0^\infty dt \int_{B_{w^*}(t)^c \cap E} \frac{|\nabla\phi|}{\phi}$$

Starting from (*),

$$= \int_0^\varepsilon d\ell \int_{\phi^{-1}(\ell)} \frac{|w - w^*|}{\phi(w)} \, d\mathcal{H}^{N-1}(w)$$

$$= \int_{\mathbb{R}} d\ell \int_{\phi^{-1}(\ell)} \mathbb{1}_E \frac{|w - w^*|}{\phi(w)} \, d\mathcal{H}^{N-1}(w)$$

$$\overset{\text{coarea}}{=} \int_E |w - w^*| \frac{|\nabla \phi|}{\phi}$$

interpret this!

We wish to compute $\displaystyle\int_{\phi^{-1}(\ell)} \frac{f(w)}{|\nabla\phi(w)|} \, d\mathcal{H}^{N-1}(w)$ for fixed ℓ and f.

Write $\phi : \mathbb{R}^N \to \mathbb{R}$ as $\phi(\theta) = f_\theta(x_1) \cdot f_\theta(x_2)$

$$\Rightarrow \nabla \phi(\theta) = f_\theta(x_1) \nabla f_\theta(x_2) + \nabla f_\theta(x_1) f_\theta(x_2)$$

$$\Rightarrow \mathbb{E}\,\phi = \int_{\mathbb{R}} dz\; z \int_{w \in \phi^{-1}(z)} \frac{\mathrm{pdf}(w)}{|\nabla \phi(w)|}\, d\mathcal{H}^{N-1}(w)$$

Now, we enforce NN structure! Write

$$f_w = W_L \circ \bigcirc_{\ell=1}^{L-1} (\sigma \circ W_\ell) \quad \text{with} \quad \sigma_i = z_i + \frac{1}{nk} z_i^k$$

We have jacobian $D[\sigma](x) = \mathrm{Diag}\left(1 + \frac{1}{n} x^{k-1}\right) = \mathbb{1} + \frac{1}{n}\mathrm{Diag}(x^{k-1})$
and so

$$\nabla f_w(x) = \ldots D[\sigma](W_2 \sigma W_1 x) W_2 \, D[\sigma](W_1 x) W_1 x$$

$$\ldots W_2\left(\mathbb{1} + \frac{1}{n}\mathrm{Diag}(x^{k-1})\right) W_1 x$$

We have

$$D[\sigma](z_j) = D[\sigma]\left(W_j \bigcirc_{\ell=1}^{j} (\sigma \circ W_\ell) x\right) = \mathbb{1} + \frac{1}{n}\mathrm{Diag}\left(z_j^{k-1}\right)$$

$$\nabla f_w(x) = \left(W_L \prod_{\ell=1}^{L-1} D[\sigma](z_\ell) W_\ell\right) x$$

$$= \left(W_L \prod_{\ell=1}^{L-1} \left(\mathbb{1} + \frac{1}{n}\mathrm{Diag}(z_\ell^{k-1})\right) W_\ell\right) x$$

With $k=2$,