PHY 521: Problem Set 3

Due on April 3, 2023

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Problem 8.1

Explain how the chessboard inequality can be used to derive a Peierls-like bound on the probability that a given simple loop is realized as a contour in the Gibbs equilibrium state of the two dimensional Ising model with periodic boundary conditions.

Solution

Proof. Consider a two dimensional Ising model with periodic boundary conditions; we know this system to exhibit reflection positivity across any axis-aligned hyperplanes. For any Peierls contour γ, we have many bonds between neighboring $(-,+)$, and we wish to bound the probability of such a contour forming. So, fix a simple loop γ for the rest of the proof. Suppose, without loss of generality, that γ has more vertical bonds $(-,+)$ across it than horizontal ones (put differently, suppose that walking along γ crosses more vertical edges of the lattice than horizontal ones). We will use the chessboard inequality to derive a bound of the form

$$\mathbb{P}\{\gamma \text{ is realized as a Peierls contour}\} \leq e^{-\beta|\gamma|}$$

To start, fix a finite volume Λ and consider a family of reflections across the following family of axis-aligned hyperplanes:

- a horizontal hyperplane along the lattice points for each possible integer y-coordinate
- a vertical hyperplane between the lattice points for each possible half-integer x-coordinate

This system will divide Λ into almost-disjoint boxes $(B_\alpha)_{\alpha\in I}$, each containing exactly one vertical edge of the lattice; conversely, each vertical edge of the lattice is contained in precisely one box B_α. In other words, there is a bijection between vertical edges of the lattice and the finite index set I, and so we can label vertical edges with our index set I. Let I_γ be the set of indices corresponding to the vertical edges that γ crosses. As such, we can define the random variable X_α for $\alpha \in I_\gamma$ to be a 1 if the vertical edge α has a $+$ outside of γ and a $-$ inside, and 0 otherwise (i.e. $X_\alpha = 1$ iff the vertical edge α helps γ be a Peierls contour with $+$ outside and $-$ inside). Then, for each $\alpha \in I$ define the observable $F_\alpha : \Omega_\Lambda \to \mathbb{R}$ by

$$F_\alpha(\sigma) = \begin{cases} X_\alpha & \alpha \in I_\gamma \\ 1 & \text{otherwise} \end{cases}$$

Note that each X_α, and therefore each observable F_α, depends only on the configuration of sites contained within each box B_α; in other words, F_α is $\sigma(B_\alpha)$-measurable for all α.

With this construction, we have boxes $(B_\alpha)_{\alpha\in I}$ (generated by reflections that the system is reflection positive across) that tile Λ and functions F_α that depend only on the sites contained in their respective box B_α; this sets us up perfectly for an application of the chessboard inequality. The theorem gives

$$\mathbb{E}\left[\prod_{\alpha\in I} F_\alpha(\sigma_{B_\alpha})\right] \leq \prod_{\alpha\in I} \mathbb{E}\left[\prod_{\alpha'\in I} F^{\#}_\alpha(\sigma_{B_{\alpha'}})\right]^{1/|I|}$$

For any $\alpha \notin I_\gamma$, note that $\mathbb{E}\left[\prod_{\alpha'\in I} F^{\#}_\alpha(\sigma_{B_{\alpha'}})\right] = 1$ trivially, since $F_\alpha \equiv 1$ for such α. Now, for $\alpha \in I_\gamma$, we know that the prospective contour γ crosses edge α, and so F_α takes the value X_α. This means that the product $\prod_{\alpha'\in I} F^{\#}_\alpha(\sigma_{B_{\alpha'}})$ takes values in $\{0,1\}$ with the value of 1 if and only if $F^{\#}_\alpha(\sigma_{B_{\alpha'}}) = 1$ for all α', which in turn happens if and only if the entire configuration σ has the pattern that across each row the spins are constant, and from row to row the constant values of the spins flip (this pattern can be seen by reflecting and conjugating F_α across each of the hyperplanes of the construction). Let σ^* denote the configuration

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exhibiting this pattern. So, for $\alpha \in I_\gamma$ we get that $\mathbb{E}\left[\prod_{\alpha' \in I} F_\alpha^\#(\sigma_{B_{\alpha'}})\right]$ is none other than the probability of this patterned configuration σ^* forming, which is

$$\mathbb{E}\left[\prod_{\alpha' \in I} F_\alpha^\#(\sigma_{B_{\alpha'}})\right] = \frac{e^{-\beta H_\Lambda(\sigma^*)}}{Z_\Lambda}$$

Note that for periodic boundary conditions and a volume with an even number of rows/columns, the energy $H_\Lambda(\sigma^*) = 0$; this is because each site has 2 neighbors with the same spin and 2 neighbors with the opposite spin, nulling its contribution to the energy. We can weakly lower bound Z_Λ as follows: note that

$$Z_\Lambda = \sum_{\sigma \in \Omega_\Lambda} e^{-\beta H_\Lambda(\sigma)} \geq e^{-\beta H_\Lambda(\sigma^+)},$$

where σ^+ is the particular configuration with every site having spin +. We compute that $H_\Lambda(\sigma^+)$ is simply equal to the (negative of) the number of adjacent pairs of sites in Λ, which is $2|\Lambda|$; this is because each edge on the lattice contributes -1 to the energy and each site in Λ yields 4 edges, though we must divide by 2 to avoid double counting. This reveals that $Z_\Lambda \geq e^{2\beta|\Lambda|}$, and so

$$\mathbb{E}\left[\prod_{\alpha' \in I} F_\alpha^\#(\sigma_{B_{\alpha'}})\right] = \frac{1}{Z_\Lambda} \leq e^{-2\beta|\Lambda|}$$

(A small caveat is that for the above reasoning, the actual pattern σ^* depends on whether B_α has a + or a − on top; either case yields the same estimate, however, so we move on). So, we get that

$$\mathbb{E}\left[\prod_{\alpha' \in I} F_\alpha^\#(\sigma_{B_{\alpha'}})\right] \leq \begin{cases} e^{-2\beta|\Lambda|} & \alpha \in I_\gamma \\ 1 & \text{otherwise} \end{cases}$$

This means that the RHS of the chessboard estimate is bounded above with

$$\prod_{\alpha \in I} \mathbb{E}\left[\prod_{\alpha' \in I} F_\alpha^\#(\sigma_{B_{\alpha'}})\right]^{1/|I|} = \prod_{\alpha \in I_\gamma} \mathbb{E}\left[\prod_{\alpha' \in I} F_\alpha^\#(\sigma_{B_{\alpha'}})\right]^{1/|I|} \leq \prod_{\alpha \in I_\gamma} \mathbb{E}\left[\prod_{\alpha' \in I} F_\alpha^\#(\sigma_{B_{\alpha'}})\right]^{1/|\Lambda|} \leq \prod_{\alpha \in I_\gamma} e^{-2\beta} = e^{-2\beta|I_\gamma|},$$

where the first inequality is because $\mathbb{E}\left[\prod_{\alpha' \in I} F_\alpha^\#(\sigma_{B_{\alpha'}})\right] \leq 1$ and $|\Lambda| \geq |I|$. Note, however, that if the contour γ is realized then it must be the case that $F_\alpha(\sigma_{B_\alpha}) = 1$ for all $\alpha \in I$ by definition of F_α. This means that

$$\mathbb{P}\{\gamma \text{ is realized as a Peierls contour}\} \leq \mathbb{E}\left[\prod_{\alpha \in I} F_\alpha(\sigma_{B_\alpha})\right]$$

Applying the chessboard estimate and our bound on the RHS from earlier,

$$\mathbb{P}\{\gamma \text{ is realized as a Peierls contour}\} \leq \prod_{\alpha \in I} \mathbb{E}\left[\prod_{\alpha' \in I} F_\alpha^\#(\sigma_{B_{\alpha'}})\right]^{1/|I|} \leq e^{-2\beta|I_\gamma|}$$

To conclude, note that since γ crosses more vertical edges than horizontal ones, $|I_\gamma| \geq |\gamma|/2$. This yields the Peierls-like estimate

$$\mathbb{P}\{\gamma \text{ is realized as a Peierls contour}\} \leq e^{-\beta|\gamma|},$$

as desired. ■

Problem 9.1

(a) For the Ising model with the nearest neighbor interactions and periodic boundary conditions on $[0, L]$ express in terms of the transfer matrix the finite volume expectation value of a product of n spins at specified sites $(x_j)_{j\in\{1,\ldots,n\}}$.

(b) Write down the corresponding expression for the + boundary conditions.

Solution

Proof of (a). We can compute in the transfer matrix formalism, using Dirac notation, completeness relations, and a trick involving the matrix S, that

$$\left\langle \prod_{j=1}^{n} \sigma_{x_j} \right\rangle_{[0,L]}^{(per)} = \frac{tr(T^{x_1} S T^{x_2-x_1} S \ldots T^{x_n-x_{n-1}} S T^{L-x_n})}{tr(T^L)},$$

where $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the z-Pauli matrix and and $T = \begin{bmatrix} e^{\beta(\mathcal{J}+h)} & e^{-\beta\mathcal{J}} \\ e^{-\beta\mathcal{J}} & e^{\beta(\mathcal{J}-h)} \end{bmatrix}$ is the transfer matrix. (Note: I do all these steps in part (b), so I didn't feel the need to repeat them here). By the cyclic property of the trace, this equals

$$= \frac{tr(S T^{x_2-x_1} S \ldots T^{x_n-x_{n-1}} S T^{L-(x_n-x_1)})}{tr(T^L)}$$

■

Proof of (b). For the + boundary conditions, we must do something a little different. Firstly, let $|+\rangle$ denote the state of positive spin. We can compute

$$\begin{aligned} Z_{\beta,L}^{(+)} &= \sum_{\sigma_1,\ldots,\sigma_L\in\{\pm\}} e^{-\beta H_L^{(+)}(\sigma)} \\ &= \sum_{\sigma_1,\ldots,\sigma_L\in\{\pm\}} \langle +|T|\sigma_1\rangle\langle\sigma_1|T|\sigma_2\rangle\ldots\langle\sigma_{L-1}|T|\sigma_L\rangle\langle\sigma_L|T|+\rangle \\ &= \langle +|T^L|+\rangle, \end{aligned}$$

where the second line comes from the definition of our Hamiltonian and the transfer matrix, and the third line comes from completness relations. With this, we find

$$\begin{aligned} \left\langle \prod_{j=1}^{n} \sigma_{x_j} \right\rangle_{[0,L]}^{(+)} &= \frac{1}{Z_{\beta,L}^{(+)}} \sum_{\sigma_1,\ldots,\sigma_L\in\{\pm\}} \left(\prod_{j=1}^{n} \sigma_{x_j} \right) e^{-\beta H_L^{(+)}(\sigma)} \\ &= \frac{1}{Z_{\beta,L}^{(+)}} \sum_{\sigma_1,\ldots,\sigma_L\in\{\pm\}} \left(\prod_{j=1}^{n} \sigma_{x_j} \right) \langle +|T|\sigma_1\rangle\langle\sigma_1|T|\sigma_2\rangle\ldots\langle\sigma_{L-1}|T|\sigma_L\rangle\langle\sigma_L|T|+\rangle) \\ &= \frac{1}{Z_{\beta,L}^{(+)}} \sum_{\sigma_{x_1},\ldots,\sigma_{x_n}\in\{\pm\}} \left(\prod_{j=1}^{n} \sigma_{x_j} \right) \langle +|T^{x_1}|\sigma_{x_1}\rangle\langle\sigma_{x_1}|T^{x_2-x_1}|\sigma_{x_2}\rangle\ldots\langle\sigma_{x_{n-1}}|T^{x_n-x_{n-1}}|\sigma_{x_n}\rangle\langle\sigma_{x_n}|T^{L-x_n}|+\rangle \\ &= \frac{1}{Z_{\beta,L}^{(+)}} \sum_{\sigma_{x_1},\ldots,\sigma_{x_n}\in\{\pm\}} \langle +|T^{x_1}S|\sigma_{x_1}\rangle\langle\sigma_{x_1}|T^{x_2-x_1}S|\sigma_{x_2}\rangle\ldots\langle\sigma_{x_{n-1}}|T^{x_n-x_{n-1}}S|\sigma_{x_n}\rangle\langle\sigma_{x_n}|T^{L-x_n}|+\rangle \\ &= \frac{1}{Z_{\beta,L}^{(+)}} \langle +|T^{x_1}ST^{x_2-x_1}S\ldots T^{x_n-x_{n-1}}ST^{L-x_n}|+\rangle = \frac{\langle +|T^{x_1}ST^{x_2-x_1}S\ldots T^{x_n-x_{n-1}}ST^{L-x_n}|+\rangle}{\langle +|T^L|+\rangle}, \end{aligned}$$

where the third line comes from using the completeness relation for $\sigma_i \notin \{\sigma_{x_j}\}_{j=1}^n$, the fourth line uses a trick via S, and the fifth line applies the rest of the completeness relations. ■

Problem 9.2

Using what is known about the spectrum of the one dimensional transfer matrix T, prove that in the one dimensional Ising model the spin-spin correlations decay exponentially in the distance; that is, they satisfy

$$\left| \langle \sigma_x \sigma_y \rangle_{[0,L]}^{(per)} - \langle \sigma_x \rangle_{[0,L]}^{(per)} \langle \sigma_y \rangle_{[0,L]}^{(per)} \right| \leq C e^{-\alpha |y-x|}$$

at some $\alpha > 0$. Identify α in terms of the spectrum of T.

Solution

Proof. Note that for the one dimensional Ising model with periodic boundary conditions, the transfer matrix takes the form

$$T = \begin{bmatrix} e^{\beta(\mathcal{J}+h)} & e^{-\beta\mathcal{J}} \\ e^{-\beta\mathcal{J}} & e^{\beta(\mathcal{J}-h)} \end{bmatrix}$$

Diagonalizing this matrix (it is Hermitian, and thus diagonalizable), let its (real) eigenvalues be λ_1, λ_2, corresponding to eigenvectors $|\psi_1\rangle, |\psi_2\rangle$ (since T is Hermitian, the left eigenvectors are $\langle\psi_1|, \langle\psi_2|$). Clearly, the entries of this matrix are all positive, and so we can apply the Perron-Frobenius Theorem. This gives us that the eigenvalues satisfy $|\lambda_2| < C\lambda_1 e^{-\alpha}$ for some $\alpha > 0$ and some C; we can compute

$$\alpha = \ln\left(\frac{\lambda_1}{\lambda_2}\right)$$

Now, let us note by the the result of Problem 9.1(a) with $x_1 = x$ that

$$\langle \sigma_x \rangle_{[0,L]}^{(per)} = \frac{tr(ST^L)}{tr(T^L)}$$

Writing this out in the eigenbasis of T, we get

$$\langle \sigma_x \rangle_{[0,L]}^{(per)} = \frac{\langle\psi_1|ST^L|\psi_1\rangle + \langle\psi_2|ST^L|\psi_2\rangle}{\langle\psi_1|T^L|\psi_1\rangle + \langle\psi_2|T^L|\psi_2\rangle} = \frac{\lambda_1^L\langle\psi_1|S|\psi_1\rangle + \lambda_2^L\langle\psi_2|S|\psi_2\rangle}{\lambda_1^L + \lambda_2^L}$$

Appling Problem 9.1(a) with $x_1 = x, x_2 = y$ that

$$\langle \sigma_x \sigma_y \rangle_{[0,L]}^{(per)} = \frac{tr(ST^{|y-x|}ST^{L-|y-x|})}{tr(T^L)}$$

Let us denote $r := |y-x|$ for notational convenience. We can compute in terms of the spectrum of T that

$$\begin{aligned} tr(ST^rST^{L-r}) &= \langle\psi_1|ST^rST^{L-r}|\psi_1\rangle + \langle\psi_2|ST^rST^{L-r}|\psi_2\rangle \\ &= \lambda_1^{L-r}\langle\psi_1|ST^rS|\psi_1\rangle + \lambda_2^{L-r}\langle\psi_2|ST^rS|\psi_2\rangle \end{aligned}$$

Note that we can express any operator in the basis of outer products of our eigenbasis; in other words, we write

$$S = S_{1,1}|\psi_1\rangle\langle\psi_1| + S_{1,2}|\psi_1\rangle\langle\psi_2| + S_{2,1}|\psi_2\rangle\langle\psi_1| + S_{2,2}|\psi_2\rangle\langle\psi_2|,$$

where $S_{i,j} = \langle\psi_i|S|\psi_j\rangle$ (to convince yourself that this is a legal move, note that in the $\{|\psi_j\rangle\}_j$ basis, each $|\psi_i\rangle\langle\psi_j|$ is simply a matrix with a 1 in the (i,j) position and 0 elsewhere; such rank one operators clearly span the space of all 2x2 operators). From this representation, we see that $S|\psi_i\rangle = S_{1,i}|\psi_1\rangle + S_{2,i}|\psi_2\rangle$ since the basis is orthonormal. Plugging this in, and noting that $S_{i,j} = \overline{S_{j,i}}$ we get

$$\begin{aligned} \langle\psi_i|ST^rS|\psi_i\rangle &= S_{1,i}\langle\psi_i|ST^r|\psi_1\rangle + S_{2,i}\langle\psi_i|ST^r|\psi_2\rangle \\ &= S_{1,i}\lambda_1^r\langle\psi_i|S|\psi_1\rangle + S_{2,i}\lambda_2^r\langle\psi_i|S|\psi_2\rangle \\ &= S_{1,i}S_{i,1}\lambda_1^r + S_{2,i}S_{i,2}\lambda_2^r = |S_{1,i}|^2\lambda_1^r + |S_{2,i}|^2\lambda_2^r \end{aligned}$$

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With this result, we can finally simplify

$$\begin{aligned} tr(ST^rST^{L-r}) &= \lambda_1^{L-r}(|S_{1,1}|^2\lambda_1^r + |S_{2,1}|^2\lambda_2^r) + \lambda_2^{L-r}(|S_{1,2}|^2\lambda_1^r + |S_{2,2}|^2\lambda_2^r) \\ &= \lambda_1^L|S_{1,1}|^2 + \lambda_2^L|S_{2,2}|^2 + (\lambda_1^{L-r}\lambda_2^r + \lambda_1^r\lambda_2^{L-r})|S_{1,2}|^2 \end{aligned}$$

So,

$$\langle\sigma_x\sigma_y\rangle_{[0,L]}^{(per)} = \frac{\lambda_1^L|S_{1,1}|^2 + \lambda_2^L|S_{2,2}|^2 + (\lambda_1^{L-r}\lambda_2^r + \lambda_1^r\lambda_2^{L-r})|S_{1,2}|^2}{\lambda_1^L + \lambda_2^L}$$

and

$$\langle\sigma_x\rangle_{[0,L]}^{(per)} = \frac{\lambda_1^L S_{1,1} + \lambda_2^L S_{2,2}}{\lambda_1^L + \lambda_2^L}$$

This allows us to compute the connected correlator

$$\begin{aligned} \langle\sigma_x;\sigma_y\rangle_{[0,L]}^{(per)} &= \langle\sigma_x\sigma_y\rangle_{[0,L]}^{(per)} - \langle\sigma_x\rangle_{[0,L]}^{(per)}\langle\sigma_y\rangle_{[0,L]}^{(per)} \\ &= \frac{\lambda_1^L|S_{1,1}|^2 + \lambda_2^L|S_{2,2}|^2 + (\lambda_1^{L-r}\lambda_2^r + \lambda_1^r\lambda_2^{L-r})|S_{1,2}|^2}{\lambda_1^L + \lambda_2^L} - \left(\frac{\lambda_1^L S_{1,1} + \lambda_2^L S_{2,2}}{\lambda_1^L + \lambda_2^L}\right)^2 \\ &= \left(\frac{1}{\lambda_1^L + \lambda_2^L}\right)^2 \cdot (\lambda_1^{2L}|S_{1,1}|^2 + \lambda_1^L\lambda_2^L|S_{1,1}|^2 + \lambda_1^L\lambda_2^L|S_{2,2}|^2 + \lambda_2^{2L}|S_{2,2}|^2 \\ &\qquad + (\lambda_1^{2L-r}\lambda_2^r + \lambda_1^{L+r}\lambda_2^{L-r} + \lambda_1^{L-r}\lambda_2^{L+r} + \lambda_1^r\lambda_2^{2L-r})|S_{1,2}|^2 \\ &\qquad - \lambda_1^{2L}|S_{1,1}|^2 - \lambda_2^{2L}|S_{2,2}|^2 - 2\lambda_1^L\lambda_2^L S_{1,1}S_{2,2}) \\ &= \left(\frac{1}{\lambda_1^L + \lambda_2^L}\right)^2 \cdot (\lambda_1^L\lambda_2^L(S_{1,1} - S_{2,2})^2 \\ &\qquad + (\lambda_1^{2L-r}\lambda_2^r + \lambda_1^{L+r}\lambda_2^{L-r} + \lambda_1^{L-r}\lambda_2^{L+r} + \lambda_1^r\lambda_2^{2L-r})|S_{1,2}|^2) \end{aligned}$$

Now, the bound that $0 \le |\lambda_2| < C\lambda_1 e^{-\alpha}$ and the triangle inequality gives

$$\left|\langle\sigma_x;\sigma_y\rangle_{[0,L]}^{(per)}\right| \le \left(\frac{1}{\lambda_1^L}\right)^2 \left(C\lambda_1^{2L}e^{-\alpha L}(S_{1,1} - S_{2,2})^2 + (C\lambda_1^{2L}e^{-\alpha r} + C\lambda_1^{2L}e^{-\alpha(L-r)} + C\lambda_1^{2L}e^{-\alpha(L+r)} + C\lambda_1^{2L}e^{-\alpha(2L-r)})|S_{1,2}|^2\right)$$

Let $d(x,y) := \min\{|y-x|, L-|y-x|\}$. Then, $d(x,y) \le a$ for all $a \in \{r, L-r, L, L+r, 2L-r\}$, and so

$$\begin{aligned} \left|\langle\sigma_x;\sigma_y\rangle_{[0,L]}^{(per)}\right| &\le \left(\frac{1}{\lambda_1^L}\right)^2 \cdot C\lambda_1^{2L}e^{-\alpha d(x,y)}\left((S_{1,1} - S_{2,2})^2 + 4|S_{1,2}|^2\right) \\ &= Ce^{-\alpha d(x,y)}\left((S_{1,1} - S_{2,2})^2 + 4|S_{1,2}|^2\right) \end{aligned}$$

Letting $c := C\left((S_{1,1} - S_{2,2})^2 + 4|S_{1,2}|^2\right)$ (note that this doesn't depend on $L, x,$ or y), we get the desired result that

$$\left|\langle\sigma_x;\sigma_y\rangle_{[0,L]}^{(per)}\right| \le ce^{-\alpha d(x,y)},$$

where $\alpha = \ln(\lambda_1/\lambda_2)$ and $d(x,y) = \min\{|y-x|, L-|y-x|\}$. ■

Problem 9.3

Consider a one dimensional system of Ising spins with nearest and next-nearest neighbor interaction, of the form

$$H_\Lambda^{(per)}(\sigma) = -\sum_{\{x,y\}\subset\Lambda} \mathcal{J}_{x-y}\sigma_x\sigma_y - h\sum_{x\in\Lambda}\sigma_x,$$

with $\mathcal{J}_u \neq 0$ only for $u \leq 2$.

(a) Express the system's pressure in terms of a finite dimensional transfer matrix.

(b) Prove, possibly relying on some of the general results quoted above, that also in this system do the truncated spin - spin correlations decay exponentially fast.

Solution

Proof of (a). Let $\mathcal{J}_1$ denote the value of $\mathcal{J}_{x-y}$ when $|x-y| = 1$, and similarly let $\mathcal{J}_2$ denote the value of $\mathcal{J}_{x-y}$ when $|x-y| = 2$. We then have

$$H_L^{(per)}(\sigma) = -\sum_{x=1}^{L}(\mathcal{J}_1\sigma_x\sigma_{x+1} + \mathcal{J}_2\sigma_x\sigma_{x+2}) - h\sum_{x=1}^{L}\sigma_x,$$

where indices σ_x are taken mod L to assert the periodic boundary conditions. For intuition, consider a window of length 2 sliding across the Ising chain. If the window currently contains (σ_x, σ_{x+1}), then the next window position will contain $(\sigma_{x+1}, \sigma_{x+2})$. The contribution to the Gibbs factor of the interaction between these two windows should be $e^{\beta(\mathcal{J}_1\sigma_x\sigma_{x+1}+\mathcal{J}_2\sigma_x\sigma_{x+2})+\beta h\sigma_x}$; note that this counts each interaction exactly once as the window is slid across. So, we require a 4×4 transfer matrix T with the property that

$$\langle\sigma_x\sigma_{x+1}|T|\sigma_{x+1}\sigma_{x+2}\rangle = e^{\beta(\mathcal{J}_1\sigma_x\sigma_{x+1}+\mathcal{J}_2\sigma_x\sigma_{x+2})+\beta h\sigma_x}$$

and that there is 0 contribution from windows without any overlap. The construction for such a matrix looks like

$$T := \begin{bmatrix} e^{\mathcal{J}_1+\mathcal{J}_2+h} & e^{\mathcal{J}_1-\mathcal{J}_2+h} & 0 & 0 \\ 0 & 0 & e^{-\mathcal{J}_1+\mathcal{J}_2+h} & e^{-\mathcal{J}_1-\mathcal{J}_2+h} \\ e^{-\mathcal{J}_1-\mathcal{J}_2+-h} & e^{-\mathcal{J}_1+\mathcal{J}_2-h} & 0 & 0 \\ 0 & 0 & e^{\mathcal{J}_1-\mathcal{J}_2-h} & e^{\mathcal{J}_1+\mathcal{J}_2-h} \end{bmatrix},$$

where the indices in order correspond to $(\sigma_x, \sigma_{x+1}) = (++), (+-), (-+), (--)$. Note that if the second value of the column index isn't equal to the first value of the row index, there is a 0 in the matrix. This is to ensure that the windows always overlap.

From here, we can repeat the usual completeness relation logic to find

$$\begin{aligned} Z_{\beta,L}^{(per)} &= \sum_{\sigma_1,\ldots,\sigma_L\in\{\pm\}} e^{-\beta H_L^{(per)}(\sigma)} \\ &= \sum_{\sigma_1,\ldots,\sigma_L\in\{\pm\}} \langle\sigma_1\sigma_2|T|\sigma_2\sigma_3\rangle\langle\sigma_2\sigma_3|T|\sigma_3\sigma_4\rangle\ldots\langle\sigma_L\sigma_1|T|\sigma_1\sigma_2\rangle \\ &= tr(T^L) \end{aligned}$$

So, if we write the eigenvalues of T as $\lambda_1, \ldots, \lambda_4$ with $|\lambda_1| \geq |\lambda_i|$ for all $i > 1$, then

$$Z_{\beta,L}^{(per)} = \lambda_1^L + \lambda_2^L + \lambda_3^L + \lambda_4^L = \lambda_1^L\left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^L + \left(\frac{\lambda_3}{\lambda_1}\right)^L + \left(\frac{\lambda_4}{\lambda_1}\right)^L\right)$$

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where $|\lambda_i/\lambda_1| < 1$ for all $i > 2$. Then,

$$\Psi(\beta, h) = \lim_{L\to\infty} \frac{1}{L} \ln Z^{(per)}_{\beta,L} = \ln \lambda_1 + \lim_{L\to\infty} \frac{1}{L} \ln \left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^L + \left(\frac{\lambda_3}{\lambda_1}\right)^L + \left(\frac{\lambda_4}{\lambda_1}\right)^L\right) = \ln \lambda_1$$

■

Proof of (b). The reasoning for this will closely follow the proof of Problem 9.2; so I will make each step a bit shorter for clarity. Firstly, note that for matrices of the form $A = \begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & c & d \\ e & f & 0 & 0 \\ 0 & 0 & g & h \end{bmatrix}$, we get $A^2 = \begin{bmatrix} a^2 & ab & bc & cd \\ ec & cf & dg & dh \\ ea & eb & cf & df \\ eg & fg & gh & h^2 \end{bmatrix}$. Since T is such a matrix with $a, b, c, d, e, f > 0$, we find that T has nonnegative elements and T^2 has strictly positive elements. This means that we can apply the Perron-Frobenius Theorem here. In particular, if the spectrum of T has eigenvalues $|\lambda_1| \geq ... \geq |\lambda_4|$ and right eigenvectors $|\psi_1\rangle, ..., |\psi_4\rangle$ and left eigenvectors $\langle\phi_1|, ..., \langle\phi_4|$, then λ_1 is real and positive and $|\lambda_2|, |\lambda_3|, |\lambda_4| < C\lambda_1 e^{-\alpha}$ for some $\alpha > 0$. As before, with the matrix

$$S := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

encoding the possible values that a window could take and their contributions to the sums in the expectations, the expectations are

$$\langle \sigma_x \rangle^{(per)}_{[0,L]} = \frac{tr(ST^L)}{tr(T^L)}$$

and

$$\langle \sigma_x \sigma_y \rangle^{(per)}_{[0,L]} = \frac{tr(ST^{|y-x|}ST^{L-|y-x|})}{tr(T^L)}$$

Once again, denoting $r := |y - x|$ and $S_{i,j} := \langle\phi_i|S|\psi_j\rangle$, we still have for some constants C_i that

$$|\langle\phi_i|ST^r S|\psi_i\rangle| = \left|\sum_{j=1}^{4} S_{j,i} S_{i,j} \lambda_j^r\right| \leq C_i \lambda_1^r (1 + 3e^{-\alpha r})$$

Letting $d(x, y) := \min\{|y - x|, L - |y - x|\}$, we again get that

$$|tr(ST^{|y-x|}ST^{L-|y-x|})| \leq \lambda_1^L (|S_{1,1}|^2 + Ce^{-\alpha d(x,y)})$$

for some constant C. So, we find that

$$\langle \sigma_x \sigma_y \rangle^{(per)}_{[0,L]} = \frac{tr(ST^{|y-x|}ST^{L-|y-x|})}{tr(T^L)} \leq \frac{1}{\left(\sum_j \lambda_j^L\right)^2} \lambda_1^{2L} (|S_{1,1}|^2 + Ce^{-\alpha d(x,y)})$$

for some new constant C. Therefore, since (with a use of big-oh notation here)

$$\langle \sigma_x \rangle^{(per)}_{[0,L]} \langle \sigma_y \rangle^{(per)}_{[0,L]} = \frac{\left(\sum_j \lambda_j^L S_{j,j}\right)^2}{\left(\sum_j \lambda_j^L\right)^2} = \frac{\lambda_1^{2L}|S_{1,1}|^2 + \lambda_1^{2L} O(e^{-\alpha L})}{\left(\sum_j \lambda_j^L\right)^2},$$

Problem 9.3 continued on next page...

the $\lambda_1^{2L}|S_{1,1}|^2$ term will fall out when we compute $\langle \sigma_x ; \sigma_y \rangle_{[0,L]}^{(per)} = \langle \sigma_x \sigma_y \rangle_{[0,L]}^{(per)} - \langle \sigma_x \rangle_{[0,L]}^{(per)} \langle \sigma_y \rangle_{[0,L]}^{(per)}$. We therefore get via the triangle inequality that

$$\left| \langle \sigma_x ; \sigma_y \rangle_{[0,L]}^{(per)} \right| \leq \frac{1}{\left(\sum_j \lambda_j^L\right)^2} \left(\lambda_1^{2L} C e^{-\alpha d(x,y)} + \lambda_1^{2L} O(e^{-\alpha L}) \right) \leq \frac{\lambda_1^{2L}}{\left(\sum_j \lambda_j^L\right)^2} C' e^{-\alpha d(x,y)}$$

for an appropriate choice of C'. However, since $\frac{\lambda_1^{2L}}{\left(\sum_j \lambda_j^L\right)^2} \leq 1$, we get that

$$\left| \langle \sigma_x ; \sigma_y \rangle_{[0,L]}^{(per)} \right| \leq C' e^{-\alpha d(x,y)}$$

as desired. ■