PHY 521: Problem Set 2

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Problem 4.1

Assuming all the relevant differentiability, prove that for energy (or energies) $E = E(V, T, N)$ at which the infimum $F(T, N, V, ...) := \inf_{E'} [E' - TS(V, E', N, ...)]$ is realized, we have

$$S(V, E, N, ...) = -\frac{\partial F(T, V, N)}{\partial T}$$

Solution

Proof. In order for E to be a minimizer, we must have that the derivative with respect to E' of the things inside the infimum must be 0 when evaluated at $E' = E$. In other words,

$$\frac{\partial}{\partial E'} [E' - TS(V, E', N, ...)] = 0 \implies 1 - T \left. \frac{\partial S(V, E', N, ...)}{\partial E'} \right|_E = 0$$

So, we get that

$$\left. \frac{\partial S(V, E', N, ...)}{\partial E'} \right|_E = \frac{1}{T}$$

From here, we note that the value of F, written in terms of the minimizing energy E (which is implicitly a function of T), is

$$F(T, N, V, ...) = E - TS(V, E, N, ...)$$

Differentiating with respect to T and applying the product rule and chain rule,

$$\frac{\partial F(T, V, N)}{\partial T} = \frac{\partial E}{\partial T} - S(V, E, N) - T \left(\frac{\partial S(V, E, N)}{\partial E} \frac{\partial E}{\partial T} + \frac{\partial S(V, E, N)}{\partial V} \frac{\partial V}{\partial T} + \frac{\partial S(V, E, N)}{\partial N} \frac{\partial N}{\partial T} \right)$$

Note that $\frac{\partial V}{\partial T}$ and $\frac{\partial N}{\partial T}$ both are simply 0 (they are fixed w.r.t. the infimum), and we have already computed that $\frac{\partial S}{\partial E} = \frac{1}{T}$ at our minimizing energy E. Plugging all of this in,

$$\frac{\partial F(T, V, N)}{\partial T} = \frac{\partial E}{\partial T} - S(V, E, N) - T \left(\frac{1}{T} \frac{\partial E}{\partial T} \right) = -S(V, E, N)$$

So, the result is proven. ■

Problem 4.2

Give an expression for the specific latent heat of water's boiling transition, at constant pressure, in terms of water's Gibbs free energy function $G(p, T, N)$.

Solution

Proof. Consider a setting where we are boiling liquid water by raising the temperature (i.e. we are raising the temperature while keeping the pressure constant in a fixed volume with a fixed number of particles). We know that

$$S(V, E, N) = -\frac{\partial G(p, T, N)}{\partial T}$$

The boiling transition (a first order phase transition) occurs where $G(p, T, N)$ has a kink singularity as a function of T, at a value that we will denote T^* (note that for boiling liquid water, $T^* = 373.15$). At this phase transition, we have a latent change in entropy given by

$$\Delta S = -\left(\frac{\partial G}{\partial T}\bigg|_{T^*_+} - \frac{\partial G}{\partial T}\bigg|_{T^*_-} \right) = \frac{\partial G}{\partial T}\bigg|_{T^*_-} - \frac{\partial G}{\partial T}\bigg|_{T^*_+}$$

We know that we can write the latent heat of this transition as

$$\Delta E = T^* \Delta S$$

So, if we denote the total mass of the water as $m_{total} = N \cdot m_{water}$, where m_{water} is the mass of a single water molecule and N is the number of water molecules, then we can say that the *specific* latent heat is

$$L = \frac{\Delta E}{m_{total}} = \frac{T^* \Delta S}{N m_{water}}$$

Plugging in our value for the difference in entropy across the phase transition, we get the result that the specific latent heat of water's boiling transition in this setting is

$$L = \frac{T^*}{N m_{water}} \left(\frac{\partial G}{\partial T}\bigg|_{T^*_-} - \frac{\partial G}{\partial T}\bigg|_{T^*_+} \right),$$

where T^* is the temperature at which liquid water boils, m_{water} is the mass of a single water molecule, and N is the number of water molecules. ■

Problem 5.1

Lemma 1. *Let H, H' be a pair of Hamiltonians with common interaction terms $\Phi_A = \Phi'_A$ but different translation invariant couplings $\mathcal{J}$ and $\mathcal{J}'$. Then, for all $\Lambda \subset \mathbb{Z}^d$ and $\sigma \in \Omega_\Lambda$,*

$$|H_\Lambda(\sigma)| \leq ||\mathcal{J}|| \cdot |\Lambda|$$

and

$$|H_\Lambda(\sigma) - H'_\Lambda(\sigma)| \leq ||\mathcal{J} - \mathcal{J}'|| \cdot |\Lambda|$$

Solution

Proof. We prove the second inequality first. Note that

$$|H_\Lambda(\sigma) - H'_\Lambda(\sigma)| = \left| \sum_{A \subset \Lambda} (\mathcal{J}_A - \mathcal{J}'_A)\, \Phi_A(\sigma_A) \right| \leq \sum_{A \subset \Lambda} |\mathcal{J}_A - \mathcal{J}'_A| \cdot |\Phi_A(\sigma_A)|,$$

where we have applied the triangle inequality. Then, since $\sup_\sigma |\Phi_A(\sigma)| = 1$ (i.e. the Φ_A's are normalized), we get that

$$|H_\Lambda(\sigma) - H'_\Lambda(\sigma)| \leq \sum_{A \subset \Lambda} |\mathcal{J}_A - \mathcal{J}'_A| = \sum_{x \in \Lambda} \sum_{A \ni x} \frac{1}{|A|} |\mathcal{J}_A - \mathcal{J}'_A|,$$

where we divide by $|A|$ because we are counting each A one time for every element of A, and so overcounting by a factor of $|A|$. Now, since the couplings are translation invariant, we see that for all $x \in \Lambda$,

$$\sum_{A \ni x} \frac{1}{|A|} |\mathcal{J}_A - \mathcal{J}'_A| = \sum_{A \ni 0} \frac{1}{|A|} |\mathcal{J}_A - \mathcal{J}'_A|,$$

since we can simply subtract the vector x from all elements of A. So, we get that

$$|H_\Lambda(\sigma) - H'_\Lambda(\sigma)| \leq \sum_{x \in \Lambda} \sum_{A \ni x} \frac{1}{|A|} |\mathcal{J}_A - \mathcal{J}'_A| = |\Lambda| \sum_{A \ni 0} \frac{1}{|A|} |\mathcal{J}_A - \mathcal{J}'_A| = ||\mathcal{J} - \mathcal{J}'|| \cdot |\Lambda|$$

This proves the second equation.

From the second equation, we can trivially derive the first by letting $\mathcal{J}'_A \equiv 0$ for all A, which means that $H'_\Lambda(\sigma) = 0$ for all σ. So, this means that $||\mathcal{J} - \mathcal{J}'|| = ||\mathcal{J}||$ and $|H_\Lambda(\sigma) - H'_\Lambda(\sigma)| = |H_\Lambda(\sigma)|$. The first equation follows. ■

Problem 5.2

Consider the d-dimensional Ising Hamiltonian with translation invariant interaction

$$H_\Lambda(\sigma) = -\sum_{\{x,y\}\subset\Lambda} \mathcal{J}_{|x-y|}\sigma_x\sigma_y - h\sum_{x\in\Lambda}\sigma_x,$$

where the $\mathcal{J}_n$ are not limited to finite range.

(i) Under what condition on the coupling constants $\{\mathcal{J}_n\}$ would the energy of a given spin's interaction with the rest be bounded uniformly in the system's size?

(ii) In the one-dimensional version of the model, under what condition on $\{\mathcal{J}_n\}$ would the total interaction of spins in $\Lambda_L := [-L, L] \subset \mathbb{Z}$ with the rest of the system (i.e. Λ_L^C) be bounded uniformly in L?

(iii) Which of the above pair of conditions suffice for the convergence of this model's pressure function $\Psi_\Lambda(\beta, \mathcal{J}) := \frac{1}{|\Lambda|}\ln Z_\Lambda(\beta)$ in the thermodynamic limit?

Solution

Proof of (i). Since $\mathcal{J}$ is translation invariant, we can suppose without loss of generality that we are interested in bounding the spin at 0's interaction energy, since it will be the same as any other. Now, let us consider the part of the Hamiltonian corresponding to the interactions of σ_0 with all other sites:

$$I_\Lambda := \left| -\sum_{y\in\Lambda}\mathcal{J}_{|y-0|}\sigma_0\sigma_y \right| \le \sum_{y\in\Lambda}|\mathcal{J}_{|y|}|,$$

where the inequality comes from the triangle inequality and the fact that $|\sigma_y| = 1$ for all y. So, if we were able to bound the quantity

$$I_\Lambda \le \sum_{y\in\Lambda}|\mathcal{J}_{|y|}| \le M$$

for some finite M independent of $|\Lambda|$, then the energy of a given spin's interaction with the rest of the system would be bounded uniformly in the system's size. Such a condition can only happen if we can bound this quantity over the entire lattice. In other words, we can ensure a uniform bound if we can find a finite M such that

$$\sum_{y\in\mathbb{Z}^d}|\mathcal{J}_{|y|}| \le M$$

■

Proof of (ii). Consider the 1D Ising model, with the definition $\Lambda_L := [-L, L] \subset \mathbb{Z}$. For any L, we can write the total interaction of spins in Λ_L with the rest of the system as

$$I_L := \left| -\sum_{x\in\Lambda_L}\sum_{y\notin\Lambda_L}\mathcal{J}_{|x-y|}\sigma_x\sigma_y \right| \le \sum_{x\in\Lambda_L}\sum_{y\notin\Lambda_L}|\mathcal{J}_{|x-y|}|,$$

where we have once again applied the triangle inequality and the fact that the spins have magnitude 1. Now, for any fixed x, we can note by a simple shift $z = y - x$ that

$$\sum_{y\notin\Lambda_L}|\mathcal{J}_{|x-y|}| = \sum_{z\notin[-L-x, L-x]}|\mathcal{J}_{|z|}|$$

When we sum this expression over different values $x \in \Lambda_L$, note that the number of times an element of the form $|\mathcal{J}_{|z|}|$ appears depends on the value $|z|$, since this denotes the number of x's such that $z \in [-L-x, L-x]$.

In particular, if $|z| \le 2L$, then there are precisely $2|z|$ such x's such that $x \in \Lambda_L$ and $z \in [-L-x, L-x]$, whereas if $|z| > 2L$ then this number is $|\Lambda_L| = 2L$; in either case, we can say that the number of times each $|\mathcal{J}_{|z|}|$ shows up is upper bounded by $|z|$. In this way,

$$I_L \le \sum_{x \in \Lambda_L} \sum_{y \notin \Lambda_L} |\mathcal{J}_{|x-y|}| \le \sum_{z \in \mathbb{Z}} 2|z||\mathcal{J}_{|z|}|$$

Note that the quantity on the right doesn't depend on L, and so it uniformly bounds I_L. So, if there were some finite M such that

$$\sum_{z \in \mathbb{Z}} 2|z\mathcal{J}_{|z|}| \le M,$$

then this would yield a uniform bound over the total interaction of spins in Λ_L with those not in Λ_L. ■

Proof of (iii). Let us note that for translationally invariant interactions that only involve pairs of sites (like the Ising Hamiltonian given), it is the case that

$$||\mathcal{J}|| := \sum_{A \ni 0} \frac{1}{|A|}|\mathcal{J}_A| = \sum_{y \in \mathbb{Z}^d} \frac{1}{2}|\mathcal{J}_{|y|}|$$

because each set $A \ni 0$ with nonzero interaction is of the form $\{0, y\}$ for some $y \in \mathbb{Z}^d$. So, if the condition from part (i) holds (the sum is finitely bounded by some M), then

$$||\mathcal{J}|| \le \frac{M}{2} < \infty$$

Under this condition, we can then apply the result of Theorem 5.3 with no boundary conditions to see that the pressure function converges in the thermodynamic limit.

We note that the condition in part (ii) is a stronger assumption, as it is the case that

$$1 \le |z| \quad \forall \text{ nonzero } z \in \mathbb{Z}^d \implies \sum_{z \in \mathbb{Z}^d} |\mathcal{J}_{|z|}| \le \sum_{z \in \mathbb{Z}^d} 2|z\mathcal{J}_{|z|}|$$

So, if the sum in the condition of (ii) is finitely bounded, then so is the sum from condition (i), which means we can apply the same earlier logic and Theorem 5.3 for the convergence of the pressure function. This tells us that **either** condition will suffice for the convergence of the model's pressure function in the thermodynamic limit. ■

Problem 5.3

Let $\epsilon > 0$. Given a translation invariant interaction with $||\mathcal{J}|| < \infty$, explain how can it be approximated by a finite range interaction $\mathcal{J}^{(R)}$ such that for every finite subset $\Lambda \subset \mathbb{Z}^d$ the corresponding energy functions satisfy

$$\left| H_\Lambda(\sigma) - H_\Lambda^{(R)}(\sigma) \right| \le \epsilon |\Lambda| \qquad \text{for all configurations } \sigma \in \Omega_\Lambda$$

By how much would the thermodynamic pressure $\Psi(\beta, \mathcal{J})$ differ? That is, state an upper bound on

$$\left| \Psi(\beta, \mathcal{J}) - \Psi(\beta, \mathcal{J}^{(R)}) \right|$$

Solution

Proof. Let $\epsilon > 0$. We have that for our original $\mathcal{J}$,

$$||\mathcal{J}|| = \sum_{A \ni 0} \frac{1}{|A|} |\mathcal{J}_A| < \infty$$

Since the sum converges, we know that there must be some R large enough that the tail gets arbitrarily small if we sort the A's that contain the origin by diameter. In particular, there must be some $R > 0$ such that

$$\sum_{\substack{A \ni 0 \\ diam(A) > R}} \frac{1}{|A|} |\mathcal{J}_A| \le \epsilon$$

For this value of R, define

$$\mathcal{J}_A^{(R)} = \mathcal{J}_A \cdot \mathbb{1}_{diam(A) \le R} \qquad \forall A \subset \Lambda,$$

where $\mathcal{J}_A$ is the coupling term for A in the original interaction. This is clearly finite range with range R, as the coupling term is 0 for all sets A with diameter larger than R. Now, we can compute that

$$||\mathcal{J} - \mathcal{J}^{(R)}|| = \sum_{A \ni 0} \frac{1}{|A|} |\mathcal{J}_A - \mathcal{J}_A^{(R)}| = \sum_{A \ni 0} \frac{1}{|A|} \left| \mathcal{J}_A (1 - \mathbb{1}_{diam(A) \le R}) \right| = \sum_{\substack{A \ni 0 \\ diam(A) > R}} \frac{1}{|A|} |\mathcal{J}_A| \le \epsilon$$

From here we can use Lemma 5.1, which states that for two different $\mathcal{J}, \mathcal{J}'$, we have that for all finite $\Lambda \subset \mathbb{Z}^d$ and all $\sigma \in \Lambda$

$$|H_\Lambda(\sigma) - H'_\Lambda(\sigma)| \le ||\mathcal{J} - \mathcal{J}'|| \cdot |\Lambda|$$

So, since our finite range interaction $\mathcal{J}^{(R)}$ has that $||\mathcal{J} - \mathcal{J}^{(R)}|| \le \epsilon$, the result follows.

We can express the partition function of the original interaction in terms of the finite range Hamiltonian to get that for all $\Lambda \subset \mathbb{Z}^d$ and *a-priori* measures $\mu(d\sigma)$,

$$Z_\Lambda(\beta) = \int_{\Omega_\Lambda} e^{-\beta H_\Lambda(\sigma)} \mu(d\sigma) = \int_{\Omega_\Lambda} e^{-\beta H_\Lambda^{(R)}(\sigma)} \cdot e^{-\beta \left(H_\Lambda(\sigma) - H_\Lambda^{(R)}(\sigma) \right)} \mu(d\sigma)$$

By the previous result, $H_\Lambda(\sigma) - H_\Lambda^{(R)}(\sigma) \le \epsilon |\Lambda|$, and so

$$e^{-\beta\epsilon|\Lambda|} \int_{\Omega_\Lambda} e^{-\beta H_\Lambda^{(R)}(\sigma)} \mu(d\sigma) \le \int_{\Omega_\Lambda} e^{-\beta H_\Lambda(\sigma)} \mu(d\sigma) \le e^{\beta\epsilon|\Lambda|} \int_{\Omega_\Lambda} e^{-\beta H_\Lambda^{(R)}(\sigma)} \mu(d\sigma)$$

$$\implies e^{-\beta\epsilon|\Lambda|} Z_\Lambda^{(R)}(\beta) \le Z_\Lambda(\beta) \le e^{\beta\epsilon|\Lambda|} Z_\Lambda^{(R)}(\beta)$$

Taking the natural log and dividing by $|\Lambda|$,

$$\Psi_\Lambda(\beta, \mathcal{J}^{(R)}) - \beta\epsilon \le \Psi_\Lambda(\beta, \mathcal{J}) \le \Psi_\Lambda(\beta, \mathcal{J}^{(R)}) + \beta\epsilon$$

Therefore, for all $\Lambda \subset \mathbb{Z}^d$,

$$\left|\Psi_\Lambda(\beta, \mathcal{J}) - \Psi_\Lambda(\beta, \mathcal{J}^{(R)})\right| \leq \beta\epsilon$$

Taking the limit as $|\Lambda| \to \infty$, we get that

$$\left|\Psi(\beta, \mathcal{J}) - \Psi(\beta, \mathcal{J}^{(R)})\right| \leq \beta\epsilon$$

for the value of R such that $\sum_{\substack{A \ni 0 \\ diam(A) > R}} \frac{1}{|A|} |\mathcal{J}_A| \leq \epsilon$. ■

Problem 6.1

Consider the d-dimensional finite-volume Ising model in $\Lambda_L = (-\lfloor L/2 \rfloor, \lfloor L/2 \rfloor]^d$ with boundary conditions τ, whose Hamiltonian is given by

$$H^{\#}_{\Lambda_L,h}(\sigma) = -J \sum_{\substack{\{x,y\}\subset\Lambda_L \\ ||x-y||=1}} \sigma_x\sigma_y - J \sum_{\substack{x\in\Lambda_L, y\in\mathbb{Z}^d\setminus\Lambda_L \\ ||x-y||=1}} \sigma_x\tau_y - h\sum_{x\in\Lambda_L}\sigma_x$$

We would like to investigate four possible choices for boundary conditions τ:

$$\tau_y = \begin{cases} 0 & \text{free b.c.} \\ +1 & + \text{ b.c.} \\ -1 & - \text{ b.c.} \\ \sigma_{per(x)}/2 & \text{periodic b.c.} \end{cases},$$

where $per(x)$ is to be summed over the sites $u \in \Lambda_L \setminus \{x\}$ which are the neighbors of x under the periodic boundary conditions.

(i) Prove that $\Psi(\beta, J, h)$ (the system's pressure in the infinite volume limit) is convex in h.

(ii) Summarize the arguments proving that at (β, h) for which $\Psi(\beta, h)$ is differentiable with respect to h, for all boundary conditions,

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|}\langle M_{\Lambda_L}\rangle^{\#}_{\beta,\Lambda_L} = -\frac{1}{\beta}\frac{\partial}{\partial h}\Psi(\beta,h)$$

where $M_\Lambda(\sigma) := \sum_{x\in\Lambda}\sigma_x$ is the bulk magnetization of the finite system.

Solution

Proof of (i). Let $h_0, h_1 \in \mathbb{R}$ be arbitrary. Define for $t \in (0,1)$ the quantity

$$h_t := th_0 + (1-t)h_1$$

Then, if we define $E_{\Lambda_L}(\sigma) := -J\sum_{\substack{\{x,y\}\subset\Lambda_L \\ ||x-y||=1}}\sigma_x\sigma_y - J\sum_{\substack{x\in\Lambda_L, y\in\mathbb{Z}^d\setminus\Lambda_L \\ ||x-y||=1}}\sigma_x\tau_y$, we get

$$\begin{aligned}
\Psi(\beta, h_t) &= \lim_{L\to\infty}\frac{1}{|\Lambda_L|}\ln\int_{\Omega_\Lambda} e^{-\beta E_{\Lambda_L}(\sigma)}\cdot e^{-\beta h_t\left(\sum_{x\in\Lambda_L}\sigma_x\right)}d\sigma \\
&= \lim_{L\to\infty}\frac{1}{|\Lambda_L|}\ln\int_{\Omega_\Lambda} e^{-\beta E_{\Lambda_L}(\sigma)}\cdot\left(e^{-\beta h_0\left(\sum_{x\in\Lambda_L}\sigma_x\right)}\right)^t\cdot\left(e^{-\beta h_1\left(\sum_{x\in\Lambda_L}\sigma_x\right)}\right)^{1-t}d\sigma \\
&= \lim_{L\to\infty}\frac{1}{|\Lambda_L|}\ln\int_{\Omega_\Lambda}\left(e^{-\beta E_{\Lambda_L}(\sigma)}\cdot e^{-\beta h_0\left(\sum_{x\in\Lambda_L}\sigma_x\right)}\right)^t\cdot\left(e^{-\beta E_{\Lambda_L}(\sigma)}\cdot e^{-\beta h_1\left(\sum_{x\in\Lambda_L}\sigma_x\right)}\right)^{1-t}d\sigma
\end{aligned}$$

Applying Holder's inequality $\left|\int f^t g^{1-t}d\mu\right| \le \left(\int |f|d\mu\right)^t\left(\int|g|d\mu\right)^{1-t}$ (set $p = \frac{1}{t}$ and $q = \frac{1}{1-t}$) and noting that everything is nonnegative and $\ln(\cdot)$ is monotonic, we get

$$\begin{aligned}
\Psi(\beta, h_t) &\le \lim_{L\to\infty}\frac{1}{|\Lambda_L|}\ln\left[\left(\int_{\Omega_\Lambda} e^{-\beta E_{\Lambda_L}(\sigma)}\cdot e^{-\beta h_0\left(\sum_{x\in\Lambda_L}\sigma_x\right)}d\sigma\right)^t\cdot\left(\int_{\Omega_\Lambda} e^{-\beta E_{\Lambda_L}(\sigma)}\cdot e^{-\beta h_1\left(\sum_{x\in\Lambda_L}\sigma_x\right)}d\sigma\right)^{1-t}\right] \\
&= \lim_{L\to\infty}\frac{1}{|\Lambda_L|}\left(t\ln\left(\int_{\Omega_\Lambda} e^{-\beta E_{\Lambda_L}(\sigma)}\cdot e^{-\beta h_0\left(\sum_{x\in\Lambda_L}\sigma_x\right)}d\sigma\right) + (1-t)\ln\left(\int_{\Omega_\Lambda} e^{-\beta E_{\Lambda_L}(\sigma)}\cdot e^{-\beta h_1\left(\sum_{x\in\Lambda_L}\sigma_x\right)}d\sigma\right)\right) \\
&= t\left(\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\ln\int_{\Omega_\Lambda} e^{-\beta H^{(\sigma)}_{\Lambda_L,h_0}d\sigma}\right) + (1-t)\left(\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\ln\int_{\Omega_\Lambda} e^{-\beta H^{(\sigma)}_{\Lambda_L,h_1}d\sigma}\right) \\
&= t\Psi(\beta, h_0) + (1-t)\Psi(\beta, h_1)
\end{aligned}$$

Problem 6.1 continued on next page. . .

Since this holds for all h_0, h_1 and all values of t, we certainly have that Ψ is convex in h. ∎

Proof of (ii). For all finite volumes Λ_L, we can compute via a simple chain rule that at values of (β, h) for which $\Psi(\beta, h)$ is differentiable with respect to h (which means Ψ_{Λ_L} is as well),

$$\begin{aligned}
\frac{\partial}{\partial h}\Psi_{\Lambda_L}(\beta,h) &= \frac{\partial}{\partial h}\left[\frac{1}{|\Lambda_L|}\ln Z_{\Lambda_L}(\beta,h)\right] = \frac{1}{|\Lambda_L|}\frac{\frac{\partial}{\partial h}\left[Z_{\Lambda_L}(\beta,h)\right]}{Z_{\Lambda_L}(\beta,h)} \\
&= \frac{1}{|\Lambda_L|}\frac{\sum_{\sigma\in\Omega_{\Lambda_L}} e^{-\beta H^{\#}_{\Lambda_L,h}(\sigma)}\cdot(-\beta)\cdot\frac{\partial}{\partial h}\left[H^{\#}_{\Lambda_L,h}(\sigma)\right]}{Z_{\Lambda_L}(\beta,h)} \\
&= \frac{1}{|\Lambda_L|}\frac{\sum_{\sigma\in\Omega_{\Lambda_L}} e^{-\beta H^{\#}_{\Lambda_L,h}(\sigma)}\cdot(-\beta)\cdot(-M_{\Lambda_L}(\sigma))}{Z_{\Lambda_L}(\beta,h)} \\
&= \frac{\beta}{|\Lambda_L|}\frac{\sum_{\sigma\in\Omega_{\Lambda_L}} e^{-\beta H^{\#}_{\Lambda_L,h}(\sigma)}\cdot M_{\Lambda_L}(\sigma)}{Z_{\Lambda_L}(\beta,h)}
\end{aligned}$$

We recognize the second fraction to be the Gibbs canonical ensemble average of $M_{\Lambda_L}(\sigma)$ (since it is multiplied by the Gibbs measure and summed over all possible states. This yields that

$$\frac{\partial}{\partial h}\Psi_{\Lambda_L}(\beta,h) = \frac{\beta}{|\Lambda_L|}\left\langle M_{\Lambda_L}\right\rangle^{\#}_{\beta,\Lambda_L}$$

We want to take the limit $L\to\infty$. Note that interchanging the limit w.r.t L and the derivative w.r.t h on the left hand side can only happen if $\lim_{L\to\infty}\frac{\partial}{\partial h}\Psi_{\Lambda_L}(\beta,h)$ exists for almost all (β,h), which it does. To see this, note that the convexity of Ψ_{Λ_L} w.r.t h from part (i) and Theorem 3.6 guarantee that the derivatives w.r.t. h converge wherever Ψ_{Λ_L} is differentiable. So, taking the limit and dividing by β, we get the desired result that

$$\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\left\langle M_{\Lambda_L}\right\rangle^{\#}_{\beta,\Lambda_L} = \frac{1}{\beta}\frac{\partial}{\partial h}\Psi(\beta,h)$$

∎

Problem 7.1

Lemma 2. *Let $1 \ll L_0 \ll L_1$ denote the lengths of two volumes, with $\Lambda_{L_0} \subset \Lambda_{L_1}$. For the 2D Ising model with the nearest neighbor interaction $\left(J_{x,y} = J \cdot \mathbb{1}_{||x-y||=1} \text{ for some } J > 0\right)$ with $h = 0$, i.e.*

$$H_{\Lambda,J}(\sigma) := -\sum_{\{x,y\}\subset\Lambda} J_{x,y}\sigma_x\sigma_y,$$

at $\beta > \widehat{\beta}$ (at which $3e^{-2\widehat{\beta}J} < 1$), we have

$$\lim_{L_0\to\infty}\lim_{L_1\to\infty}\left\langle \mathbb{1}\left[\mathcal{L}^-_{L_0,L_1}\right]\right\rangle^+_\beta = 0$$

$$\lim_{L_0\to\infty}\lim_{L_1\to\infty}\left\langle \mathbb{1}\left[\mathcal{L}^+_{L_0,L_1}\right]\right\rangle^+_\beta = 1$$

Solution

Proof. Suppose without loss of generality that $\sigma_0 \in \Lambda_{L_0}$. To prove the first equation, note that $\mathcal{L}^-_{L_0,L_1}$ is the event that there exists a path of $-$ spins from the boundary of Λ_{L_0} to the boundary of Λ_{L_1}. Since the boundary conditions are $+$, such a path *must* have a maximal contour enveloping it with $+$'s on the outside and $-$'s on the inside. Since the length of the path must be at least $L_1 - L_0$, the length of any contour enveloping this path is at least $2(L_1 - L_0)$. So, we seek the probability of such a contour existing as a function of L_0, L_1, J, and β. Using Lemma 7.3 (the Peierls contour estimate), we know that for any contour γ we have that the probability of its existence is at most

$$\mathbb{P}^+_\beta[\gamma] \le e^{-2\beta J|\gamma|}$$

For a contour of length l drawn from any arbirtary start point, if we always traverse the contour going above the horizontal line going through the start, there are at most $l/2$ choices for the first edge (otherwise there won't be enough edges below the horizontal line to complete the closed path). Now, since there are 4 neighbors to each site, each subsequent step will only have 3 possible choices since we aren't going backward (note that this weak bound doesn't force the path to be closed, but it will suffice). This yields that there are at most $\frac{l}{2}3^l$ possible contours of length l. So, we can apply a union bound and the fact that any contour γ containing a path of $-$'s from Λ_0 to Λ_1 must have $|\gamma| \ge 2(L_1 - L_0)$ to see that

$$\begin{aligned}
\left\langle \mathbb{1}\left[\mathcal{L}^-_{L_0,L_1}\right]\right\rangle^+_\beta &\le \sum_{\gamma \text{ containing path of } -\text{'s}} \mathbb{P}^+_\beta[\gamma] \le \sum_{l=2(L_1-L_0)}^{\infty}\sum_{\{\gamma:|\gamma|=l\}} \mathbb{P}^+_\beta[\gamma] \\
&\le \sum_{l=2(L_1-L_0)}^{\infty}\sum_{\{\gamma:|\gamma|=l\}} e^{-2\beta Jl} = \sum_{l=2(L_1-L_0)}^{\infty} e^{-2\beta Jl}\left|\{\gamma \;:\; |\gamma| = l\}\right| \\
&\le \sum_{l=2(L_1-L_0)}^{\infty} e^{-2\beta Jl}\cdot\frac{l}{2}3^l = \sum_{l=2(L_1-L_0)}^{\infty}\frac{l}{2}\left(3e^{-2\beta j}\right)^l
\end{aligned}$$

Since $\beta > \widehat{\beta}$, we know that the term inside the parenthesis is < 1 and the sum is therefore convergent. This means that if we take the limit as $L_1 \to \infty$, this tail of the sum must go to 0 (it is a convergent sum of nonnegative things). So, since $\left\langle \mathbb{1}\left[\mathcal{L}^-_{L_0,L_1}\right]\right\rangle^+_\beta$ is nonnegative, the above bound gives us

$$\lim_{L_1\to\infty}\left\langle \mathbb{1}\left[\mathcal{L}^-_{L_0,L_1}\right]\right\rangle^+_\beta = 0 \implies \lim_{L_0\to\infty}\lim_{L_1\to\infty}\left\langle \mathbb{1}\left[\mathcal{L}^-_{L_0,L_1}\right]\right\rangle^+_\beta = 0$$

Problem 7.1 continued on next page. . .

For the second equation, note that $\left\langle \mathbb{1}\left[\mathcal{L}^+_{L_0,L_1}\right]\right\rangle^+_\beta$ is the event that there is a path of $+$'s between L_0 and L_1. The complement of this event (i.e. the event that there is *no* such path) is equivalent to the event that there is a loop of $-$'s within Λ_{L_1} encircling Λ_{L_0}. In other words, if we let $E^+_{L_0,L_1}$ be the event that there is such a contour encircling Λ_{L_0}, then we have that $\left\langle \mathbb{1}\left[\mathcal{L}^+_{L_0,L_1}\right]\right\rangle^+_\beta = 1 - \left\langle \mathbb{1}\left[E^+_{L_0,L_1}\right]\right\rangle^+_\beta$. Any possible contour γ that could satisfy $E^+_{L_0,L_1}$ must have a length of at least $|\gamma| \geq 2L_0$, since it has to envelop Λ_{L_0}. Using the exact above logic, we can apply a similar union bound to see that

$$\begin{aligned}
\left\langle \mathbb{1}\left[E^+_{L_0,L_1}\right]\right\rangle^+_\beta &\leq \sum_{\gamma \text{ containing } \Lambda_0} \mathbb{P}^+_\beta\left[\gamma\right] \leq \sum_{l=2L_0}^{\infty} \sum_{\{\gamma:|\gamma|=l\}} \mathbb{P}^+_\beta\left[\gamma\right] \\
&\leq \sum_{l=2L_0}^{\infty} \sum_{\{\gamma:|\gamma|=l\}} e^{-2\beta J l} = \sum_{l=2L_0}^{\infty} e^{-2\beta J l}\left|\{\gamma \ : \ |\gamma| = l\}\right| \\
&\leq \sum_{l=2L_0}^{\infty} e^{-2\beta J l} \cdot \frac{l}{2} 3^l = \sum_{l=2L_0}^{\infty} \frac{l}{2}\left(3e^{-2\beta j}\right)^l
\end{aligned}$$

As before, this sum is convergent. Note that this value is independent of L_1; so, taking the limit as $L_1 \to \infty$ yields the same bound. Therefore, taking the subsequent limit $L_0 \to \infty$ causes this tail to go to 0, yielding

$$\lim_{L_0\to\infty} \lim_{L_1\to\infty} \left\langle \mathbb{1}\left[E^+_{L_0,L_1}\right]\right\rangle^+_\beta = 0$$

$$\implies \lim_{L_0\to\infty} \lim_{L_1\to\infty} \left\langle \mathbb{1}\left[\mathcal{L}^+_{L_0,L_1}\right]\right\rangle^+_\beta = 1 - \lim_{L_0\to\infty} \lim_{L_1\to\infty} \left\langle \mathbb{1}\left[E^+_{L_0,L_1}\right]\right\rangle^+_\beta = 1$$

■