# PHY 521: Final Written Assignment

Due on May 9, 2023

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# Part I

## Problem 12.1

**Theorem 1** (Monotonicity in the dependence on the boundary condition)**.** *For Ising spin systems in any fixed volume $\Lambda$, the finite volume Gibbs measures conditioned on different exterior spin configurations are monotone in the boundary spins. More explicitly, for any pair of exterior configurations $\widetilde{\sigma}, \sigma'$ (here, exterior configurations need only be defined on $\Lambda^C$), the conditional distributions satisfy:*

$$\widetilde{\sigma} \preceq \sigma' \implies \rho^{\widetilde{\sigma}}_{\Lambda,\beta} \preceq \rho^{\sigma'}_{\Lambda,\beta}$$

*In particular, each such finite volume Gibbs distribution with an exterior spin configuration $\widetilde{\sigma}$ is bounded between the $-$ and $+$ finite volume states*

$$\rho^{-}_{\Lambda,\beta} \preceq \rho^{\widetilde{\sigma}}_{\Lambda,\beta} \preceq \rho^{+}_{\Lambda,\beta}$$

**Proof.** Let $\Lambda$ be a finite volume, and let $\widetilde{\sigma}, \sigma'$ be two arbitrary exterior spin configurations with

$$\widetilde{\sigma} \preceq \sigma'$$

Let $f \in \mathscr{B}_0$ be a local and monotone function. Then, letting $\mu$ denote the *a priori* measure over spin configurations, we have the regular conditional expectations

$$\rho^{\widetilde{\sigma}}_{\Lambda,\beta}[f] = \int_{\Omega_\Lambda} f(\sigma, \widetilde{\sigma}) \frac{e^{-\beta H_\Lambda(\sigma|\widetilde{\sigma})}}{Z^{\widetilde{\sigma}}_\Lambda} \mu_\Lambda(d\sigma) \qquad \text{and} \qquad \rho^{\sigma'}_{\Lambda,\beta}[f] = \int_{\Omega_\Lambda} f(\sigma, \sigma') \frac{e^{-\beta H_\Lambda(\sigma|\sigma')}}{Z^{\sigma'}_\Lambda} \mu_\Lambda(d\sigma)$$

Define the tilting function

$$G(\sigma, \widetilde{\sigma}, \sigma') := \left( \frac{e^{-\beta H_\Lambda(\sigma|\widetilde{\sigma})}}{Z^{\widetilde{\sigma}}_\Lambda} \right)^{-1} \frac{e^{-\beta H_\Lambda(\sigma|\sigma')}}{Z^{\sigma'}_\Lambda} = \frac{Z^{\widetilde{\sigma}}_\Lambda}{Z^{\sigma'}_\Lambda} \cdot e^{-\beta[H_\Lambda(\sigma|\sigma') - H_\Lambda(\sigma|\widetilde{\sigma})]}$$

**Lemma 2.** *$G(\cdot, \widetilde{\sigma}, \sigma') : \Omega_\Lambda \to \mathbb{R}$ is nonnegative and monotonically increasing.*

**Proof of Lemma 2.** Nonnegativity of $G$ is manifest. For monotonicity, suppose that $\sigma^{(1)}, \sigma^{(2)} \in \Omega_\Lambda$ are such that $\sigma^{(1)} \preceq \sigma^{(2)}$. Then, since the Hamiltonian takes the form

$$H_\Lambda(\sigma^{(j)}|\tau) = -\sum_{\substack{\{x,y\} \subset \Lambda \\ ||x-y||=1}} \mathcal{J}_{x-y} \sigma^{(j)}_x \sigma^{(j)}_y - \sum_{\substack{x \in \Lambda, y \in \Lambda^C \\ ||x-y||=1}} \mathcal{J}_{x-y} \sigma^{(j)}_x \tau_y - h \sum_{x \in \Lambda} \sigma^{(j)}_x,$$

we know that

$$H_\Lambda(\sigma^{(j)}|\sigma') - H_\Lambda(\sigma^{(j)}|\widetilde{\sigma}) = -\sum_{\substack{x \in \Lambda, y \in \Lambda^C \\ ||x-y||=1}} \mathcal{J}_{x-y} \sigma^{(j)}_x (\sigma'_y - \widetilde{\sigma}_y)$$

Now, note that since $\sigma^{(2)}_x \geq \sigma^{(1)}_x$ for all $x \in \Lambda$ and $\sigma'_y - \widetilde{\sigma}_y \geq 0$ for all $y \in \Lambda^C$ and $\mathcal{J}_{x-y} \geq 0$, we have that

$$\sum_{\substack{x \in \Lambda, y \in \Lambda^C \\ ||x-y||=1}} \mathcal{J}_{x-y} \sigma^{(2)}_x (\sigma'_y - \widetilde{\sigma}_y) \geq \sum_{\substack{x \in \Lambda, y \in \Lambda^C \\ ||x-y||=1}} \mathcal{J}_{x-y} \sigma^{(1)}_x (\sigma'_y - \widetilde{\sigma}_y)$$

$$\implies H_\Lambda(\sigma^{(1)}|\sigma') - H_\Lambda(\sigma^{(1)}|\widetilde{\sigma}) \geq H_\Lambda(\sigma^{(2)}|\sigma') - H_\Lambda(\sigma^{(2)}|\widetilde{\sigma})$$

$$\implies -\beta[H_\Lambda(\sigma^{(1)}|\sigma') - H_\Lambda(\sigma^{(1)}|\widetilde{\sigma})] \leq -\beta[H_\Lambda(\sigma^{(2)}|\sigma') - H_\Lambda(\sigma^{(2)}|\widetilde{\sigma})]$$

Problem 12.1 continued on next page. . . 

By monotonicity of $e^t$, this means that

$$e^{-\beta[H_\Lambda(\sigma^{(1)}|\sigma')-H_\Lambda(\sigma^{(1)}|\widetilde{\sigma})]} \leq e^{-\beta[H_\Lambda(\sigma^{(2)}|\sigma')-H_\Lambda(\sigma^{(2)}|\widetilde{\sigma})]}$$

Since $\frac{Z_\Lambda^{\widetilde{\sigma}}}{Z_\Lambda^{\sigma'}} \geq 0$ (partition functions are positive), we get

$$\frac{Z_\Lambda^{\widetilde{\sigma}}}{Z_\Lambda^{\sigma'}} \cdot e^{-\beta[H_\Lambda(\sigma^{(1)}|\sigma')-H_\Lambda(\sigma^{(1)}|\widetilde{\sigma})]} \leq \frac{Z_\Lambda^{\widetilde{\sigma}}}{Z_\Lambda^{\sigma'}} e^{-\beta[H_\Lambda(\sigma^{(2)}|\sigma')-H_\Lambda(\sigma^{(2)}|\widetilde{\sigma})]}$$

By definition of $G$, this is precisely the same statement as

$$G(\sigma^{(1)}, \widetilde{\sigma}, \sigma') \leq G(\sigma^{(2)}, \widetilde{\sigma}, \sigma')$$

Since this result holds for all $\sigma^{(1)} \preceq \sigma^{(2)}$, monotonicity follows. ■

Now, we know by construction that $G$ satisfies a tilting property

$$\rho_{\Lambda,\beta}^{\widetilde{\sigma}}[f(\cdot)\; G(\cdot, \widetilde{\sigma}, \sigma')] = \rho_{\Lambda,\beta}^{\sigma'}[f]$$

By Lemma 12.4, since $\rho_{\Lambda,\beta}^{\widetilde{\sigma}}$ is a Gibbs measure of the ferromagnetic Ising model with a fixed boundary condition, it obeys the FKG lattice condition and therefore enjoys a positive association property. This means that since both $f$ and $G(\cdot, \widetilde{\sigma}, \sigma')$ are monotone,

$$\rho_{\Lambda,\beta}^{\widetilde{\sigma}}[f] \cdot \rho_{\Lambda,\beta}^{\widetilde{\sigma}}[G(\cdot, \widetilde{\sigma}, \sigma')] \leq \rho_{\Lambda,\beta}^{\widetilde{\sigma}}[f(\cdot)\; G(\cdot, \widetilde{\sigma}, \sigma')]$$

Observe that by the tilting property of $G$,

$$\rho_{\Lambda,\beta}^{\widetilde{\sigma}}[G(\cdot, \widetilde{\sigma}, \sigma')] = \rho_{\Lambda,\beta}^{\sigma'}[1] = 1$$

This means that

$$\rho_{\Lambda,\beta}^{\widetilde{\sigma}}[f] \leq \rho_{\Lambda,\beta}^{\widetilde{\sigma}}[f(\cdot)\; G(\cdot, \widetilde{\sigma}, \sigma')] = \rho_{\Lambda,\beta}^{\sigma'}[f]$$

Since this holds for all local, monotone functions $f$, we find that

$$\rho_{\Lambda,\beta}^{\widetilde{\sigma}} \preceq \rho_{\Lambda,\beta}^{\sigma'},$$

as desired. The final conclusion of the lemma comes from observing that the exterior configurations $\sigma^{\pm}$ that produce the $\pm$ boundary conditions satisfy

$$\sigma^- \preceq \widetilde{\sigma} \preceq \sigma^+$$

for all possible fixed exterior configurations $\widetilde{\sigma}$. ■

# Problem 12.2

**Theorem 3** (Translation invariance of the $\pm$ infinite volume Gibbs states)**.** *Suppose we have a model on a transitive, locally finite graph, with a translation invariant a priori measure. If for all finite volumes $\Lambda$ the Hamiltonian $H_\Lambda^\pm$ is translation-invariant, then the infinite volume Gibbs states $\rho_\beta^\pm$ are also translation invariant for all $\beta$.*

*More precisely, fix a vector $a$ that respects the transitivity of the graph (such as with integer coordinates on $\mathbb{Z}^d$) and define a translation map $T_a : \Omega \to \Omega$ that sends $\sigma_x \mapsto \sigma_{x+a}$. If $H_\Lambda^\pm(\sigma) = H_\Lambda^\pm(T_a(\sigma))$ for all $\sigma \in \Omega_\Lambda$ and all finite $\Lambda$, then we have*

$$\int_\Omega f(\sigma)\rho_\beta^\pm(d\sigma) = \int_\Omega f(T_a(\sigma))\rho_\beta^\pm(d\sigma) \qquad \text{for all local, measurable functions } f \in \mathscr{B}_0 \text{ and } \forall \beta$$

**Proof.** Let $\beta$ be arbitrary. Suppose that $H_\Lambda^\pm(\sigma) = H_\Lambda^\pm(T_a(\sigma))$ for all $\sigma \in \Omega_\Lambda$ and all finite $\Lambda$. Let $f \in \mathscr{B}_0$ be an arbitrary local measurable function. We want to show that

$$\rho_\beta^\pm[f] = \rho_\beta^\pm[f \circ T_a]$$

Denote by $\rho_{\Lambda_L,\beta}^\pm$ a finite volume Gibbs state with $\pm$ boundary conditions over $\Lambda_L^C$. We start with a helpful.

**Lemma 4.** *Let $\epsilon > 0$ be arbitrary and suppose that the finite volume Hamiltonian and a priori measures are translation invariant. Let $f \in \mathscr{B}_0$ be an arbitrary local measurable function. Then, there exists an $M$ large enough that for all $L > M$ and all $\beta$,*

$$\left|\rho_{\Lambda_L,\beta}^\pm[f] - \rho_{\Lambda_L,\beta}^\pm[f \circ T_a]\right| < \epsilon$$

**Proof of Lemma 4.** Define for each volume $\Lambda$ the induced shifted volume map $T_a(\Lambda) := \{x + a : x \in \Lambda\}$. We first claim that by translation invariance of the finite volume Hamiltonian,

$$\rho_{T_{-a}(\Lambda_L),\beta}^\pm[f] = \rho_{\Lambda_L,\beta}^\pm[f \circ T_a] \qquad \forall L < \infty$$

(The above statement boils down to the fact that shifting the function right is the same as shifting the volume left). To see this, note that for a translation invariant *a priori* measure $\mu(d\sigma)$, we can express

$$\begin{aligned}
\rho_{\Lambda_L,\beta}^\pm[f \circ T_a] &= \int_{\Omega_{\Lambda_L}} f(T_a(\sigma))\frac{e^{-\beta H_{\Lambda_L,\beta}^\pm(\sigma)}}{Z_{\Lambda_L,\beta}^\pm}\mu(d\sigma) \\
&= \int_{\Omega_{\Lambda_L}} f(T_a(\sigma))\frac{e^{-\beta H_{\Lambda_L,\beta}^\pm(T_a(\sigma))}}{Z_{\Lambda_L,\beta}^\pm}\mu(d\sigma) \\
&= \int_{\Omega_{T_{-a}(\Lambda_L)}} f(\sigma)\frac{e^{-\beta H_{T_{-a}(\Lambda_L),\beta}^\pm(\sigma)}}{Z_{T_{-a}(\Lambda_L),\beta}^\pm}\mu(d\sigma) \\
&= \rho_{T_{-a}(\Lambda_L),\beta}^\pm[f],
\end{aligned}$$

where the second line comes from translation invariance of the Hamiltonian and the third line comes from translation invariance of the integral w.r.t. $\mu$ (note that the volume over which we compute the Hamiltonian and partition function shifts by $-a$ between the second and third lines, since translating the integral shifts the boundary conditions).

From here, we can note that for all $L > ||a||$,

$$\Lambda_{L-||a||} \subset \Lambda_L \subset \Lambda_{L+||a||} \qquad \text{and} \qquad \Lambda_{L-||a||} \subset T_{-a}(\Lambda_L) \subset \Lambda_{L+||a||}$$

by definition of $T_{-a}(\Lambda_L) = \{x - a : x \in \Lambda_L\}$. From Problem 12.1, we know that the measures $\rho^{+}_{\Lambda_\ell,\beta}$ and $\rho^{-}_{\Lambda_\ell,\beta}$ are monotonically decreasing and increasing, respectively, as we increase $\ell$ (the $\pm$ boundary conditions get relaxed as we increase $\ell$). So, monotonicity grants

$$\min\left\{\rho^{\pm}_{\Lambda_{L-||a||},\beta}[f],\ \rho^{\pm}_{\Lambda_{L+||a||},\beta}[f]\right\} \leq \rho^{\pm}_{\Lambda_L,\beta}[f],\ \rho^{\pm}_{T_{-a}(\Lambda_L),\beta}[f] \leq \max\left\{\rho^{\pm}_{\Lambda_{L-||a||},\beta}[f],\ \rho^{\pm}_{\Lambda_{L+||a||},\beta}[f]\right\}$$

This immediately tells us that

$$\left|\rho^{\pm}_{\Lambda_L,\beta}[f] - \rho^{\pm}_{T_{-a}(\Lambda_L),\beta}[f]\right| \leq \left|\rho^{\pm}_{\Lambda_{L-||a||},\beta}[f] - \rho^{\pm}_{\Lambda_{L+||a||},\beta}[f]\right|$$

Lastly, since the infinite Gibbs states exist for the $\pm$ boundary conditions, the sequence $(\rho^{\pm}_{\Lambda_\ell,\beta}[f])_\ell$ is Cauchy, and so there exists an $M'$ such that for all $L > M' + ||a||$,

$$\left|\rho^{\pm}_{\Lambda_{L-||a||},\beta}[f] - \rho^{\pm}_{\Lambda_{L+||a||},\beta}[f]\right| < \epsilon$$

Therefore, with $M := M' + ||a||$, we get that for all $L > M$,

$$\left|\rho^{\pm}_{\Lambda_L,\beta}[f] - \rho^{\pm}_{\Lambda_L,\beta}[f \circ T_a]\right| = \left|\rho^{\pm}_{\Lambda_L,\beta}[f] - \rho^{\pm}_{T_{-a}(\Lambda_L),\beta}[f]\right| < \epsilon$$

as desired. ■

Now, by definition of the infinite Gibbs states, we know that as $L \to \infty$,

$$\rho^{\pm}_{\Lambda_L,\beta}[f] \to \rho^{\pm}_{\beta}[f] \qquad \text{and} \qquad \rho^{\pm}_{\Lambda_L,\beta}[f \circ T_a] \to \rho^{\pm}_{\beta}[f \circ T_a]$$

Let $\epsilon > 0$ be arbitrary. Then, there exists an $N$ large enough that for all $L > N$,

$$\left|\rho^{\pm}_{\Lambda_L,\beta}[f] - \rho^{\pm}_{\beta}[f]\right| < \epsilon \qquad \text{and} \qquad \left|\rho^{\pm}_{\Lambda_L,\beta}[f \circ T_a] - \rho^{\pm}_{\beta}[f \circ T_a]\right| < \epsilon$$

Furthermore, by Lemma 4 there exists an $M$ large enough that for all $L > M$,

$$\left|\rho^{\pm}_{\Lambda_L,\beta}[f] - \rho^{\pm}_{\Lambda_L,\beta}[f \circ T_a]\right| < \epsilon$$

From these bounds, a simple application of the triangle inequality grants that for all $L > \max\{N, M\}$,

$$\begin{aligned}
\left|\rho^{\pm}_{\beta}[f] - \rho^{\pm}_{\beta}[f \circ T_a]\right| &= \left|\rho^{\pm}_{\beta}[f] + \rho^{\pm}_{\Lambda_L,\beta}[f] - \rho^{\pm}_{\Lambda_L,\beta}[f] + \rho^{\pm}_{\Lambda_L,\beta}[f \circ T_a] - \rho^{\pm}_{\Lambda_L,\beta}[f \circ T_a] - \rho^{\pm}_{\beta}[f \circ T_a]\right| \\
&\leq \left|\rho^{\pm}_{\beta}[f] - \rho^{\pm}_{\Lambda_L,\beta}[f]\right| + \left|\rho^{\pm}_{\Lambda_L,\beta}[f] - \rho^{\pm}_{\Lambda_L,\beta}[f \circ T_a]\right| + \left|\rho^{\pm}_{\Lambda_L,\beta}[f \circ T_a] - \rho^{\pm}_{\beta}[f \circ T_a]\right| \\
&< \epsilon + \epsilon + \epsilon = 3\epsilon
\end{aligned}$$

Since this holds for arbitrary $\epsilon$, we find that

$$\left|\rho^{\pm}_{\beta}[f] - \rho^{\pm}_{\beta}[f \circ T_a]\right| = 0 \implies \rho^{\pm}_{\beta}[f] = \rho^{\pm}_{\beta}[f \circ T_a]$$

Since this holds for all local and measurable $f$, we find that $\rho^{\pm}_{\beta}$ is translation invariant, and the desired result is proven. ■

# Problem 12.3

**Theorem 5** (Criterion for uniqueness of the infinite Gibbs state)**.** *Consider an Ising spin system on any countable infinite graph $\mathcal{V}$. For any $\beta > 0$, we have the following criterion for uniqueness of the infinite Gibbs state:*

$$\text{the Gibbs state is unique} \iff \rho_\beta^-[\sigma_x] = \rho_\beta^+[\sigma_x] \quad \forall x \in \mathcal{V},$$

*where in the above expression, $\sigma_x$ is interpreted as an observable from $\Omega \to \{-1, 1\}$ mapping $\sigma \mapsto \sigma_x$.*

**Proof.** ( $\implies$ ) Suppose first that the infinite Gibbs state is unique. Then, we know that $\rho_\beta^- = \rho_\beta^+$, which means that for all local, measurable $f \in \mathscr{B}_0$,

$$\rho_\beta^-[f] = \rho_\beta^+[f]$$

This trivially implies that

$$\rho_\beta^-[\sigma_x] = \rho_\beta^+[\sigma_x] \quad \forall x \in \mathcal{V}$$

( $\impliedby$ ) Suppose now that $\rho_\beta^-[\sigma_x] = \rho_\beta^+[\sigma_x] \quad \forall x \in \mathcal{V}$. We know from the result of Problem 12.1 that $\rho_{\Lambda,\beta}^- \preceq \rho_{\Lambda,\beta}^+$ for all finite volumes $\Lambda$, which means that in the limit,

$$\rho_\beta^- \preceq \rho_\beta^+$$

Applying Holley's Theorem, this tells us that there exists a monotone coupling $\mu(d\sigma, d\sigma')$ on the product space $\Omega \times \Omega$ that satisfies the following properties:

1. $\mu(d\sigma, d\sigma')$ is supported on the collection of pairs $(\sigma, \sigma')$ for which $\sigma \preceq \sigma'$
2. For all measurable observables $f : \Omega \to \mathbb{R}$,
$$\int_{\Omega\times\Omega} f(\sigma)\mu(d\sigma, d\sigma') = \int_\Omega f(\sigma)\rho_\beta^-(d\sigma)$$
3. For all measurable observables $f : \Omega \to \mathbb{R}$,
$$\int_{\Omega\times\Omega} f(\sigma')\mu(d\sigma, d\sigma') = \int_\Omega f(\sigma)\rho_\beta^+(d\sigma)$$

Since the observable $\sigma \mapsto \sigma_x$ is certainly measurable, we find that by properties 2 and 3, for all $x \in \mathcal{V}$ we have

$$\rho_\beta^-[\sigma_x] = \int_{\Omega\times\Omega} \sigma_x \mu(d\sigma, d\sigma') \qquad \text{and} \qquad \rho_\beta^+[\sigma_x] = \int_{\Omega\times\Omega} \sigma'_x \mu(d\sigma, d\sigma')$$
$$\implies 0 = \rho_\beta^+[\sigma_x] - \rho_\beta^-[\sigma_x] = \int_{\Omega\times\Omega} (\sigma'_x - \sigma_x)\mu(d\sigma, d\sigma')$$

Let

$$E_x := \{(\sigma, \sigma') \subset \Omega \times \Omega : \sigma_x < \sigma'_x\} \qquad \text{and} \qquad F_x := \{(\sigma, \sigma') \subset \Omega \times \Omega : \sigma_x > \sigma'_x\}$$

Over $E_x$, we know that $\sigma'_x - \sigma_x = 2$ since each $\sigma_x \in \{-1, 1\}$; similarly, over $F_x$ we know that $\sigma'_x - \sigma_x = -2$. Lastly, over $(E_x \cup F_x)^C$, we know that $\sigma'_x - \sigma_x = 0$ trivially. So,

$$\begin{aligned}
0 &= \int_{\Omega\times\Omega} (\sigma'_x - \sigma_x)\mu(d\sigma, d\sigma') \\
&= \int_{E_x} (\sigma'_x - \sigma_x)\mu(d\sigma, d\sigma') + \int_{F_x} (\sigma'_x - \sigma_x)\mu(d\sigma, d\sigma') + \int_{(E_x\cup F_x)^C} (\sigma'_x - \sigma_x)\mu(d\sigma, d\sigma') \\
&= \int_{E_x} 2\mu(d\sigma, d\sigma') + \int_{F_x} (-2)\mu(d\sigma, d\sigma') + \int_{(E_x\cup F_x)^C} 0\mu(d\sigma, d\sigma') \\
&= 2\mu(E_x) - 2\mu(F_x)
\end{aligned}$$

Problem 12.3 continued on next page. . .

However, note that if $\sigma_x > \sigma'_x$, then it cannot be that $\sigma \preceq \sigma'$. By property 1 of $\mu$, this means that $\mu(F_x) = 0$. Thus, $\mu(E_x) = 0$ for all $x \in \mathcal{V}$. Now, let

$$E := \{(\sigma, \sigma') \subset \Omega \times \Omega : \sigma \neq \sigma'\}$$

It is clear that

$$E \subset \left( \bigcup_{x \in \mathcal{V}} E_x \right) \cup \left( \bigcup_{x \in \mathcal{V}} F_x \right),$$

as in order for two states $\sigma \neq \sigma'$ to be distinct, they must differ at some $\sigma_x \neq \sigma'_x$, which means that either $\sigma_x > \sigma'_x$ or $\sigma_x < \sigma'_x$. However, applying the countable subadditivity of the measure $\mu$, we find that

$$\mu(E) \leq \sum_{x \in \mathcal{V}} \mu(E_x) + \mu(F_x) = 0 \implies \mu(E) = 0$$

Now, let $\nu_\beta^{(1)}, \nu_\beta^{(2)}$ be two arbitrary infinite volume Gibbs states. For any local, monotone function $f$, we know by the taking the limit of the result of Problem 12.1 that

$$\rho_\beta^-[f] \leq \nu_\beta^{(1)}[f] \leq \rho_\beta^+[f] \qquad \text{and} \qquad \rho_\beta^-[f] \leq \nu_\beta^{(2)}[f] \leq \rho_\beta^+[f]$$

However, using the coupling $\mu$, we find that

$$\begin{aligned} \rho_\beta^+[f] - \rho_\beta^-[f] &= \int_{\Omega \times \Omega} f(\sigma) - f(\sigma')\mu(d\sigma, d\sigma') \\ &= \int_E f(\sigma) - f(\sigma')\mu(d\sigma, d\sigma') + \int_{E^C} f(\sigma) - f(\sigma')\mu(d\sigma, d\sigma') \end{aligned}$$

Clearly, $f(\sigma) - f(\sigma') = 0$ over $E^C$ since $\sigma = \sigma'$ on $E^C$. Also, our earlier discussion revealed that $\mu(E) = 0$. This means that

$$\rho_\beta^+[f] - \rho_\beta^-[f] = 0 + 0 = 0$$

Therefore, the result that $\rho_\beta^-[f] \leq \left\{ \nu_\beta^{(1)}[f],\ \nu_\beta^{(2)}[f] \right\} \leq \rho_\beta^+[f]$ implies that for all local, monotone $f$,

$$\nu_\beta^{(1)}[f] = \nu_\beta^{(2)}[f]$$

This means that

$$\nu_\beta^{(1)} \succeq \nu_\beta^{(2)} \quad \text{and} \quad \nu_\beta^{(1)} \preceq \nu_\beta^{(2)} \implies \nu_\beta^{(1)} = \nu_\beta^{(2)}$$

Since this holds for all pairs of arbitrary infinite volume Gibbs states, we find that the infinite volume Gibbs state is unique. ■

# Part II

# Problem 1

Express for what other boundary conditions on the Ising model (expressed in terms of Ising spins) can one now conclude convergence of the Ising model's state in the infinite volume limit ($\Lambda_L$ with $L \to \infty$).

## Solution

**Theorem 6.** *The Ising model's infinite Gibbs state converges for both the free (no edges outside of $\Lambda$) and wired (all edges outside of $\Lambda$) boundary conditions to $\rho_\beta^f$ and $\rho_\beta^w$, respectively. Furthermore, for all infinite Gibbs states $\nu_\beta$, we find that*

$$\rho_\beta^f \preceq \nu_\beta \preceq \rho_\beta^w$$

**Proof.** We work with the natural partial order on FK cluster models, given by

$$\widetilde{n} \succeq n \iff \widetilde{n}_b \geq n_b \quad \forall b \in \mathscr{E}$$

Let $\Lambda$ be a finite volume. The result from the first part of Problem 12.1 still applies in this case, and so we find that for any pair of exterior configurations $\widetilde{n}, n'$, we know that

$$\widetilde{n} \preceq n' \implies \rho_{\Lambda,\beta}^{\widetilde{n}} \preceq \rho_{\Lambda,\beta}^{n'}$$

Now, the exterior configuration (defined on $\Lambda^C$) for the free boundary condition is that $n_b^f = 0$ for all edges $b$ that are not fully contained in $\Lambda$. Certainly, for any exterior configuration $n$, we have that $n^f$ and $n$ agree within $\Lambda$ and that $n^f$ is minimal outside of $\Lambda$, and so

$$n^f \preceq n \implies \rho_{\Lambda,\beta}^f \preceq \rho_{\Lambda,\beta}^n$$

Similarly, the wired boundary conditions are defined via $n_b^w = 1$ for all edges $b$ that are not fully contained in $\Lambda$. Again, for any exterior configuration $n$, we have that $n^w$ and $n$ agree within $\Lambda$ and that $n^w$ is maximal outside of $\Lambda$, and so

$$n \preceq n^w \implies \rho_{\Lambda,\beta}^n \preceq \rho_{\Lambda,\beta}^w$$

From this, we can conclude that for all finite $\Lambda$ and all exterior configurations $n$ that are defined on $\Lambda^C$, we have

$$\rho_{\Lambda,\beta}^f \preceq \rho_{\Lambda,\beta}^n \preceq \rho_{\Lambda,\beta}^w$$

An important thing to note is that this immediately implies that $\rho_{\Lambda,\beta}^f$ (resp. $\rho_{\Lambda,\beta}^w$) is monotonically increasing (resp. decreasing) in $\Lambda$, since a larger $\Lambda$ relaxes the constraints on $\Lambda^C$ and can only serve to make the resulting exterior configuration less minimal (resp. maximal). So, the sequence $(\rho_{\Lambda_L,\beta}^f)_L$ is monotone increasing and bounded above by $\rho_{\Lambda,\beta}^w$ for any $\Lambda$, which means that it converges as $L \to \infty$. Inversely, the sequence $(\rho_{\Lambda_L,\beta}^w)_L$ is monotone decreasing and bounded below by $\rho_{\Lambda,\beta}^f$ for any $\Lambda$, which means it converges as $L \to \infty$. From this, we conclude that the infinite Gibbs state **always converges for the free and wired boundary conditions**. Furthermore, for any infinite Gibbs state induced by a boundary configuration sequence $(n_L)_L$, the bounds

$$\rho_{\Lambda_L,\beta}^f \preceq \rho_{\Lambda_L,\beta}^{n_L} \preceq \rho_{\Lambda_L,\beta}^w$$

will also hold in the limit, and so the infinite state is bounded between the infinite free and wired states. An important corollary of this is that if the infinite free and wired Gibbs states agree, then there is a unique infinite Gibbs state and all boundary conditions will converge to this state. ■

# Problem 2

For what sequence of boundary conditions # can one conclude convergence of the pressure, defined as

$$\Psi(\beta)^{\#} = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \log Z^{\#}_{\Lambda_L,\beta},$$

if/when the limit exists?

## Solution

**Proof.** We follow the reasoning in Chapter 5 of the course notes to determine when this limit exists. Let us first denote the term that a boundary interaction along the boundary of a finite volume $\Lambda$ with conditions # contributes to the Hamiltonian by

$$\phi^{\#}_{\Lambda}(\sigma)$$

If it is the case that the sequence of volumes $(\Lambda^{(n)})_n$ is a van Hove sequence (which means the ratio of outer and inner packing numbers converges to 1) such as cubes of increasing side length, then for boundary conditions # satisfying the condition that

$$\sup_{\sigma} \left|\phi^{\#}_{\Lambda^{(n)}}(\sigma)\right| = o\left(\left|\Lambda^{(n)}\right|\right) \qquad \forall n, \tag{$\star$}$$

Corollary 5.5 guarantees that the pressure function

$$\Psi(\beta)^{\#} = \lim_{n\to\infty} \frac{1}{\left|\Lambda^{(n)}\right|} \log Z^{\#}_{\Lambda^{(n)},\beta}$$

converges.

Now, in general it holds for the Ising Hamiltonian that

$$\sup_{\sigma} \left|\phi^{\#}_{\Lambda^{(n)}}(\sigma)\right| = O\left(\left|\partial\Lambda^{(n)}\right|\right) \qquad \forall n,$$

since the number of terms in that contribute to the boundary term equals the size of the boundary, and each term contributes at most a value of the coupling $\mathcal{J}$. So, on graphs where we have that van Hove sequences satisfy

$$\frac{|\partial\Lambda^{(n)}|}{|\Lambda^{(n)}|} \to 0,$$

such as on lattices $\mathbb{Z}^d$ (in $\mathbb{Z}^d$, the numerator scales to the $(d-1)^{th}$ power while the denominator scales to the $d^{th}$ power), the condition $(\star)$ is satisfied for any choice of boundary conditions #. Therefore, for such lattices, any boundary condition will yield a convergent pressure function when the limit is taken along a van Hove sequence. ■

# Problem 3

(a) What partial order can be introduced on the configuration space $\Omega$ which does not respect the $A - B$ connectivity symmetry and instead gives preference to one over the other, and for which you expect (or may prove) the FKG lattice condition?

(b) Assuming (without necessarily spelling the proof) that the order you propose does satisfy the FKG condition – for what collection of the model's selected states what would that allow to conclude convergence in the double limit $L_n, \beta \to \infty$ for at least some pre-specified $(L_n)_n$?

## Solution

**Proof of (a).** Consider the $A-B$ continuum model of the antiferromagnetic quantum spin chain, with the space split into strips going along the $\beta$ dimension. We can define a partial order on the configuration space $\Omega$ as follows. First, for each position $n \in \mathbb{Z}$, define the maps $A_n, B_n : \Omega \to 2^{[0,\beta]}$ that map a configuration $\sigma$ to the set of rungs that it contains in the $n^{th}$ $A$ and $B$ strips, respectively. Define a partial ordering as follows:

$$\sigma \preceq \sigma' \iff A_n(\sigma) \subseteq A_n(\sigma') \quad \text{and} \quad B_n(\sigma) \supseteq B_n(\sigma') \quad \forall n \in \mathbb{Z}$$

This partial ordering certainly does not respect the $A - B$ connectivity symmetry, since configurations that connect $A$'s and disconnect $B$'s dominate; this setup is inherently not symmetric. We expect this partial ordering to yield an FKG inequality because $\sigma \preceq \sigma'$ only happens when $\sigma'$ makes all $A$ strips more connected *and* makes all $B$ strips less connected. The maximal state is the state with all possible rungs across every $B$ strip so that every pair of neighboring $A$ strips is connected at all heights $t \in [0, \beta]$, and *no* rungs connecting any $B$ strips. The minimal state is the exact opposite. Note that these two configurations correspond to the maximal possible number of loops, which yields the largest Gibbs factor because of the equation

$$\langle \Phi | e^{\beta H_a} | \Phi \rangle = e^{\beta |\mathcal{J}|} \int_{\Omega_{\mathcal{J},\beta}} (2S+1)^{N_\ell(\sigma)} \rho_{\mathcal{J},\beta}(d\sigma)$$

This tells us that configurations with more loops are preferred by the Hamiltonian, and we know that the maximal and minimal states generate the maximum possible number of loops. Furthermore, we expect that the minimum operation induced by this partial ordering forces states more toward the minimal state, and the maximum operation pushes states toward the maximal state. Since the loop-counting argument reveals that more minimal/maximal states are more preferred by the Hamiltonian, it is intuitive that we would have the FKG lattice condition

$$\rho(\sigma \wedge \sigma') \cdot \rho(\sigma \vee \sigma') \geq \rho(\sigma) \cdot \rho(\sigma')$$

when the measure $\rho$ is a Gibbs state. ■

**Proof of (b).** Assume that the partial order from part (a) satisfies the FKG condition, and pre-specify the sequence $L_n$ to be either the even natural numbers or the odd natural numbers (this will enforce dimerization and allow for convergence toward the minimal and maximal states, respectively). If we have preset evens, fix $A$ to be the strip type that we will add rungs to; if odd, fix $B$. Then, if we set the sequence of states $(\sigma)_\beta$ to be that which adds a new rung at $\beta$ on the strips of the type we just fixed, we see that the infinite states will converge toward the maximal state if $A$ was fixed and the minimal state if $B$ was fixed. ■