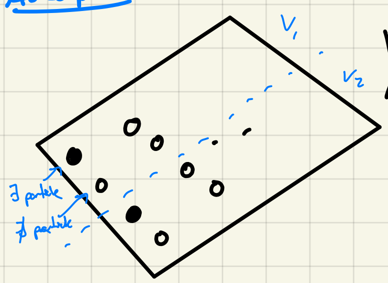
Aizenman

Lecture 1/31 - First day babeyy!

Adsorption



V - # sites
N - # particles

$n_1 = \frac{N_1}{V_1}$, $n_2 = \frac{N_2}{V_2}$

local density variation

> **Q:** If every configuration is equally likely, what is $\mathbb{P}\{|n_1 - n_2| > \varepsilon\}$?

local entropy density = entropy/vol

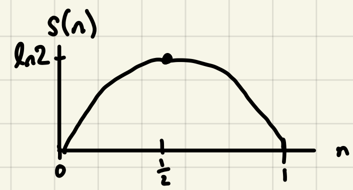
We start by noting that the # of states is $W(N,V) = \binom{V}{N} = \frac{V!}{N!(V-N)!} \approx e^{s(n)V}$

Boltzmann's result that # of states is exponential in volume

Using the Sterling approx. and a bunch of algebra, $\log N! = N(\ln N - 1) + \ln(2\pi N) + O(\frac{1}{N})$

$$\boxed{s(n) = -\left[n \ln n + (1-n)\ln(1-n)\right]}$$

local entropy density



(This is just like Shannon entropy $S(\{p_i\}_{i=1}^{N}) = -\sum_{i=1}^{N} p_i \ln p_i$)!

We can say that, for some density difference Δn,

$$\mathbb{P}\{n_1 - n_2 = \Delta n\} = \frac{W(V_1,N_1)\,W(V_2,N_2)}{W(V,N)} = \frac{e^{\left[s(n+\frac{\Delta n}{2}) + s(n-\frac{\Delta n}{2})\right]\frac{V}{2}}}{e^{s(n)V}}$$

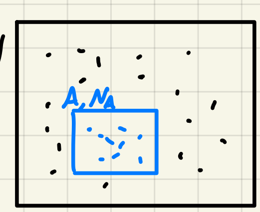
Taylor 1st order $\approx e^{\frac{1}{2} s''(n) (\frac{\Delta n}{2})^2 \cdot \frac{V}{2}}$ (<0) very small for large volume!

equivalence of ensemble

> **Q:** If we have a subvolume A, what is the distribution of the # of particles in A?

It turns out that it follows a Poisson distribution that is identical for different A's and depends only on |A|.

V, N



Stat Mech Setup

On the macroscopic level:
- divisibility
- additivity of V, N, E, ... ← E-additivity is because interactions happen on a very short range in the macro view
- many DOFs (spin states of constituents, etc.)

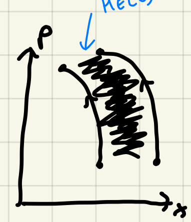
∃ a natural notion of counting microstates

examples of microstate counting

Recall from PHY205 that phase space evolutions of $((\vec{x}_1, \dots, \vec{x}_n), (\vec{p}_1, \dots, \vec{p}_n))$ preserve the Liouville measure $\int dx \dots dp \dots$,
⟺ the volume in phase space ⟺ # of states

$\Rightarrow W(V,N) = \int_{V^N} \int_{\mathbb{R}^{d \cdot N}} \mathbb{1}\{H_N(\vec{x},\vec{p}) \in (E, E+\Delta E)\}\, d\vec{x}^{\,d} d\vec{p}^{\,d}$ Liouville integral

Shell for which $H \in [E, E+\Delta E]$



PHY 208 would say $W(V,N) = \mathrm{Tr}\, P_{(E, E+\Delta E)}$ trace of density matrix

Note: Since s is convex and $S = e^{sV}$, S is also convex.
So, variational formulations of stat mech. allow states that maximize the convex objective S.

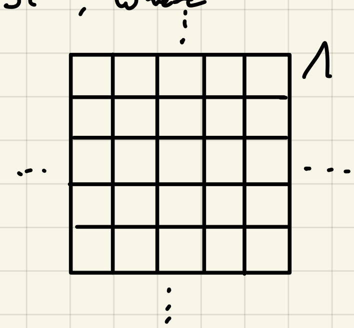
local entropy density of energy density

We arrive at the fact that $$W([E, E+\Delta E]) \propto e^{s\left(\frac{E}{V}\right)\cdot |V|}$$

Lecture 2/2 - Partition Fns + Ensembles

We consider both discrete & continuous models. We describe the configuration of a model by defining on a domain G, which is often a lattice. At each site, we have possible values that depend on the model. More formally, $w: G \to \Omega$, where

$\Omega = \{0,1\}$ adsorption $\quad \Omega = \{-1,1\}$ Ising model $\quad \Omega = \mathbb{R}^d$ continuum



Partition Functions

For a given space, we define the partition function by

$$Z_\Lambda(\beta) = \int_{\Lambda^N}\int_{\mathbb{R}^{d\cdot N}} e^{-\beta H_N(\vec{x},\vec{p})} \underbrace{\prod_{j=1}^{N} d\vec{x}_j\, d\vec{p}_j}_{\text{Liouville Measure}}$$

Liouville Measure preserved during time evolution

the "canonical ensemble" allows us to not exclude configurations, but to fine-tune the desirable config. space via β (or chemical potential μ if we use $e^{-\mu N}$)

$$= \int_{\mathbb{R}} e^{-\beta E} e^{S(E)}\, dE \approx |\Lambda| \int_{\mathbb{R}} e^{-\beta u |\Lambda|} e^{s(u)|\Lambda|}\, du$$

let $u = \frac{E}{|\Lambda|}$

$$\approx |\Lambda| \int_{\mathbb{R}} e^{[s(u)-\beta u]|\Lambda|}\, du$$

(Were we in a discrete model, we define Z_Λ as a discrete sum over the discrete phase space

$$Z_\Lambda(\beta) = \sum_{w\in\Omega} e^{-\beta H_\Lambda(w)}$$

energy of certain cell; inverse temp)

In such integrals, since u is normally smooth + bounded, we expect it to be dominated by $\sup_u (s(u) - \beta u)$.

Assuming entropy actually behaves as (thermodynamic limit) $W([E, E+\Delta E]) \approx e^{s(\frac{E}{|\Lambda|})\cdot|\Lambda|}$, then

$$\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \ln Z_\Lambda(\beta) = \sup_u \{s(u) - \beta u\}$$

Legendre transform of $s(\cdot)$

"winner takes all"

Types of ensembles

"microcanonical"

- $H_n \in [E, E+\Delta E]$

"Canonical"

- $e^{-\beta H_n}$
- releases bounds on E

"grand canonical"

- $e^{\beta[-H + \mu N + hM]}$ (energy; chem. potential; # particles; magnetization)
- releases bounds on other extensive properties like N, M

Example: Adsorption on a lattice

$\Omega_0 = \{0,1\}$

$$Z_\Lambda(\mu) = \underbrace{\sum_{\omega\in\Omega_\Lambda}}_{\text{sum over all configurations}} e^{\mu \overbrace{N(\omega)}^{\text{\# of particles in config.}}} = \sum_{\omega\in\Omega_\Lambda} \underbrace{\prod_j}_{\text{prod. over sites}} e^{\mu\omega_j} = (1+e^{\mu})^{|\Lambda|}$$

Let $n = \frac{N}{|\Lambda|}$ be the particle density. Then,

$$\sup_\mu \{ s(n) - \mu(n) \} = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \ln(1+e^{\mu})^{|\Lambda|} = \underbrace{\ln(1+e^{\mu})}_{\text{Legendre transform of } s(\cdot)}$$

This matches the result $s(n) = -n\ln(n) - (1-n)\ln(1-n)$ that we found for adsorption via the Stirling approx.

Convexity

Def: A set $D \subseteq \mathbb{R}^{\nu}$ is convex if $\forall x,y\in D$,

$$tx + (1-t)y \in D \quad \forall t\in[0,1].$$

D contains line between x and y

A function $f: D \to \mathbb{R}$ is convex if $\forall x,y \in D$,

$$f(tx+(1-t)y) \leq t f(x) + (1-t) f(y)$$

f lies below line between $f(x)$ and $f(y)$

Equivalently, $\forall x\in D$ and a fixed $y\in D$, $\frac{f(y)-f(x)}{y-x}$ is monotone increasing.

If f is twice differentiable, a sufficient condition of convexity is

$$f''(x) \geq 0 \quad \forall x \in D.$$

Theorem: Let $\{G_\alpha(x)\}_\alpha$ be a family of linear functions of x. Then,

$$F(x) = \sup_\alpha G_\alpha(x) \quad \text{is convex.}$$

Proof: Intersection of closed half-spaces, which are all convex, is itself convex. $\square$

Note that the Legendre Transform looks similar: it is indeed the case that Legendre transforms <u>are convex.</u>

Theorem: $\forall$ convex $F: [a,b] \to \mathbb{R}$,

① F is differentiable everywhere except at a countable number of points.

② On the set where $F'(x)$ exists, it is monotone increasing.

③ $\forall x \in [a,b]$, both $F'_-(x) = \lim\limits_{\varepsilon \to 0} \dfrac{F(x-\varepsilon) - F(x)}{-\varepsilon}$ and $F'_+(x) = \lim\limits_{\varepsilon \to 0} \dfrac{F(x+\varepsilon) - F(x)}{\varepsilon}$

exist and satisfy $\lim\limits_{\varepsilon \to 0} F'_\pm(x-\varepsilon) \leq F'_-(x) \leq F'_+(x) \leq \lim\limits_{\varepsilon \to 0} F'_\pm(x+\varepsilon)$

④ The right derivatives are continuous a.e., and where this happens, F' exists.

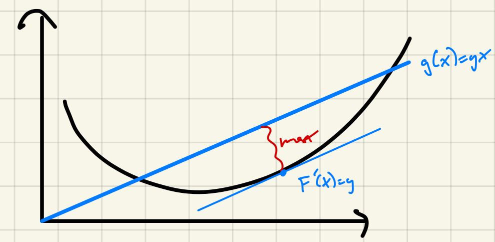
Lecture 2/7 - Convexity + Legendre Transform

Def. The Legendre Transform of a convex function F is

$$\boxed{(TF)(y) = \sup_{x} \left\{ y \cdot x - F(x) \right\}}$$

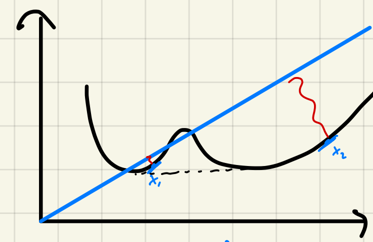
← inf for concave

In a sense, varying y explores the values of F for which F' takes the value y.



The transform T is itself convex, since it is the max of linear functions.

If F is not convex, T computes the Legendre Transform of the convex hull of F.



Both points have same F', but x_2 is selected by the sup.

Theorem: (Inverted Property of the Legendre Transform)

$$\forall \text{ convex } F : \mathbb{R} \to \mathbb{R}, \quad T(TF) = F$$

Proof: We prove this assuming F differentiable, but the result holds generally. Let $x(y)$ be the point where $F'(x) = y$. Then,

$$(TF)(y) = y \cdot x(y) - G(x(y)) \quad \text{and} \quad G'(x(y)) = y$$

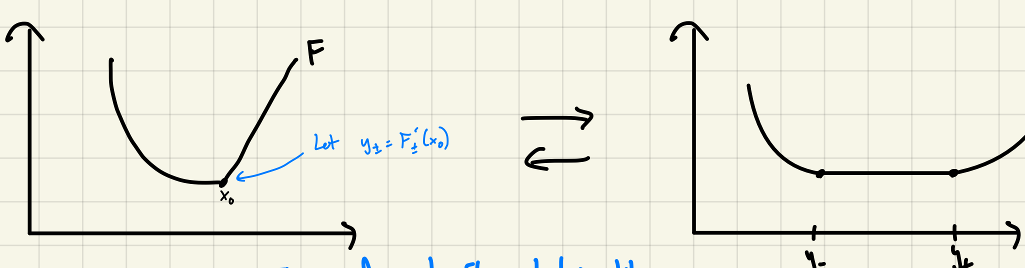
So, $(T^2F)(x) = \sup_y \left\{ xy - (TF)(y) \right\} = x \cdot y(x) - (TF)(y(x))$ for $y(x)$ s.t. $(TF)'(y) = x$

We can compute $(TF)'(y) = x(y) + \cancel{y \cdot x'(y)} - \underbrace{G'(x(y))}_{y} \cancel{x'(y)} = x(y)$

$$\Rightarrow (T^2F)(x) = x \cancel{y(x)} - \cancel{y(x) \cdot x(y(x))} + G(x(y(x))) = G(x).$$

$\square$

If we have a F with a kink, Legendre Transform maps the kink to a flat region, and vice versa.



This discontinuity looks like a first-order phase transition.

Lecture 2/9 - Time Evolution + Ergodicity

We define phase space to be $(\vec{\underline{x}}, \vec{\underline{p}})$ under the Hamiltonian

$$H(\vec{\underline{x}}, \vec{\underline{p}}) = \sum_j \frac{1}{2m} \|\vec{p}_j\|^2 + V(\vec{\underline{x}})$$

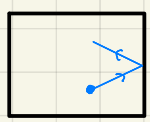
via

$$\dot{x}_j(t) = \frac{\partial H}{\partial p_j} = \frac{p}{2m} \quad , \quad \dot{p}_j(t) = -\frac{\partial H}{\partial x_j} = -\vec{\nabla} V$$

$$\Rightarrow \frac{d}{dt} Q(\vec{\underline{x}}(t), \vec{\underline{p}}(t)) = \underbrace{\{Q, H\}}_{\text{Poisson bracket}} = Q\left[\frac{\overleftarrow{\partial}}{\partial x_j}\frac{\overrightarrow{\partial}}{\partial p_j} - \frac{\overleftarrow{\partial}}{\partial p_j}\frac{\overrightarrow{\partial}}{\partial x_j}\right] H \quad \forall Q$$

Evolution under this mechanics preserves the Liouville measure $\prod_j d\vec{x}_j d\vec{p}_j$

Constants of Motion

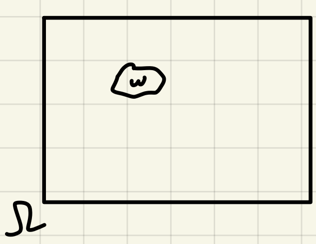
- We always have $\frac{d}{dt} H = 0 \Rightarrow$ energy is conserved.
- If $V(\vec{\underline{x}}) = 0$, then $\vec{\underline{p}}$ is also a constant of motion.
- Under boundary conditions like , a particle reflecting on a flat, (axis-aligned) wall only flips one coordinate at a time.
- Reflection on curved boundaries may mix things.

This leads us to the concept of ergodicity.

Ergodicity

Consider a probability space $(\underbrace{\Omega}_{\text{state space}}, \underbrace{\mathcal{B}}_{\text{Borel algebra}}, \underbrace{\rho(d\omega)}_{\text{measure}})$ $\omega \in \Omega$

Our state space is bounded as



Def: An invertible measure-preserving transformation $T: \Omega \to \Omega$, $\omega \mapsto T\omega$ satisfies

$$\mu(T^{-1}(E)) = \mu(E) \quad \forall\, E \subseteq \Omega \text{ measurable.}$$

they differ by a set of measure 0!

Def: A measure-preserving transformation T is ergodic if $f(Tx) \overset{a.s.}{=} f(x)$ holds only for constant f. In other words, if E is s.t. $T^{-1}(E)$ and E differ by a set of measure 0, then either $\mu(E)=0$ or $\mu(E^c)=0$.

Thm: (Birkhoff)

If $\{T_i\}_{i\in\mathbb{N}}$ is a collection of ergodic, measure-preserving transformations, then $\forall$ bounded, measurable $f: \Omega \to \mathbb{R}$, the limit exists and equals

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(T^m x) \overset{a.e.}{=} \int_\Omega f(x)\,\mu(dx)$$

If $\{T_t\}_{t\geq 0}$ is a collection of ergodic, measure-preserving transformations, then $\forall$ bounded, measurable $f: \Omega \to \mathbb{R}$, the limit exists and equals

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau f(T_t)\,d\tau \overset{a.e.}{=} \int_\Omega f(x)\,\mu(dx)$$

Time averages are equivalent to probability averages!

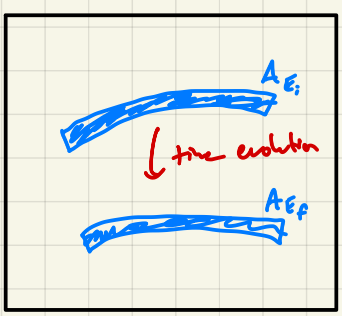
The ergodic hypothesis is, in general, not true. Not all microstates are equally probable after a long time, as we see below.

Thm: (Poincaré Recurrence)

Let T be a measure-preserving transformation on $(\Omega, \mathcal{B}, \mu)$ and $A \subseteq \Omega$ measurable with $\mu(A) > 0$. Then, for a.e. $x \in A$, $\exists\, m \in \mathbb{N}$ s.t. $T^m x \in A$.

For sets of pos. measure, we eventually return to the set.

Boltzmann statistics is based on the equidistribution hypothesis:



$A_E := \{\omega \mid E \leq H(\omega) \leq E + \Delta E\}$ where $\Delta E = o(E)$ is tiny

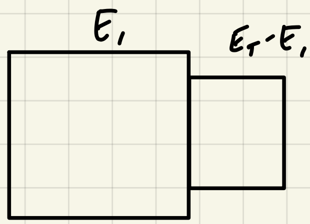
We know that under Hamiltonian mechanics, Liouville measure is preserved.

only μ that's robustly preserved by the evolution

The equilibrium hypothesis states that the system equilibrates to a distribution in Ω with same Liouville measure and the correct E.

Thembodynamics

Fun fact: gravity is not thermodynamically stable



We have two systems in equilibrium states with total energy E_T. When we combine the two systems, the equilibrium energies are such that the total entropy

$$S = \underbrace{k_B \log W_1(E_1)}_{S_1(E_1)} + \underbrace{k_B \log W_2(E_T - E_1)}_{S_2(E_T - E_1)}$$

entropy is additive

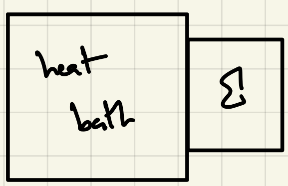
is maximized over E_1.

$$\Rightarrow \text{ in equilibrium, } \quad \left.\frac{dS_1}{dE_1}\right|_{E_1} = \left.\frac{dS_2}{dE_1}\right|_{E_T - E_1}$$

Since we know equality of temperature is the condition for equilibrium, we define the temperature T, in Kelvin, to be

$$\frac{\partial S}{\partial E} = \beta = \frac{1}{k_B T}$$

Consider a heat bath with constant temperature T, and a system Σ.



The variational principle maximizes w.r.t. E_Σ

$$S_\Sigma(V, E_\Sigma, N, \dots) + S_{Bath}(E_{Tot} - E_\Sigma)$$

Since T & β constant for the bath, this equals

$$\approx S_\Sigma(V, E_\Sigma, N, \dots) + \underbrace{S_{Bath}(E_{Tot})}_{\text{constant}} - \beta E_\Sigma$$

So, we in effect maximize Legendre transform!

$$-\beta F(V, \beta, N) = \sup_{E_\Sigma} \left[S_\Sigma(V, E_\Sigma, N) - \beta E_\Sigma \right]$$

F is the Helmholtz free energy, and is the available energy when held to a constant temp. β. So, β is the Legendre transform dual to energy!

If we hold T and P fixed the thermodynamic potential is

pressure

$$-\beta G(p, T, N_1, \dots, N_r) = -\frac{\beta}{k_B} \inf_E \left[E - TS(V, E, N) \right]$$

$$\Rightarrow G(p, T, \mu) = \inf_{E, V} \left[E + pV + \sum_j \mu_j N_j - TS(V, E, N_1, N_2, \dots) \right]$$

This is the Gibbs free energy.

Inverting the Legendre Transform yields

$$\boxed{S(V,E,N) = \frac{-\partial G(T,p,N)}{\partial T}}$$

Places where G has a kink singularity correspond to first-order phase transitions.

Lecture 2/14 -

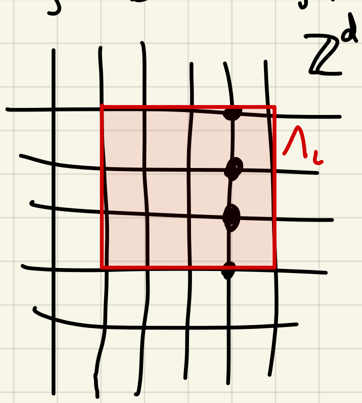
From the definition of $G(\cdot)$, we can write the differential form as

$$dG = -S\,dT + V\,dp + \sum_j \mu_j \, dN_j$$

Statistical Mechanics

We would like to investigate the nature of entropy.

We work with finite-dim graph (meaning as size diverges, the size of boundary is $o(\text{volume})$), say $\mathbb{Z}^d$. This graph is homogenous to translation, and is tilable (by cubes in this case).



$\{\theta_x\}_{x\in\mathbb{Z}^d}$ - each lattice point

Ω_0 - measurable set of outcomes for each θ_x

$\mu_0(d\theta)$ - measure on outcomes in Ω_0

$\Omega_A = \Omega_0^A = \{w : A \to \Omega_0\}$ - all possible lattice configurations of $A \subset \mathbb{Z}^d$

$w = (\theta_x)_{x\in\mathbb{Z}^d}$

$\Lambda_L = \{1,\dots,L\}^d$ - linear box of size L

(In the example of the Ising model, a single spin can take values $-1,0,1$ with equal probability. In this case, $\Omega_0 = \{-1,0,1\}$, and μ assigns equal weight.)

We turn to the extensive energy function, also called the Hamiltonian. As an example, in the Ising model, $\theta_u \in \{-1,1\}$ and

$$H_\Lambda(w) = -\sum_{(u,v)\in\Lambda^2} \underset{\uparrow \text{ interaction term / coupling constant}}{J_{u,v}}\,\theta_u\theta_v - \underset{\uparrow \text{ external field orientation}}{h} \sum_{u\in\Lambda}\theta_u$$

This is an example where energy is given to pairs and singletons.

More generally, $H_\Lambda(\vec{\sigma}) = \sum_{A \subset \Lambda} J_A \, \Phi_A(\vec{\sigma}_A)$ is a framework to describe interactions among all subsets.

We can easily bound by

infinity norm of Φ_A. We assume this $=1$ s.t. J is a good notion of coupling

$$\max_{w \in \Omega_\Lambda} |H_\Lambda(w)| \leq \sum_{x \in \Lambda} \sum_{A \ni x} \frac{1}{|A|} |J_A| \max_{\sigma_A} |\Phi_A(\sigma_A)|$$

$$\Rightarrow \boxed{\|J\| = \sum_{A \ni 0} \frac{1}{|A|} |J_A|} \quad \text{and} \quad \max_{w \in \Omega_\Lambda} |H_\Lambda(w)| \leq |\Lambda| \, \|J\|$$

Now, let us write out the **partition function**

$$\boxed{Z_\Lambda(\beta, J) = \int_{\Omega_\Lambda} e^{-\beta H_\Lambda(\overset{w}{\sigma})} \, \mu(d\sigma)}$$

$\prod_x \mu_0(d\sigma_x)$

Taking the <u>thermodynamic limit</u> $|\Lambda| \gg 1$, we split it into a sum of slices of configuration space sharing the same energy $E = u|\Lambda|$, where u is the energy density.

$$\Rightarrow Z_\Lambda(\beta) \approx \int \underbrace{e^{-\beta u |\Lambda|}}_{\text{integrand}} \; \underbrace{e^{S(u)|\Lambda|}}_{\#\text{ of states w/ energy density } u} du = \int e^{\overbrace{[-\beta u + S(u)]}^{\text{convex, has a max}} |\Lambda|} du$$

Legendre transform

$$\Rightarrow \underbrace{\frac{1}{|\Lambda|} \log Z_\Lambda}_{\text{can be thought of as thermodynamic pressure}} \xrightarrow[|\Lambda| \to \infty]{} \max_u [S(u) - \beta u] = -\beta F(\rho)$$

because rate of exponential growth dominates the integral

free energy

<u>Theorem:</u> (Existence of Pressure Function)

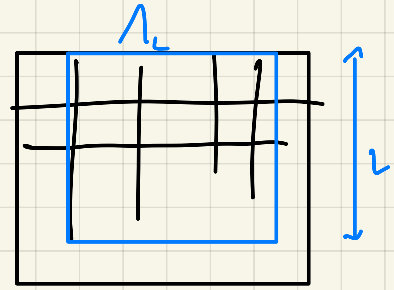
$J_A = J_{A+u}$ and $\Phi_{A+u}((\sigma_x)) = \Phi_A((\sigma_{x-u}))$

For any translation-invariant system, the following limit exists

$$\Psi_\Lambda(\beta, J) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \log Z_{\Lambda_L}(\beta) \qquad (\|J\| < \infty)$$

<u>Important:</u> this links the stat mech construction to things that are useful for thermo!

Lecture 2/16 -



Pressure Function in Thermodynamic Limit

The object of interest is

$$Z_{\Lambda_L}(\beta, \dots) = \int_{\Omega_{\Lambda_L}} e^{-\beta H_{\Lambda_L}(\overset{\text{microstates}}{\omega})} \mu(d\omega)$$

Letting

$$\Psi(\beta, \dots) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \log Z_{\Lambda_L} \qquad (*)$$

we see a "winner takes all" principle, where the ω with the smallest H is most likely to be observed.

We prove that the thermodynamic limit of $(*)$ exists with a box-chopping argument: draw smaller boxes of size k, ignore interaction terms along the boundaries such that energy is additive and μ is multiplicative. Taking $k, L \to \infty$ together, the $\frac{\text{surface area}}{\text{Volume}}$ ratio of the k-boxes converges to 0, and so the sequence

$$\left\{ \frac{1}{|\Lambda_L|} \log Z_{\Lambda_L} \right\}_L$$

is Cauchy and has a limit.

$$\left(\text{In fact, } \quad \frac{1}{|\Lambda_L|} \log Z_{\Lambda_L} \approx \frac{1}{|\Lambda_k|} \log Z_{\Lambda_k} + O\Big(\underbrace{\tfrac{1}{k}}_{\text{surface/vol.}}\Big) \right)$$

This only works for Hamiltonians with short range interactions across the boundary. For the general case, it helps to truncate long-range interactions and bound the error.

Let
$$\|J\| = \sum_{A \ni 0} \frac{1}{|A|} J_A \qquad \text{and} \qquad \overbrace{J_A^{(R)} = J_A \cdot \mathbb{1}_{\{\text{diam}(A) \leq R\}}}^{\text{truncated to } R}$$

We wish to also prove the existence of the thermodynamic limit for $J^{(R)}$.

Now,
$$\left| H_\Lambda(\omega) - H_\Lambda^{(R)}(\omega) \right| \leq \left| \sum_x \sum_{\substack{A \subset \Lambda \\ \text{diam} A > R \\ A \ni x}} \frac{1}{|A|} J_A \, \Phi_A(\sigma_A) \right| \leq |\Lambda| \underbrace{\left\| J - J^{(R)} \right\|}_{\leq \varepsilon}$$

If $\|J\| < \infty$, then $\forall \varepsilon > 0 \quad \exists R > 0$ s.t. $\displaystyle \sum_{\substack{A \ni x \\ \text{diam} A > R}} \frac{1}{|A|} |J_A| < \varepsilon$ (bound the tail w.r.t. R)

So,

$$Z_\Lambda = \int_{\Omega_\Lambda} e^{-\beta H_\Lambda^{(R)}(\omega) - \beta \underbrace{(H_\Lambda - H_\Lambda^{(R)})(\omega)}_{\leq \varepsilon |\Lambda|}} \mu(d\omega)$$

$$\Rightarrow e^{-\beta \varepsilon |\Lambda|} Z_\Lambda^{(R)} \leq Z_\Lambda \leq e^{\beta \varepsilon |\Lambda|} Z_\Lambda^{(R)}$$

$$\Rightarrow \left| \frac{1}{|\Lambda|} \log Z_\Lambda^{(R)} - \frac{1}{|\Lambda|} \log Z_\Lambda \right| \leq \beta \varepsilon$$

Since $\left\{ \frac{1}{|\Lambda_L|} \log Z_{\Lambda_L} \right\}_L$ converge and we can arbitrarily approximate with large enough R, then the finite-interaction approx. also converges!

In fact, we can bound the distance between these limits by

$$\left| \psi(\beta, J) - \psi(\beta, J^{(R)}) \right| \leq \beta \| J - J^{(R)} \|$$

$$\Rightarrow \psi(\beta, J) = \lim_{R \to \infty} \psi(\beta, J^{(R)})$$

However, since the ψ's are convex, the PSET problem 3.1 reveals that the derivatives also converge. This is a very useful property because we can view ψ as a generating function!

mean of H over states

$$\frac{d}{d\beta} \psi_\Lambda(\beta, \ldots) = \frac{-1}{|\Lambda|} \frac{\int H_\Lambda(\omega) e^{-\beta H(\omega)} \mu(d\omega)}{\int e^{-\beta H(\omega)} \mu(d\omega)} = \frac{1}{|\Lambda|} \langle H \rangle_{\Lambda, \beta}$$

variance of H over states

$$\frac{d^2}{d\beta^2} \psi_\Lambda(\beta, \ldots) = \ldots = \frac{1}{|\Lambda|} \left(\langle H^2 \rangle_{\Lambda,\beta} - \langle H \rangle_{\Lambda,\beta}^2 \right) = \frac{1}{|\Lambda|} \langle [H - \langle H \rangle] \rangle$$

$$\vdots$$

In general, $\psi_\Lambda(\beta, \ldots) = \frac{1}{|\Lambda|} \log \int e^{-\beta H_\Lambda(\omega)} \mu(d\omega)$ is the cumulant generating function of H_Λ!

This yields several properties of ψ!

1. $\psi_\Lambda(\beta)$ is convex in β ($\psi_\Lambda'' \geq 0$ since variance ≥ 0)
2. $\psi(\beta)$ is convex in β (pointwise limit is convex)
3. At a.e. β, $\psi(\beta)$ is differentiable and $\lim_{L \to \infty} \frac{\langle H_{\Lambda_L} \rangle_{\Lambda_L, \beta}}{|\Lambda_L|} = \frac{d}{d\beta} \psi(\beta)$

Also, $\int_{\beta_1}^{\beta_2} \overbrace{\psi''(\beta)}^{\geq 0} d\beta = \psi'(\beta_2) - \psi'(\beta_1) \Rightarrow m\left(\left\{ \beta : \overset{?}{\left\langle \left(\frac{H_\Lambda - \langle H_\Lambda \rangle}{\sqrt{n}} \right)^2 \right\rangle} > b \right\} \right) \leq \frac{\beta_2 - \beta_1}{b}$

So, the regions where ψ'' is large are rather small.

Gibbs Equilibrium States

Def: We have **microstates** ω, which are classically configurations in our configuration space Ω and quantumly vectors in the Hilbert space.

Def: **Observables** are classically functions $F(\omega)$ over Ω and quantumly are operators on our Hilbert space.

Def: **States** are expectation-value functionals

$$\rho : F \to \langle F \rangle_\rho, \qquad F \mapsto \int_\Omega F(\omega)\, \rho(d\omega)$$

given by a probability measure on Ω.

So, $\langle F \rangle_\beta = \int_\Omega F(\omega) \underbrace{e^{-\beta H_n(\omega)} \frac{\mu(d\omega)}{Z_n(\beta)}}_{\text{Gibbs measure } \rho(d\omega)}$ is the expectation in the Gibbs canonical ensemble.

The measure $\rho(d\omega) = \frac{e^{-\beta H(\omega)}}{Z(\beta)} \mu(d\omega)$ is a tilted version of the a priori distribution μ over configuration space.

We can generalize this "tilting" via the following measure theory lingo.

Def: Given a (finite) measure space $(\Omega, \mathcal{B}, \mu)$ and a $f : \Omega \to \mathbb{R}$ that is normalized $\left(\int f(\omega)\mu(d\omega) = 1\right)$, then

$\boxed{\rho(d\omega) = f(\omega)\,\mu(d\omega)}$ is a measure and $f = \frac{\delta \rho}{\delta \mu}$ is the **Radon-Nikodym derivative**.

Furthermore, the **entropy** of ρ over μ is given by

$$S(\rho|\mu) = -\int_\Omega f(\omega) \log(f(\omega))\, \mu(d\omega) = -\int \log(f(\omega))\, \rho(d\omega)$$

Recall the general Jensen's inequality:

Theorem (Probability Jensen):

For a measure space $(X, \mathcal{B}, \mu)$ of positive measure, any integrable $g: X \to \mathbb{R}$, and any concave $F: \mathbb{R} \to \mathbb{R}$,

$$\int F(g(x))\,\mu(dx) \le F(\langle g \rangle_\mu)\,\mu(X)$$

where $\langle g \rangle_\mu$ is the normalized mean of g and is given by

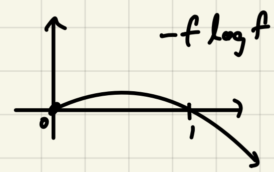
$$\int_X \langle g \rangle_\mu \,\mu(dx) = \int_X g(x)\,\mu(dx)$$

With this, we can prove:

Theorem:

$$S(\rho|\mu) \le 0, \quad \text{with equality iff} \quad f(\omega) = \frac{1}{\int \mu(d\omega)}$$

Proof: The Jensen inequality on $\overbrace{g(f) = -f \log f}^{\text{concave}}$ leads



$$S(\rho|\mu) = \int g(f(\omega))\,\mu(d\omega) \le g\Big(\underbrace{\int f(\omega)\,\mu(d\omega)}_{\text{normalized}}\Big) = 0$$

$g(1) = 0$

The bound $S(\rho|\mu) \le 0$ (in fact $\le \log \mu(\Omega_\Lambda)$ for unnormalized) yields a variational characterization of Gibbs states.

Theorem: (Variational Gibbs)

For a finite system with a-priori measure μ and energy function $H_\Lambda(\omega)$, the Gibbs measure $\rho_{\Lambda,\beta}(d\omega) = \dfrac{e^{-\beta H_\Lambda(\omega)}\mu(d\omega)}{Z_\Lambda(\beta)}$ minimizes the state function

$$\mathcal{F}(\rho) := \int_{\Omega_\Lambda} H_\Lambda(\omega)\,\rho(d\omega) - \frac{1}{\beta} S(\rho\|\mu) = \langle H_\Lambda \rangle_\rho - \frac{1}{\beta} S(\rho\|\mu)$$

Note that β controls the weight of the energy minimization and entropy maximization.

Large β (small T) prefer ground states while small β (high T) prefer high entropy!

Pressure as Gibbs Measure's Generating Function

Recall the pressure function given by

$$\psi_\Lambda(\beta, J) := \frac{1}{|\Lambda|} \log Z_\Lambda(\beta, J)$$

We saw that

expected energy density

$$\frac{\partial}{\partial \beta} \psi_\Lambda(\beta, J) = -\frac{1}{|\Lambda|} \langle H_\Lambda \rangle_{\rho_{\beta,\Lambda}} \quad \text{and} \quad \frac{\partial^2}{\partial \beta^2} \psi_\Lambda(\beta, J) = \frac{1}{|\Lambda|} \operatorname{Var}(H_\Lambda)_{\rho_{\beta,\Lambda}} \geq 0$$

More generally, $\forall A \subseteq \Lambda$ we have

$$\frac{\partial}{\partial J_A} \psi_\Lambda(\beta, J) = -\frac{\beta}{|\Lambda|} \left\langle \frac{\partial}{\partial J_A} H_\Lambda \right\rangle_{\rho_{\beta,\Lambda}}$$

Convexity arguments can also give

$$\lim_{L \to \infty} \frac{1}{|\Lambda_L|} \langle H_{\Lambda_L} \rangle_{\rho_{\beta,\Lambda_L}} = -\frac{\partial}{\partial \beta} \psi(\beta, J)$$

"limiting expected energy density" is derivative of $\psi(\beta)$

This holds true for <u>all choices of b.c.'s</u> for the finite volumes Λ_L.

However, for β at which $\frac{\partial}{\partial \beta} \psi(\beta, J)$ is discontinuous, values of $\frac{1}{|\Lambda_L|} \langle H_\Lambda \rangle_{\rho_{\beta,\Lambda}}$ <u>depend on boundary conditions</u>, yielding a first-order phase transition!

For such β's, the range of observable energy densities collapses as $L \to \infty$ to the interval

$$-\left[\frac{\partial}{\partial \beta} \psi(\beta+0, J), \frac{\partial}{\partial \beta} \psi(\beta-0, J) \right]$$

Concentration of Measure

(google: (1) Cramér large deviation expansion for martingales
(2) Donsker-Varadhan theory of large deviations)

Theorem: (Concentration of energy density)

For any extensive system with Hamiltonian of the form

$$H_\Lambda(\sigma) = \sum_{A \subseteq \Lambda} J_A \Phi_A(\sigma) = \sum_{x \in \Lambda} \left(\sum_{A \ni x} \frac{1}{|A|} J_A \Phi_A(\sigma) \right)$$

for each $\beta < \infty$ there are functions of the form $\delta_{\beta,\pm}$ s.t. $\forall \varepsilon > 0$, at large enough L we have

b.c.

$$\mathbb{P}^{\#}_{\rho_{\beta,\Lambda_L}} \left\{ \frac{1}{|\Lambda_L|} H^{\#}_{\Lambda_L} \le -\frac{\partial \psi}{\partial \beta}(\beta+0, J) - \varepsilon \right\} \le e^{-\delta_{\beta,+}(\varepsilon)|\Lambda_L|}$$

$\mathbb{P}$ over Gibbs-tilted a-priori measure

b.c.

$$\mathbb{P}^{\#}_{\rho_{\beta,\Lambda_L}} \left\{ \frac{1}{|\Lambda_L|} H^{\#}_{\Lambda_L} \ge -\frac{\partial \psi}{\partial \beta}(\beta-0, J) + \varepsilon \right\} \le e^{-\delta_{\beta,-}(\varepsilon)|\Lambda_L|}$$

"probability of energy density deviation is exponentially small in volume"

Lecture 2/23-

Recap

Recall Gibbs states given by measure $\rho_{\Lambda,\beta}(d\omega) = \dfrac{e^{-\beta H_\Lambda(\omega)}\mu_\Lambda(d\omega)}{Z_\Lambda(\beta,\dots)}$

If we define Free Energy to be

$$\mathcal{F}_\beta(\rho) = \int_\Omega H_\Lambda(\omega)\rho(d\omega) - \frac{1}{\beta}\overbrace{S(\rho\|\mu)}^{\text{entropy}} = \langle H_\Lambda\rangle_\rho - \frac{1}{\beta}S(\rho\|\mu)$$

A minimizer of $\mathcal{F}$ minimizes $\langle H_\Lambda\rangle_\rho - \frac{1}{\beta}S(\rho\|\mu)$, or equivalently it maximizes

$$S(\rho\|\mu) - \beta\langle H_\Lambda\rangle_\rho = -\int_\Omega \log\left(\frac{\delta\rho}{\delta\mu}(\omega)\right)\frac{\delta\rho}{\delta\mu}(\omega)\mu(d\omega) - \langle H_\Lambda\rangle_\rho$$

$$= \dots$$

$$= S(\rho\|\rho_\beta) \quad + \text{constant?}$$

Gibbs measure

Since $S(\rho\|\rho_\beta) \le 0$ with equality iff $\rho = \rho_\beta$, we see that ρ_β maximizes this.
Equivalently,

$$\boxed{\text{Gibbs states are the minimizers of } \mathcal{F}!}$$

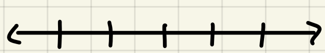
Note that β controls the relative weights of energy $\langle H_\Lambda\rangle_\rho$ and entropy $S(\rho\|\mu)$ in this optimization. Temperature controls the balance between energy & entropy!

Phase Transitions

Consider an Ising model on $\mathbb{Z}^d$, where $\forall x \in \mathbb{Z}^d$, $\sigma_x \in \{-1,+1\}$, with an energy given by $H_\Lambda = -\frac{J}{2}\sum_{u\sim v}\sigma_u\sigma_v$

neighbors

1D Ising Model:



$x \in \mathbb{Z}$

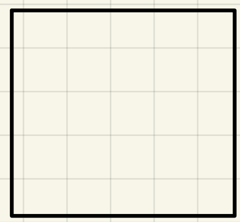
Note that as $T \to 0$, we only want to minimize H_Λ, and so there are two ground states: $++++++$ and $------$ (this is an example of discrete symmetry breaking, where a symmetry of the Hamiltonian (spin flip) leads to multiple distinct states).

We can show that for $T > 0$, there is no phase transition. We can manufacture a Markov chain where each flip is $\sim$ Bernoulli(p), and the length of that flip is $\sim$ Exponential(μ): there is no phase transition in 1D!

Evan: go over this proof?

2D Ising Model:

We'd like to study the infinite limit. First, though, let's discuss things for a finite volume.



$\Omega_0 = [-L, L]^2 \cap \mathbb{Z}^2$

$\sigma_x \in \{-1, +1\}$

$H = -\frac{J}{2} \sum_{u \sim v} \sigma_u \sigma_v$

We expect majority to be the same sign, with some occasional clusters of flips.

Q: What would symmetry breaking look like?

A: We ask two questions:

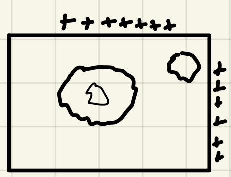
(1) if we apply external magnetic field $h \in \{\pm 1\}$, does the system $\mu_\beta(h)$ have a discontinuity

$\Longleftrightarrow$

(2) do the boundary conditions (+ or − along boundary of box) affect μ_β in the interior as $L \to \infty$?

We work with the second of these two formulations.

Def: A Peierls contour is a closed path on $\mathbb{Z}^2$ s.t. the spins on its interior are the same, and are opposite the spins on the exterior.



Theorem: For the Ising model on $\mathbb{Z}^2$, there $\exists$ a β_c s.t. $\forall \beta > \beta_c$,

$$\mathbb{P}^{(+)\,\leftarrow \text{boundary condition}}_{\beta, L}\{\sigma_0 = -1\} \le p_0 \quad \forall L,$$ where $p_0 < \frac{1}{2}$ doesn't depend on L.

Proof: Let γ = a polygonal path w. $\sigma_x = \begin{cases} + & \text{on outside} \\ - & \text{on inside} \end{cases}$ be a Peierls contour.

We claim that any arbitrary polygonal path has this property with probability $\le e^{-2\beta|\gamma|}$.

To see this, note that

indicator for whether the state σ makes γ a Peierls contour

$$\mathbb{P}^{(+)}_{\beta, L}\{\gamma \text{ satisfies above}\} = \sum_{\sigma \in \Omega} \frac{\mathbb{1}_\gamma\{\sigma\}\, e^{-\beta H_L(\sigma)}}{Z_L}$$

configuration

Let us use a rescaled and shifted Hamiltonian

form a geometric Hamiltonian!

$$H_L(\sigma) = 2J \sum_{(u,v) \text{ pairs}} \left(\frac{1-\sigma_u \sigma_v}{2}\right) + \text{const} = 2J \sum_{\gamma} \mathbb{1}_\gamma\{\sigma\} + \text{const}$$

s.t. the energy counts the # of deviations from uniform +.

We can say $\quad Z_L \geq \sum_{\sigma \in \Omega} \mathbb{1}_\gamma\{\sigma\} \left[e^{-\beta H_L(\sigma)} + e^{-\beta H_L(R_\gamma \sigma)}\right]$

why?

where $(R_\gamma \sigma)_x = \begin{cases} -\sigma_x & \text{if } x \text{ is inside } \gamma \\ \sigma_x & \text{otherwise} \end{cases}$ is the mapping that flips spins inside γ.

$$\Rightarrow Z_L \geq \left(1 + e^{\beta H_L(\sigma) - \beta H_L(R_\gamma \sigma)}\right) \sum_{\sigma \in \Omega} \mathbb{1}_\gamma\{\sigma\}\, e^{-\beta H_L(\sigma)}$$

Each contour satisfies $H_L(\sigma) - H_L(R_\gamma \sigma) = 2|\gamma|$ by our rewritten Hamiltonian.

$$\Rightarrow Z_L \geq \left(1 + e^{2\beta|\gamma|}\right) \sum_{\sigma \in \Omega} \mathbb{1}_\gamma\{\sigma\}\, e^{-\beta H_L(\sigma)}$$

$$\Rightarrow \boxed{\mathbb{P}^{(+)}_{\beta,L}\{\gamma \text{ satisfies above}\} \leq \frac{1}{1 + e^{2\beta|\gamma|}}}$$

the "energy estimate" of each γ; they are unlikely, but there are many

We can now form the "entropy estimate", or the # of closed polygonal paths encircling the origin (or any interior point).

The number of γ's encircling the origin s.t. $|\gamma| = m$ is $\frac{m}{2} 3^m$, why? and so by a union bound,

$$\mathbb{P}^{(+)}_{\beta,L}\{\sigma_0 = -1\} = \mathbb{P}^{(+)}_{\beta,L}\{\text{a contour encircles } x=0\}$$

$$\leq \sum_{\gamma \text{ encircling } x=0} \mathbb{P}^{(+)}_{\beta,L}\{\gamma \text{ satisfies above}\} \leq \sum_{\gamma \text{ encircles } 0} e^{-2\beta|\gamma|}$$

$$= \sum_{m=4}^{\infty} e^{-2\beta m} \cdot \frac{m}{2} 3^m = \sum_{m=4}^{\infty} \frac{m}{2} \cdot \left(3e^{-\beta}\right)^m$$

length must be ≥ 4 to encircle a point!

we finished this proof on a pset

We just saw that when $\beta > \beta_c$,

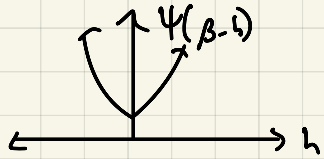
$$\langle \theta_x \rangle^{(+)}_{\Lambda_L,\beta} \geq (1-p_0) - p_0 = 1-2p_0 > 0 \quad \text{and} \quad \langle \theta_x \rangle^{(-)}_{\Lambda_L,\beta} = -\langle \theta_x \rangle^{(+)}_{\Lambda_L,\beta} < 0$$

Also, we had

magnetization

$$m(\beta) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle \theta_x \rangle^{(+)}_{\Lambda,\beta} = \beta \frac{\partial}{\partial h} \psi(\beta, h)$$

$\Rightarrow$



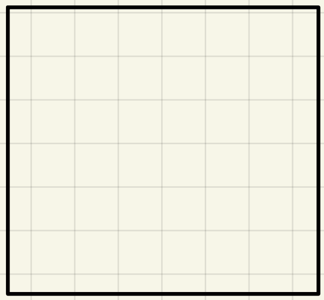
Lecture 2/28 - Continuous Symmetry Breaking

Note that Peierls argument of flip contours no longer works for vector-valued spins. Generalizing from systems with $\sigma_x \in \{-1,1\}$ with global spin flip symmetry, we discuss N-dimensional Ising model with $O(N)$ symmetry and $\vec{\sigma}_x = (\sigma_{x,1}, \dots, \sigma_{x,N}) \in S^N$.

orthogonal N×N matrices

unit N-sphere

O(N)-Symmetric Model


Λ

$$H_\Lambda^{(B.C.)} = -\sum_{(x,y)\in\Lambda^2} J_{x-y}\, \vec{\sigma}_x \cdot \vec{\sigma}_y - \sum_x \vec{h}\cdot\vec{\sigma}_x$$

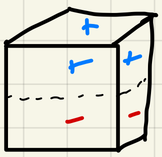
$$= \frac{1}{2}\sum_{(x,y)\in\Lambda^2} J_{x-y} \| \underbrace{\vec{\sigma}_x - \vec{\sigma}_y}_{} \|_2^2 - \sum_x \vec{h}\cdot\vec{\sigma}_x$$

$= 2 - 2\vec{\sigma}_x\cdot\vec{\sigma}_y$

Boundary Conditions

B.C.'s of Λ can be:

- free
- uniform: $\sigma_x = (1,0,\dots)\ \forall x \in \partial\Lambda$
- periodic: In 1D, $\overbrace{H = -\sum_{n=1}^{L-1} \vec{\sigma}_n\cdot\vec{\sigma}_{n+1} - \vec{\sigma}_L\cdot\vec{\sigma}_1}^{\text{this corresponds to gluing the chain into a ring!}}$

These all generate translation-invariant states in the thermodynamic limit! In fact, all B.C.'s in 2D yield translation-invariant states. In 3D, we can construct  which won't be.

$\boxed{\text{We select periodic B.C.'s.}}$

Fourier Transform

For $\Lambda_L = (-\frac{L}{2}, \frac{L}{2}]^d$, let $\Lambda_L^* = (-\pi,\pi]^d \cap \frac{\pi}{L}\mathbb{Z}^d$.

We have the transform and its inverse given by

$$\hat{\sigma}(\vec{p}) = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{\vec{x}\in\Lambda_L} e^{-i\vec{p}\cdot\vec{x}}\, \vec{\sigma}_x \qquad \vec{\sigma}_x = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{p\in\Lambda_L^*} e^{i\vec{p}\cdot\vec{x}}\, \hat{\sigma}(\vec{p})$$

All arbitrary spin configurations can be seen as superpositions of plane waves!

We can verify that they are inverses.

$$\text{RHS} = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{p\in\Lambda_L^*} e^{i\vec{p}\cdot\vec{x}} \frac{1}{\sqrt{|\Lambda_L|}} \sum_{\vec{u}\in\Lambda_L} e^{-i\vec{p}\cdot\vec{u}}\, \vec{\sigma}_u = \sum_{u\in\Lambda_L} \vec{\sigma}_u \overbrace{\frac{1}{|\Lambda_L|} \sum_{p\in\Lambda_L^*} e^{i\vec{p}\cdot(\vec{x}-\vec{u})}}^{=\delta_{x-u}} = \sum_{u\in\Lambda_L} \vec{\sigma}_u\, \delta_{x-u} = \vec{\sigma}_x$$

since if $x \neq u$, $\sum_p e^{ip(x-u)} = e^{ip_1} \sum_p e^{ip(x-u)}$ by shifting $\Rightarrow \sum_p e^{ip(x-u)} = 0$

Suppose that $\vec{h}=0$ (no external field).

Now, we want to write H in terms of $\hat{\theta}(\vec{p})$. Note that $\vec{h}=0 \Rightarrow H_L = (\theta, A\theta) = \sum_{x,y} \vec{\theta}_y^T J_{y-x} \vec{\theta}_x$ is simply a matrix product. Since A is translation-invariant, we know that it is simultaneously diagonalizable with the translation operator. These are the plane waves, i.e. $\varphi_p(\vec{x}) = \frac{1}{\sqrt{|\Lambda_L|}} e^{i\vec{p}\cdot\vec{x}} \Rightarrow \varphi_{\vec{p}}(\vec{x}+\vec{a}) = e^{i\vec{p}\cdot\vec{a}}\varphi_{\vec{p}}(\vec{x})$

Consider the function $\psi(x)=\vec{\theta}_x \Rightarrow |\psi\rangle = \sum_{p\in\Lambda_L^*} \langle \varphi_p|\psi\rangle |\varphi_p\rangle$ (eigenfunctions of H) (project to orthonormal basis $\{|\varphi_p\rangle\}_p$ of L^2)

Then, $H = \langle\psi|\hat{H}|\psi\rangle = -\frac{1}{2}\langle\psi|J|\psi\rangle = -\frac{1}{2}\sum_{p,p'}\langle\psi|\varphi_p\rangle \underbrace{\langle\varphi_p|J|\varphi_{p'}\rangle}_{\delta_{p-p'}\langle J\rangle_{\varphi_p}} \langle\varphi_{p'}|\psi\rangle = \sum_p |\langle\psi|\varphi_p\rangle|^2 \cdot -\frac{1}{2}\langle J\rangle_{\varphi_p}$

Equivalently, if we write H in diagonal form, we get

$$H = -\frac{1}{2}\sum_{x,y\in\Lambda_L} \vec{\theta}_x^T J_{x-y}\vec{\theta}_y \qquad J_{x-y} = \begin{cases} 1 & \|x-y\|=1 \\ -2d & \|x-y\|=0 \\ 0 & \text{otherwise}\end{cases}$$

$$= -\frac{1}{2}(\theta, J\theta)$$

J is basically a discrete version of the Laplacian (discrete differences)

We have $$J\varphi_{\vec{p}}(\vec{x}) = \frac{1}{\sqrt{|\Lambda_L|}} e^{i\vec{p}\cdot\vec{x}} \overbrace{\sum_{\|\vec{u}\|=1}\left(1-e^{i\vec{p}\cdot\vec{u}}\right) J_{\vec{u}}}^{\xi(\vec{p})}$$

Computing,

$$\xi(\vec{p}) = \frac{1}{2}\sum_{\|\vec{u}\|=1}\left(1-e^{i\vec{p}\cdot\vec{u}}\right) = -\frac{1}{2}\sum_{j=1}^{d} 2(1-\cos(p_j)) = 2\sum_{j=1}^{d}\sin^2\left(\frac{p_j}{2}\right)$$

For small p_j's, this behaves like $\xi(\vec{p}) \approx \frac{1}{2}\sum_j p_j^2 = \frac{1}{2}\|\vec{p}\|^2$ (kinetic energy is $\frac{p^2}{2}$!)

shaky stuff. his "nose was too close to the blackboard"

The energy is the sum of the energies of the plane waves, yielding

$$H_L = \sum_{p\in\Lambda_L^*} \underset{-\frac{1}{2}\langle J\rangle_{\varphi_p}}{\mathcal{E}(\vec{p})}\, \underset{\langle\psi|\varphi_p\rangle}{|\hat{\theta}(\vec{p})|^2} - h\sqrt{|\Lambda_L|}\,\hat{\theta}_0 \text{ if } h\neq 0 \quad \text{with} \quad \mathcal{E}(\vec{p}) = \sum_{u\in\Lambda_L} e^{i\vec{p}\cdot\vec{u}} J_{\vec{u}}$$

F.T. of coupling energy

This agrees with our bra-ket stuff! In total, we get that the energy decomposes to the sum of plane wave energies! We also know that if $\xi(p) = \frac{p^2}{2m}$, $\langle \xi(p)\rangle = \frac{1}{2}k_B T$

Now, Parseval (f and $\hat{f}$ have same L^2 norm) yields

$$1 = \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}|\theta_x|^2 = \frac{1}{|\Lambda_L|}\sum_{p\in\Lambda_L^*}|\hat{\theta}(\vec{p})|^2$$

$$\Rightarrow 1 = \frac{1}{|\Lambda_L|}\sum_{p\in\Lambda_L^*}\langle|\hat{\theta}(\vec{p})|^2\rangle \approx \underbrace{\frac{1}{2}\frac{1}{|\Lambda_L|}\sum_{p\neq 0}\frac{1}{|p|^2}\langle\|\vec{\hat{\theta}}(p)\|^2\xi(\vec{p})\rangle}_{\sim \int_{[-\pi,\pi]^d} dp \frac{1}{|p|^2}\text{, diverges if } d=2} + \frac{1}{2|\Lambda_L|}\langle\|\vec{\hat{\theta}}(0)\|^2\rangle$$

Lastly, note that the Fourier transform of spin-spin correlation function reappears as

$$\hat{S}_\Delta^{(L)}(\vec{p}) := \sum_{x\in\Lambda_L} e^{i\vec{p}\cdot\vec{x}} S_\Delta^{(L)}(\vec{x}) = \sum_{x\in\Lambda_L} e^{i\vec{p}\cdot\vec{x}}\langle\vec{\theta}_0\cdot\vec{\theta}_x\rangle_{\Lambda_L} = \langle\|\vec{\hat{\theta}}(p)\|^2\rangle_{\Lambda_L}$$

intensity of the p^{th} mode

Symmetry Breaking as a Condensation Phenomenon

The above reasoning, together with the equipartition law, allow us to give a sufficient condition for symmetry breaking much akin to condensation into macroscopic occupation of the ground state (a lá Bose-Einstein Condensation).

Prop 8.1:

needed for local integrability of $\frac{1}{|\vec{p}|^2}$

Let $d \geq 2$. Suppose that in a system of bounded spins with nearest-neighbor interaction, we have the Gaussian domination bound

$$\mathcal{E}(p)\, \hat{S}^{(L)}_{\Lambda,\beta}(p) \leq \frac{1}{2\beta} \qquad \forall p$$

Define

$$C_d := \frac{1}{(2\pi)^d} \int_{[-\frac{\pi}{2},\frac{\pi}{2}]^d} \frac{1}{\mathcal{E}(\vec{p})}\, d\vec{p}$$

Then, $\forall \beta > C_d/2$, the following hold

(i) $\liminf\limits_{L\to\infty} \left\langle \left\| \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \theta_x \right\|^2 \right\rangle \geq 1 - C_d/2\beta$ (expected magnitude of bulk magnetization increases with β)

(ii) at $\vec{h}=0$, $\Psi(\beta,\vec{h})$ has discontinuous derivative (cone singularity) (phase transition!)

(iii) in the infinite limit, the system has Gibbs states of nonzero magnetization, i.e. the spin-rotation symmetry is broken

Proof: (i) Parseval-Plancherel yields that when spins are unit magnitude ($\|\vec{\theta}(x)\|=1 \ \forall x$),

$$\underbrace{\frac{1}{|\Lambda_L|}\|\hat{\theta}(0)\|^2}_{=\left\|\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\vec{\theta}(x)\right\|^2} + \frac{1}{|\Lambda_L|}\sum_{p\in\Lambda_L^*\setminus\{0\}} \|\hat{\vec{\theta}}(p)\|^2 = 1$$

just like how $\hat{f}(0)$ is avg. value of f

Taking an expectation,

$$\left\langle \left\|\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\vec{\theta}(x)\right\|^2\right\rangle = 1 - \frac{1}{|\Lambda_L|}\sum_{p\in\Lambda_L^*\setminus\{0\}}\langle\|\hat{\vec{\theta}}(p)\|^2\rangle = 1 - \frac{1}{|\Lambda_L|}\sum_{p\in\Lambda_L^*\setminus\{0\}}\hat{S}^{(L)}_{\beta}(p)$$

The Gaussian domination bound leads

$$\geq 1 - \frac{1}{2\beta}\cdot\underbrace{\left[\frac{1}{|\Lambda_L|}\sum_{p\in\Lambda^*\setminus\{0\}}\frac{1}{\mathcal{E}(p)}\right]}_{\text{Riemann approx. for } C_d} \xrightarrow{L\to\infty} 1 - \frac{1}{2\beta}C_d$$

(ii) From (i), we know that $\exists B>0$ s.t. $\forall L$ large enough and all b.c.'s,

$$\left\langle \left\| \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \vec{\theta}(x) \right\|^2 \right\rangle^{(b.c.)}_{\Lambda_L,\beta,\vec{h}=0} \geq B^2$$

The finite-volume pressure function satisfies (with $\vec{h}=(1,0,\dots)$) ↙ suffices to consider this by rotational symmetry

$$e^{(\psi(\beta,\vec{h})-\psi(\beta,0))|\Lambda_L|} = \left\langle e^{\beta\vec{h}\cdot\sum_{x\in\Lambda_L}\vec{\theta}_x} \right\rangle \geq e^{\beta\|\vec{h}\||\Lambda_L|B(1-\epsilon)}\,\mathbb{P}\left\{\sum_{x\in\Lambda_L}\theta_x^{(1)} \geq B(1-\epsilon)\right\}$$

A Chebyshev-type estimate gives

$$B^2 \leq \left\langle \left| \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\theta_x^{(1)} \right|^2 \right\rangle_{\vec{h}=0} \leq B^2(1-\epsilon)^2 + B^2\,\mathbb{P}\left\{\sum_{x\in\Lambda_L}\theta_x^{(1)} \geq B(1-\epsilon)\right\}$$

$$\Rightarrow e^{(\psi(\beta,\vec{h})-\psi(\beta,0))|\Lambda_L|} \geq e^{\beta\|\vec{h}\||\Lambda_L|B(1-\epsilon)} \cdot \frac{B^2(1-(1-\epsilon)^2)}{B^2}$$

$$\overset{?}{\Longrightarrow} \psi(\beta,\vec{h})-\psi(\beta,0) \geq \beta B\|\vec{h}\|$$ (Lemma 8.2 in notes)

In particular, this implies a conical singularity at $\vec{h}=0$.

(iii) As always, discontinuous derivative of $\psi \Rightarrow$ symmetry breaking. More explicitly, we have the relation

$$\left\langle \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\vec{\theta}(x) \right\rangle = \frac{1}{\beta}\vec{\nabla}_{\vec{h}}\,\psi_L(\beta,\vec{h})$$

When $\vec{\nabla}_{\vec{h}}\psi$ has different directional derivatives (singularity), to each direction corresponds at least one translation-invariant Gibbs state ρ for which $\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\vec{\theta}(x) \overset{\rho\text{-a.s.}}{=}$ the value of $\vec{\nabla}_{\vec{h}}\psi$ in this direction.

Each such ρ exhibits rotational symmetry breaking.

□

The condition $\mathcal{E}(p)\,\hat{S}^{(i)}_{\rho,\beta}(p) \leq \frac{1}{2\beta}$ is in general not well-understood, and we only know it holds for reflection-positive systems.

Lecture 3/7

"what's a factor of 2 among friends?"

Remarks on Symmetry Breaking

If a system's Hamiltonian H and a priori distribution $\mu(d\theta)$ are invariant under a certain transformation $\theta_x \mapsto R(\theta_x)$, this is a symmetry.

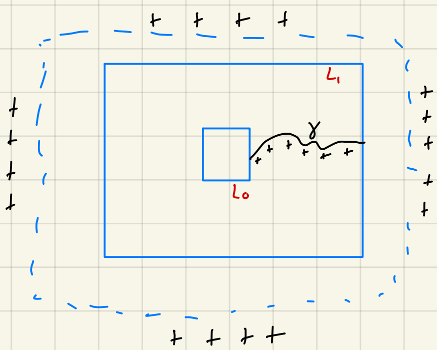
We say that a symmetry is broken if there exist β, an observable F, and a pair of boundary conditions bc_1, bc_2 such that

$$\langle F(\theta)\rangle_\beta^{bc_1} = \lim_{L\to\infty} \langle F(\theta)\rangle^{bc_1}_{\Lambda_L,\beta} \neq \lim_{L\to\infty}\langle F(R\theta)\rangle^{bc_2}_{\Lambda_L,\beta} := \langle F(R\theta)\rangle^{bc_2}_\beta$$

F is asymmetric under R in large volume limit

With $F(\theta)=\theta_0$, this translates to asking whether an interior point remembers far away boundary conditions.

Note: The boundary conditions are very far away and observables we construct to prove symmetry breaking (Peierls argument) is



$1 \ll L_0 \ll L_1$

$$F(\theta) = \mathbb{1}\{\exists \text{ a path with + spins connecting } L_0 \text{ and } L_1\}$$

$$F(\theta) = \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} \vec{\theta}(x) \quad \text{for continuous case}$$

Back to Continuous Symmetry Breaking

Def: A vector space $\mathcal{H}$ (over $\mathbb{C}$) is a Hilbert space if it has a positive inner product $\langle\cdot,\cdot\rangle$ s.t. $\forall f,g,h \in \mathcal{H}$,

(i) $\langle f,g\rangle = \overline{\langle g,f\rangle}$ (ii) $\langle h, f+ag\rangle = \langle h,f\rangle + a\langle h,g\rangle$ (iii) $\langle f,f\rangle \geq 0$

Theorem: (Schwartz Inequality)

$$\forall f,g \in \mathcal{H}, \quad \langle f,g\rangle \leq \langle f,f\rangle^{\frac{1}{2}} \cdot \langle g,g\rangle^{\frac{1}{2}}$$

Proof: $\forall \lambda, \quad \langle \lambda f + g, \lambda f + g \rangle \geq 0 \Rightarrow |\lambda|^2 \underbrace{\langle f,f\rangle}_{A^2} + \underbrace{\lambda\langle f,g\rangle + \bar{\lambda}\langle g,f\rangle}_{=2\,\mathrm{Re}(\lambda \underbrace{\langle f,g\rangle}_{C})} + \underbrace{\langle g,g\rangle}_{B^2} \geq 0$

$$\Rightarrow \forall \lambda, \quad A^2\left(|\lambda|^2 + 2\lambda \frac{C}{A^2}\right) + B^2 \geq 0 \Rightarrow A^2\left(\lambda + \frac{C}{A^2}\right)^2 - A^2\frac{C^2}{A^4} + B^2 \geq 0$$

Letting $\lambda = -\frac{C}{A^2}$, $\quad -\frac{C^2}{A^2} + B^2 \geq 0 \Rightarrow |C| \leq \sqrt{AB}$

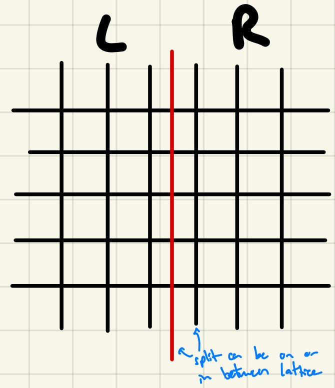
$\square$

Def: Let $\mathcal{H}_+ = \{ f : f \in \mathcal{B}_+ \text{ w. } \mathbb{E}[f^2] < \infty \}$

be a Hilbert space of measurable functions that depend only on states on the right side of a split, (generalize w/ half-spaces and hyperplanes) with inner product

reflection operator, Gibbs measure

$$\langle F, G \rangle = \int \bar{F}(\theta)\, \mathcal{R}\, G(\theta)\, \rho_{\Lambda_L}(d\theta) = \mathbb{E}[\bar{F}\mathcal{R}G]$$



We say a system has reflection positivity about a reflection $\mathcal{R}$ iff $\forall F, G : \Omega \to \mathbb{C}$ observables in $\mathcal{B}_+$,

$$\langle F, F \rangle = \mathbb{E}[\bar{F}\mathcal{R}F] \geq 0 \quad \text{and} \quad \langle F, G \rangle = \overline{\langle G, F \rangle}$$

To characterize which systems have RP, a sufficient condition is

Prop 8.4:

A sufficient condition for the Gibbs states to be RP w.r.t. a reflection $\mathcal{R}$ is that its Hamiltonian can be written as

$$-H = A + \mathcal{R}A + \sum_{j=1}^{K} B_j \mathcal{R} B_j \quad \text{where } A, B_j \in \mathcal{H}_+ \text{ depend only on spins on one side of } \mathcal{R}$$

Proof: check the notes :) $\square$

Some other conditions for reflection positivity and examples of long-range RP interactions are presented in Friedli/Velenik and Fröhlich/Zegarlinski:

For $1 \leq d \leq 4$, this class includes two-body spin-spin interactions with power-law decay like $J_{x-y} = \frac{1}{\|x-y\|^{\tau}}$, $\tau \geq |d-2|_+$

The Chessboard Inequality

Note first that Schwartz + RP gives that $\forall F, G \in \mathcal{H}_+$,

$$\mathbb{E}[\bar{F}\,\mathcal{R}G] \leq \mathbb{E}[\bar{F}\mathcal{R}F]^{\frac{1}{2}}\ \mathbb{E}[\bar{G}\mathcal{R}G]^{\frac{1}{2}}$$

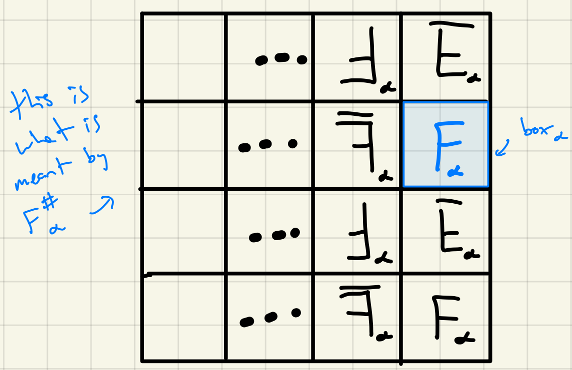
Consider a reflection $\mathcal{R}$, and let $\mathcal{B}_\pm$ be the collection of functions depending only on spins in $\Lambda_\pm$ (each side of $\mathcal{R}$); i.e. $\mathcal{B}_\pm$ are functions measurable w.r.t. $\sigma(\Omega_{\Lambda_\pm})$.

Then, $F_+ \in \mathcal{B}_+$ and $F_- \in \mathcal{B}_-$ gives through CS+RP that, since $F_- = \mathcal{R}G$ for some $G \in \mathcal{B}_+$,

$$|\mathbb{E}[F_+F_-]|^2 = |\mathbb{E}[F_+\mathcal{R}G]|^2 \leq \mathbb{E}[F_+\overline{\mathcal{R}F_+}]\,\mathbb{E}[\bar{G}\mathcal{R}G]$$
$$= \mathbb{E}[\bar{F}_+\mathcal{R}F_+]\ \mathbb{E}[\overline{\mathcal{R}F_-}\,F_-] = \mathbb{E}[\bar{F}_+\mathcal{R}F_+]\ \mathbb{E}[\bar{F}_-\mathcal{R}F_-]$$

Suggestively written, this means that the expectation of a product $F_+(\sigma_+)F_-(\sigma_-)$ is bounded by the geometric mean of $\mathbb{E}[\bar{F}_\pm \mathcal{R}F_\pm]$, done by reflecting and conjugating throughout both domains $\Omega_{\Lambda_\pm}$. Generalizing to more domains,

Theorem (Chessboard Inequality)



If a spin system in Λ is RP w.r.t. a family of reflections across perpendicular hyperplanes that divide Λ into almost-disjoint boxes $\Lambda = \bigcup_{\alpha=1}^{N} \Lambda_\alpha$, then $\forall$ functions $\{F_\alpha\}$ with $F_\alpha \in \mathcal{B}_\alpha$, ← only depends on things in box Λ_α

$$\left|\mathbb{E}\left[\prod_\alpha F_\alpha(\sigma_\alpha)\right]\right| \leq \prod_\alpha \underbrace{\mathbb{E}\left[\prod_\gamma F^\#_\alpha(\sigma_\gamma)\right]}_{\geq 0 \text{ by RP}}{}^{1/\#\text{ boxes}}$$

"The product is dominated by what happens if you take them one at a time and duplicate them everywhere."

Proof: WOLOG (by scaling), suppose that $\mathbb{E}[\prod_\gamma F^\#_\alpha(\sigma_\gamma)] = 1 \quad \forall \alpha$.

This reduces the task to proving the following statement:
Let $\mathcal{S} = \{F_j\}_{j=1,...,K} \subset \mathcal{B}_{\alpha_0}$ be a collection of functions measurable in a common box Λ_0, each normalized by (8.38), and let $\kappa : \{1, ..., K\} \to \{1, ..., K\}$ represent assignments of functions from $\mathcal{S}$ to the cells. Then the following maximum

$$\max_{\kappa:\{1,...,K\}\to\{1,...,K\}} \left|\langle \prod_\alpha F^\sharp_{\kappa(\alpha)}(\sigma_\alpha)\rangle\right| \tag{8.39}$$

(which need not be unique) is attained by a configuration for which $\kappa(\alpha)$ is constant.

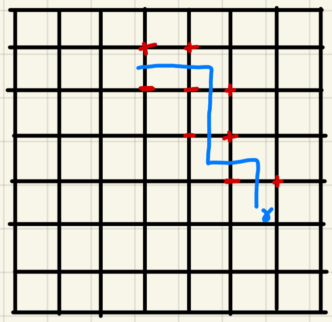
By the Cauchy-Schwarz inequality if κ is maximizer then so is each of the two configurations which are obtained by symmetrizing κ with respect to an arbitrarily chosen reflection plane. Such reflections can be used to decrease the amount of disagreement in the nearest neighbor assignments while staying within the collection of optimizing assignments. The only maximizing configurations whose nearest neighbor disagreement cannot be further reduced corresponds to κ such that $\kappa(\alpha) = \kappa(\alpha')$ for each pair of neighboring boxes. This condition implies that among the maximizer there is one for which $\kappa(\alpha)$ takes a common value for all α, and the claim follows. $\square$

?? huh??

"Read FSS and FILS for more"

We can gain some intuition behind the chessboard inequality by using it to derive a Peierls-type estimate $\mathbb{P}\{\gamma\} \le e^{-\beta|\gamma|}$

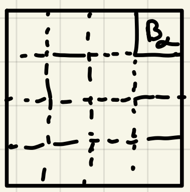
Consider an Ising model with periodic b.c.'s



If γ is a Peierls contour, then we have many bonds between − and +, as shown. We wish to bound this probability

Suppose WLOG that γ has more vertical than horizontal bonds across it.

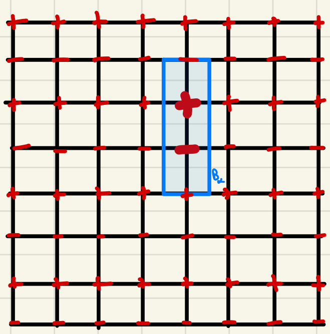
Recall the chessboard inequality:



For observables F_α and boxes B_α, we have

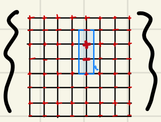
$$\mathbb{E}\left[\prod_{\alpha\in A} F_\alpha(\theta_{B_\alpha})\right] \le \prod_{\alpha\in A} \mathbb{E}\left[\prod_{\alpha'\in A} F^{\#}_\alpha(\theta_{B_{\alpha'}})\right]^{\frac{1}{|A|}}$$

Now, consider a vertical bond between a (+, −) pair.



Since the system is reflection positive, reflections along horizontal hyperplanes along the lattice and vertical ones between the lattice will duplicate this setup as shown.

This is a bulk effect, for which we can compute

$\mathbb{P}\Big\{$  $\Big\}$ $= \dfrac{e^{-\beta H(\theta_{+/-})}}{e^{-\beta H(\theta_{+})}{}_{+\dots}} \le e^{-\beta[H(\theta_{\pm}) - H(\theta_{+})] - 2|\Lambda|}$

$\le e^{-2\beta|\Lambda|}$

$$F_\alpha(\theta_{B_\alpha}) := \begin{cases} 1 & \text{if } B_\alpha \text{ disjoint from } \gamma \\ \mathbb{1}\left[\begin{smallmatrix}+\\|\\-\end{smallmatrix}\right] & \text{else} \end{cases}$$

$$\Rightarrow \mathbb{E}\Big[\overbrace{\prod_{\alpha'\in A} \underset{\substack{\uparrow\\ F_\alpha \text{ reflected}\\ \text{all over the}\\ \text{system}}}{F^{\#}_\alpha}\big(\underset{\substack{\uparrow\\ \text{different}\\ \text{boxes}\\ B_{\alpha'} \text{ of}\\ \text{the system}}}{\theta_{B_{\alpha'}}}\big)}^{\text{probability that whole system looks like } F_\alpha\text{'s reflected stuff}}\Big]^{\frac{1}{|A|}} \le \left(e^{-2\beta|\Lambda|}\right)^{\frac{1}{|\Lambda|}} = e^{-2\beta} \overset{\text{Chessboard ineq.}}{\Rightarrow} \mathbb{E}\left[\prod_{\alpha\in A} F_\alpha(\theta_{B_\alpha})\right] \le \prod_{\alpha\in A} e^{-2\beta}$$

Focusing only on $\{\alpha : B_\alpha \text{ intersects } \gamma\}$ since F_α is $\equiv 1$ elsewhere,

$$\mathbb{P}\left\{\exists \text{ all vertical } \begin{smallmatrix}+\\|\\-\end{smallmatrix} \text{ bonds for } \gamma\right\} \le e^{-2\beta|\gamma|/2} \overset{\text{we supposed more than half of the bonds across } \gamma \text{ were vertical!}}{=} e^{-\beta|\gamma|}$$

□

The Gaussian Domination Bound

Working once again in an $O(N)$ spin model with

$$H_\Lambda^{per}(\sigma) = \frac{1}{2} \sum_{\{x,y\} \subseteq \Lambda} J_{x-y} \|\sigma_x - \sigma_y\|^2 \quad \left(-\vec{h} \cdot \sum_{x \in \Lambda} \vec{\sigma}_x\right)$$

we suppose further that corresponding Gibbs states are RP (this is automatically the case for such models)

Recall our partition function

$$Z_{\Lambda_L} = \sum_{\sigma \in \Omega_{\Lambda_L}} e^{-\frac{\beta}{2} \sum_{x-y} J_{xy} (\sigma_x - \sigma_y)^2}$$

Consider a modified partition function

$$Z_{\Lambda_L}(z) = \sum_{\sigma \in \Omega_{\Lambda_L}} e^{-\frac{\beta}{2} \sum_{x-y} J_{xy} ((\sigma_x + z_x) - (\sigma_y + z_y))^2}$$

Here, z denotes a stress/biasing of the spins at each site.

<u>Theorem:</u>

$$\forall z, \quad Z_\Lambda(z) \leq Z_\Lambda$$

translate measure!

<u>Proof:</u> $$\frac{Z_\Lambda(z)}{Z_\Lambda} = \int_{\mathbb{R}^{|\Lambda|}} e^{-\frac{\beta}{2} \sum (\sigma_x - \sigma_y)^2} \frac{\prod_x (T_{z_x} \mu_0)(d\sigma_x)}{Z_\Lambda}$$

$$= \mathbb{E}\Big[\prod_{x \in \Lambda_L} \delta$$

finish proof that $O(N)$ models over $\mathbb{Z}^d$ with interactions of the n.n. form (which are RP) satisfy

$$\hat{S}^{(L)}_{\Lambda,\beta}(\vec{p})\, \mathcal{E}(\vec{p}) \leq \frac{1}{2\beta}$$

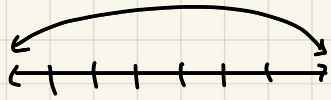
Lecture ??? - Transfer Matrices

Finish this

Corrections from 4/4 -

"as the Chinese proverb says, there are a dozen ways to cook rice"

Consider a system of spins with periodic b.c.'s :



in 1D

Writing $T_{\theta\theta'}$ as a transfer matrix and diagonalizing

$$T = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_k\end{pmatrix} \Rightarrow T^L = \begin{pmatrix}\lambda_1^L & & \\ & \ddots & \\ & & \lambda_k^L\end{pmatrix}$$

let $\lambda_1 > \lambda_j \quad \forall j \geq 2$

and so for $k=2$ with n.n. interactions,

$$Z_L^{per} = \sum_{\theta_1,\dots,\theta_n} \qquad = \operatorname{tr}(T^L) = \sum_{i=1}^{k} \lambda_i^L \overset{k=2}{=} \lambda_1^L\left(1+e^{-\alpha L}\right)$$

and

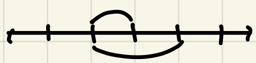
for periodic b.c.'s, $d(x,y) = \min\{|x-y|, L-|x-y|\}$

$$\langle\theta_x;\theta_y\rangle_L^{per} := \langle\theta_x\theta_y\rangle_L^{per} - \langle\theta_x\rangle_L^{per}\langle\theta_y\rangle_L^{per} = |\langle\varphi_1|S|\varphi_2\rangle|^2 e^{-\alpha d(x,y)}\left[1+O\left(e^{-\frac{\alpha L}{2}}\right)\right]$$

truncated correlator

connected correlator decays exponentially fast!

If we no longer have n.n. interactions, such as



In this case, we define $\tau_m := (\theta_m^{(1)}, \theta_m^{(2)})$ and the transfer matrix

$$T_{m,m+1} = \mathbb{1}\left[\tau_m^{(2)}, \tau_{m+1}^{(1)}\right] e^{-\beta V_{m,m+1}} \Rightarrow T \in \mathbb{R}^{4\times 4} \Rightarrow T^2 \text{ is positive?}$$

(ensures the overlapping windows make sense)

Alternatively, we can group each two sites into a single site and writes a more complex $T \in \mathbb{R}^{2\times 2}$

Lecture 3/21 - Infinite-Volume Gibbs States

(If you didn't get the right answer for 4.2, you can do it again and email it to the grader and CC Aizenman.)

Note that upon taking a limit, we both gain and lose information. We may lose boundary conditions, and we may gain translation invariance, etc. So, it makes sense to consider Gibbs states in the infinite-volume limit.

Recall: For finite volumes Λ, Gibbs states form probability measures on Ω_Λ with density

configuration space

$$\mu(d\omega) = \frac{e^{-\beta H_\Lambda(\omega)}}{Z_\Lambda} \underbrace{\mu_\Lambda(d\omega)}_{\text{a-priori measure}}$$

the infinite limit of this doesn't make sense ∴

In the Ising model, $\Omega_\Lambda = \{-1,1\}^\Lambda \Rightarrow \omega \in \Omega_\Lambda$ is a map $\omega : \Lambda \to \{-1,1\}$

In the infinite limit: For the Ising model, $\Omega = \{-1,1\}^{\mathbb{Z}^d}$, where $\omega \in \Omega$ is a map $\omega : \mathbb{Z}^d \to \{-1,1\}$ and $\sigma_x = \omega(x)$.

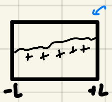
Note that this infinite sequence of binary choices is exactly like how we describe $[0,1]$ via binary expansion. So, $\Omega \cong [0,1]$. Now, let us investigate the topology of Ω.

First, we will need a crash course in some measure theory and conditional probability.

Some measure theory

In the σ-algebra of measurable sets, we most certainly have all local sets - sets which are describable by a local characterization.

examples

- collections of configurations $\{\omega \mid \exists$  $\}$

- any set for which inclusion can be verified by looking at a finite region

So, we can define $\mathcal{B}$ (measurable sets) to be the minimal σ-algebra containing the local sets.

Def: A function $f : \Omega \to \mathbb{R}$ is measurable w.r.t. a σ-algebra $\mathcal{B}$ if and only if $\{\omega \in \Omega : f(\omega) < \lambda\} \in \mathcal{B}$ for all $\lambda \in \mathbb{R}$ (preimages of $f < \lambda$ are measurable)

We can ask the following question:

$$\text{is} \quad \underset{\substack{\text{sequence} \\ (\sigma_x)_x}}{f(\omega)} = \limsup_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \sigma_x \quad \text{measurable?}$$

Note that this function is very nonlocal: no finite set $A \subseteq \Lambda$ determines the value of f. However, we can show that f is measurable!

The condition $\{f(\omega) < \lambda\}$ is equivalent to the event that $\forall \epsilon = \frac{1}{k} > 0$, $\exists N(k)$ s.t. $\forall L \geq N(k)$,
$$\underbrace{\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} \sigma_x < \lambda + \frac{1}{k}}_{\substack{\text{this is a local} \\ \text{condition!} \\ \Rightarrow \in \mathcal{B}}}$$

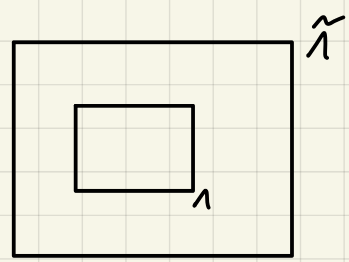
Let $A_{L,k}$ be the set $\quad A_{L,k} := \left\{\omega\in\Omega : \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sigma_x < \lambda + \frac{1}{k}\right\} \in \mathcal{B}$

Then, we can write $\quad A_\lambda := \left\{\omega\in\Omega : f(\omega) < \lambda\right\} = \bigcap_{k=1}^{\infty} \bigcap_{L=N(k)}^{\infty} A_{L,k}$

By closure of $\mathcal{B}$ under countable intersection, $A_\lambda \in \mathcal{B} \Rightarrow f$ is measurable.

Lastly, let us define $\mathcal{B}_\infty := \mathcal{B} \bigcap_{\substack{\Lambda\subseteq\mathbb{Z}^d \\ |\Lambda|<\infty}} \big(\underset{\substack{\text{events measurable} \\ \text{within } \Lambda}}{\mathcal{B}_\Lambda}\big)^c$. In words, $\mathcal{B}_\infty$ denotes the measurable sets that don't depend on any finite region.

Some conditional probability



Consider two finite volumes $\Lambda \subseteq \tilde{\Lambda}$, and define $\Lambda^c := \tilde{\Lambda}\setminus\Lambda$.

Suppose that we'd like to know the distribution of configurations in Λ given the configuration in Λ^c.

Note that $\quad H(\omega_\Lambda, \omega_{\Lambda^c}) = H_\Lambda(\omega_\Lambda ; \omega_{\Lambda^c}) + H_{\Lambda^c}(\omega_{\Lambda^c})$

$H_\Lambda(\omega_\Lambda;\omega_{\Lambda^c})$ contains things inside Λ and interactions between Λ and Λ^c; in a sense, ω_{Λ^c} determines the boundary conditions. We can write out the conditional Gibbs measure

$$\boxed{\rho(d\omega_\Lambda \mid \omega_{\Lambda^c}) = \frac{e^{-\beta H_\Lambda(\omega_\Lambda;\omega_{\Lambda^c})}}{Z_{\Lambda;\omega_{\Lambda^c}}}\,\mu(d\omega_\Lambda)}$$

The above expression led DLR to define the infinite Gibbs measure for $|\tilde{\Lambda}| \to \infty$ as:

Def: An infinite Gibbs state for a Hamiltonian

$$H(\omega) = \sum_{A \subset \mathbb{Z}^d} J_A \, \phi_A(\omega_A)$$

is any probability measure ρ on $(\Omega, \mathcal{B})$ whose finite volume conditional probability is

$$\rho(d\omega_\Lambda \mid \omega_{\Lambda^c}) = \frac{e^{-\beta H_\Lambda(\omega_\Lambda; \omega_{\Lambda^c})}}{Z_{\Lambda; \omega_{\Lambda^c}}} \mu(d\omega_\Lambda)$$

This formulation give a good characterization of symmetry breaking! We say that there is symmetry breaking if there are infinite Gibbs states whose densities don't have symmetries that the system (H, μ) have.

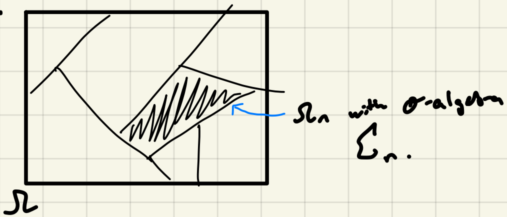
Lecture 3/23 -

Regular Conditional Expectation

Recall from probability theory the following discussion on regular conditional expectation:

In the beginning, we had $\mathbb{P}\{A|B\} := \dfrac{\mathbb{P}\{A \cap B\}}{\mathbb{P}\{B\}}$

More generally, consider a probability space Ω partitioned into finite $(\Omega_n)_n$ that are disjoint. For each n, define Σ_n as its σ-algebra.



Then, we define

$$\mathbb{E}[f \mid \Sigma_n](\omega) := \frac{\int_{\Omega_n} f(\omega)\, \mu(d\omega)}{\int_{\Omega_n} \mu(d\omega)}$$

From here, we generalize to general σ-algebras.
More formally, we have the existence of regular conditional expectation

Prop 10.7:

Let (Ω, Σ, μ) be a probability space and $\Sigma_0 \subseteq \Sigma$ a sub-σ-algebra. Then, $\exists$ a unique linear map associating to each bounded, Σ-measurable function $f \in \underbrace{L^\infty(\Omega, \Sigma)}_{\text{bounded}}$ the function $\mathbb{E}[f|\Sigma_0] : \Omega \to \mathbb{C}$ s.t.

(i) $\mathbb{E}[f|\Sigma_0] \in L^\infty$

(ii) $\forall f \in L^\infty(\Omega, \Sigma)$ and all $g \in L^\infty(\Omega, \Sigma)$, ($g$: Σ_0-measurable)

$$\int_\Omega f(\theta) g(\theta)\, \mu(d\theta) = \int \mathbb{E}[f|\Sigma_0](\theta)\, g(\theta)\, \mu(d\theta)$$

Remarks:

① In the L^2 perspective, the mapping
$$P_{\Sigma_0} : L^2(\Omega, \Sigma) \longrightarrow L^2(\Omega, \Sigma), \quad f \longmapsto \mathbb{E}[f|\Sigma_0]$$
extends in $L^2(d\mu)$ into an orthogonal projection onto the subspace

$$\operatorname{Range} P_{\Sigma_0} = \{ f \in L^2(d\mu) : f \text{ is } \Sigma_0\text{-measurable} \}$$

② $\forall$ monotone decreasing sequences of σ-algebras $\Sigma_1 \supset \dots \supset \Sigma_n \supset \dots$, the corresponding projections commute and have the towering property

$$P_{\Sigma_n} P_{\Sigma_k} = P_{\Sigma_n} \quad \forall n \ge k \qquad \text{i.e.,} \quad \mathbb{E}\big[\mathbb{E}[f|\Sigma_k] \,\big|\, \Sigma_n\big] = \mathbb{E}[f|\Sigma_n] \text{ for } \Sigma_n \subseteq \Sigma_k$$

In probabilistic terms, for bounded f, $\{P_{\Sigma_n} f\}_n$ forms a martingale.

③ By the martingale convergence theorem, $\forall f \in L^\infty(\Omega, \Sigma)$ the pointwise limit $\lim_{n\to\infty} P_{\Sigma_n} f(\theta)$ exists μ-a.s. and yields the function $P_{\Sigma_\infty} f$, $\Sigma_\infty = \bigcap_n \Sigma_n$.

We have the following theorem:

Theorem: (Dobrushin-Lanford-Ruelle Condition)

For all finite $\Lambda \subseteq \mathbb{Z}^d$,

energy of σ_Λ given the values of σ_{Λ^c} outside Λ

a priori (product) measure on Λ

$$\mathbb{E}\left[f(\sigma_\Lambda, \sigma_{\Lambda^c}) \,\middle|\, \sigma_{\Lambda^c}\right] = \int_{\Omega_\Lambda} f(\sigma_\Lambda, \sigma_{\Lambda^c}) \frac{e^{-\beta H_\Lambda(\sigma_\Lambda | \sigma_{\Lambda^c})}}{Z_\Lambda} \mu_\Lambda(d\sigma_\Lambda)$$

Z_Λ ~ normalization of measure

In the probability theory notation,

$$\mathbb{E}\left[f \,\middle|\, \Sigma_{\Lambda^c}\right](\sigma) = \int_{\Omega_\Lambda} f(\sigma_\Lambda, \sigma_{\Lambda^c}) \frac{e^{-\beta H_\Lambda(\sigma_\Lambda | \sigma_{\Lambda^c})}}{Z_\Lambda} \mu_\Lambda(d\sigma_\Lambda)$$

In other words, the regular conditional expectation is given by a skewed measure.

Note: we can use the tower rule on top of this!

$$\mathbb{E}[f(\sigma)] = \mathbb{E}\left[\mathbb{E}\left[f(\sigma_\Lambda, \sigma_{\Lambda^c}) \,|\, \sigma_{\Lambda^c}\right]\right]$$

Example

Consider Ising model w/ translation-invariant $H = -\sum_{\{x,y\}} J_{x-y} \sigma_x \sigma_y - h \sum_x \sigma_x$

We'd like to compute $\mathbb{E}[\sigma_0]$.

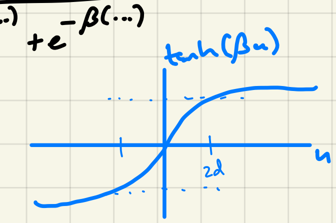
First, note that

$$H(\sigma_0 | \sigma_{\{0\}^c}) = -\sum_y J_{0-y} \sigma_0 \sigma_y - h\sigma_0$$
$$= -\sigma_0 \left[\sum_y J_{0-y} \sigma_y + h\right]$$

So, DLR with the uniform a priori measure μ gives

$$\mathbb{E}[\sigma_0 | \sigma_{\{0\}^c}] = \sum_{\sigma_0 \in \{\pm 1\}} \frac{\sigma_0 e^{\beta\sigma_0 (\sum_y J_{0-y}\sigma_y + h)}}{\underbrace{\sum_{\sigma_0 \in \{\pm 1\}} e^{\beta \sigma_0 (\sum_y J_{0-y}\sigma_y + h)}}_{\text{partition function}}} = \frac{e^{\beta(\dots)} - e^{-\beta(\dots)}}{e^{\beta(\dots)} + e^{-\beta(\dots)}}$$

$$= \tanh\left(\beta\left(\sum_y J_{0-y}\sigma_y + h\right)\right)$$



By the tower rule, in any Gibbs state,

$$\mathbb{E}[\sigma_x] = \mathbb{E}\left[\mathbb{E}\left[\sigma_x | \sigma_{\{x\}^c}\right]\right] = \mathbb{E}\left[\tanh\left(\beta\left(\sum_y J_{0-y}\sigma_y + h\right)\right)\right]$$

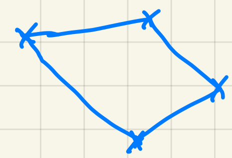
In nearest-neighbor interaction, $2d\beta < 1 \Rightarrow$ bounded tanh that allows us to converge the expanding range

We can (and eventually will) compute this outer expectation by repeatedly conditioning on a $\{y\}^c$. Then, for β small enough, as we move farther away, the bounds on tanh compound and things are nice.

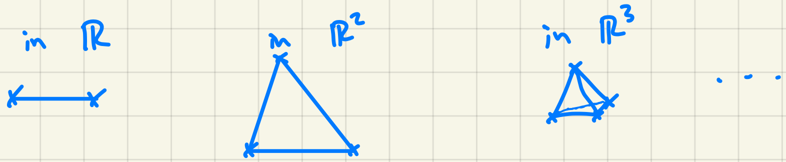
We would like to characterize the set of possible Gibbs measures for a certain (H, β) combination, as the coexistence of infinite Gibbs states is the hallmark of first order phase transitions! First, some vocabulary.

Def. The extremal points of a convex set K are the points $x \in K$ s.t. $\nexists\, a, b \in K,\ t \in (0,1)$ s.t. $x = at + b(1-t)$

(basically the vertices)



Def. A simplex is a convex set K s.t. $\forall x \in K$, x has a unique representation as a convex sum (or integral) of the extremal points of K.



For finite dim K, there are $|\dim K| + 1$ extremal points on the simplex, and all points $x \in K$ are expressible as a unique convex combination of them.

In infinite-dim K, all $x \in K$ are a unique integral over normalized measure, or an <u>expectation</u>.

<u>Theorem</u>: (Properties of infinite Gibbs measures)

For specified (H, β), we have

① The set of Gibbs measures is closed under convex combination (and so the set is convex).

② In fact, the set of Gibbs measures is a simplex

<u>Remarks:</u>

① $\Rightarrow$ if you have more than one Gibbs state, you have an infinite number

② $\Rightarrow$ that every Gibbs state admits a <u>unique extremal value decomposition.</u>

Lecture 3/28.

Moving on, we now inspect the relationship between uniqueness of Gibbs states and symmetry breaking.

Consider a probability space (Ω, Σ, μ). μ induces a Gibbs measure $\rho(d\omega)$ and we have the DLR characterization of the Gibbs measure

$$\mathbb{E}_\rho[f] = \mathbb{E}_\Lambda\left[\mathbb{E}_\rho[f \mid \sigma_{\Lambda^c}]\right] = \int \mathbb{E}_\rho[f \mid \sigma_{\Lambda^c}]\,\rho(d\sigma)?$$

$$\text{with} \quad \mathbb{E}_\rho[f \mid \sigma_{\Lambda^c}] = \int_{\Omega_\Lambda} f(\sigma_\Lambda, \sigma_{\Lambda^c}) \frac{e^{-\beta H_\Lambda(\sigma_\Lambda \mid \sigma_{\Lambda^c})}}{Z_\Lambda(\sigma_{\Lambda^c})}\,\mu(d\sigma_\Lambda)$$

We have the following theorem:

<u>Theorem:</u> (limit exists as we condition at ∞)

Given a prob. space (Ω, Σ, μ) and a monotone decreasing sequence of sub-σ-algebras $\Sigma_m \searrow$, then for any bounded measurable f measurable w.r.t. Σ

$$\mathbb{E}[f \mid \Sigma_m](\omega) \xrightarrow[\text{for } \mu\text{-a.e. } \omega]{\text{a.s.}} \mathbb{E}[f \mid \Sigma_\infty](\omega)$$

where $\Sigma_\infty := \overline{\bigcap_m \Sigma_m}$.

<u>Proof:</u> uses the martingale convergence theorem. Look it up :) $\square$

To apply this to our uses,

$\Sigma_\Lambda :=$ the σ-algebra of measurable sets of configurations within Λ

$\mathcal{B}_\Lambda :=$ the set of measurable functions induced by Σ_Λ.

$$\left(f(\sigma) \in \mathcal{B}_\Lambda \Rightarrow f \text{ depends only on } \Lambda\right)$$

We define $\Sigma_\infty := \overline{\bigcap_{\substack{\Lambda \subset \mathbb{Z}^d \\ |\Lambda| < \infty}} \Sigma_{\Lambda^c}}$ and $\mathcal{B}_\infty := \bigcap_{\substack{\Lambda \subset \mathbb{Z}^d \\ |\Lambda| < \infty}} \mathcal{B}_{\Lambda^c}$

Then, $f \in \mathcal{B}_\infty$ if f doesn't depend on the states inside <u>any finite volume</u>.

<u>Examples</u>

- An example $f \in \mathcal{B}_\infty$ is $f(\sigma) := \limsup_{L \to \infty} \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \sigma_x$, $\quad \Lambda_L := [-\frac{L}{2}, \frac{L}{2}]^d$

- Let $\mu_p(d\sigma)$ be defined s.t. $\sigma_j \sim \text{Bernoulli}(p)$ i.i.d.
  The LLN $\Rightarrow \frac{1}{L}\sum_{j=1}^{L} \sigma_j \xrightarrow{\mu_p\text{a.s.}} p \Rightarrow f(\sigma) \equiv p$ a.e. and Σ_∞ is trivial $(\Sigma_\infty = \{\emptyset, \Omega\})$

  For $\mu := \lambda \mu_{\frac{1}{2}} + (1-\lambda)\mu_{\frac{1}{3}}$, $\quad f(\sigma) = \begin{cases} \frac{1}{2} & \text{w.p. } \lambda \\ \frac{1}{3} & \text{w.p. } 1-\lambda \end{cases}$ and Σ_∞ is not trivial!

Let $\mu(d\sigma)$ be a Gibbs state. As before, but in the probabilistic notation,

$$\mathbb{E}_\mu[f] = \int \mathbb{E}[f \mid \Sigma_{\Lambda^c}](\sigma)\, \mu(d\sigma) \underset{\text{(Theorem)}}{\overset{\text{a.s.}}{=}} \int \mathbb{E}[f \mid \Sigma_\infty](\sigma)\, \mu(d\sigma)$$

the measure really only matters at ∞

So, for μ-a.e. σ, $f(\sigma) \longmapsto \mathbb{E}[f \mid \Sigma_\infty](\sigma)$ is a Gibbs measure since it satisfies DLR.

Theorem:

(i) Any Gibbs state can be presented as a convex combination of extremal Gibbs state.

(ii) A Gibbs state μ is extremal $\Longleftrightarrow$ Σ_∞ is trivial w.r.t. μ (functions measurable at ∞ are constant a.e.; they're only supported on one type of configuration only)

Corollary:

If μ_1, μ_2 are extremal Gibbs states (for the same Hamiltonian), then either

(i) $\mu_1 = \mu_2$ or (ii) $\mu_1 \perp \mu_2$ (mutually singular; the measures are supported on different sets)

Lecture 3/30-

Infinite Gibbs States + Symmetry Breaking

Theorem (10.11):

Given an extensive Hamiltonian with finite energy per site and a $\mathcal{B}$, a sufficient condition for uniqueness of its infinite volume Gibbs state is:

for any pair of Gibbs measures μ_1, μ_2, $\exists C < \infty$ s.t. $\forall$ positive $f: \Omega \to \mathbb{R}_+$,

$$\mathbb{E}_{\mu_1}[f] \leq C\, \mathbb{E}_{\mu_2}[f] \quad \text{(absolutely continuous)}$$

Proof: It suffices to prove that there exists a unique extremal Gibbs states. This does not allow an $A \subseteq \Omega$ s.t. $\mu_1(\mathbb{1}_A) = 1, \quad \mu_2(\mathbb{1}_A) = 0$

So, there cannot be two mutually-singular extremal Gibbs states, since otherwise there would be such an A.

$\square$

Ex/

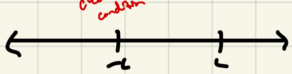
$$H(\sigma) = -\sum J_{x-y}\, \sigma_x \sigma_y$$

Theorem:

For 1D arrays of (bounded) spins $\{\sigma_m\}$ with $\sum_{m>0} m|J_m| < \infty$, the Gibbs state is unique $\forall \beta < \infty$.

In particular, $J_m = \frac{1}{m^2}$ and $d > 2 \Rightarrow$ no 1st order phase transition.

check this condition

Proof: Consider  For any x and y on opposite sides of u,

$$\sum_{\substack{x \geq u \\ y < u}} |J_{|x-y|}| \leq \sum_{m \geq 1} m\, |J_m| < C_0 < \infty.$$

Now, for any $f \in \mathcal{B}_{[-L,L]}$,

$$\mathbb{E}_\mu[f] \overset{\text{DLR}}{=} \int \mathbb{E}_\mu\left[f \mid \sigma_{\Lambda_L^c}\right] \mu(d\sigma_{\Lambda_L^c})$$

We have

$$\mathbb{E}_\mu\left[f \mid \sigma_{\Lambda_L^c}\right] = \int f(\sigma)\, \frac{e^{-\beta H_L(\sigma_{\Lambda_L} \mid \sigma_{\Lambda_L^c})}}{Z_{\Lambda_L}}\, \mu_{\Lambda_L}(d\sigma_{\Lambda_L})$$

and

$$H_L(\sigma_{\Lambda_L} \mid \sigma_{\Lambda_L^c}) = H_{\Lambda_L}(\sigma_{\Lambda_L}) + R_L(\sigma_{\Lambda_L}, \sigma_{\Lambda_L^c}) \quad \text{and} \quad |R_L(\sigma)| \leq 4C_0$$

from left + right

$$\Rightarrow \mathbb{E}_\mu\left[f \mid \sigma_{\Lambda_L^c}\right] \begin{cases} \leq e^{4\beta C_0} \int f(\sigma_{\Lambda_L}) \dfrac{e^{-\beta H_L(\sigma_{\Lambda_L})}}{Z_{\Lambda_L}}\, \mu_{\Lambda_L}(d\sigma_{\Lambda_L}) \\[2ex] \geq e^{-4\beta C_0} \underbrace{\int f(\sigma_{\Lambda_L}) \dfrac{e^{-\beta H_L(\sigma_{\Lambda_L})}}{Z_{\Lambda_L}}\, \mu_{\Lambda_L}(d\sigma_{\Lambda_L})}_{\text{same for any } \mu_1, \mu_2, \text{ call it } \mathbb{E}^{\text{free}}[f \mid \sigma_{\Lambda_L^c}]} \end{cases}$$

- estimate doesn't depend on size of the volume!
- we approximate measurable functions by local functions

$$\Rightarrow \mathbb{E}_{\rho_1}[f|\theta_{\Lambda_L^c}] \le e^{4\beta C_0}\mathbb{E}^{free}[f|\theta_{\Lambda_L^c}] \quad \text{and} \quad \mathbb{E}_{\rho_2}[f|\theta_{\Lambda_L^c}] \ge e^{-4\beta C_0}\mathbb{E}^{free}[f|\theta_{\Lambda_L^c}]$$

$$\Rightarrow \mathbb{E}_{\rho_1}[f] \le e^{4\beta C_0}\int \underbrace{\mathbb{E}^{free}[f|\theta_{\Lambda_L^c}]}_{\text{does not depend on } \theta_{\Lambda_L^c}} \rho_1(d\theta_{\Lambda_L^c}) = e^{4\beta C_0}\mathbb{E}^{free}_{\Lambda_L}[f]\underbrace{\int \rho_1(d\theta_{\Lambda_L^c})}_{=1}$$

$$\le e^{8\beta C_0}\int \mathbb{E}_{\rho_2}[f|\theta_{\Lambda_L^c}]\rho_2(d\theta_{\Lambda_L^c}) = e^{8\beta C_0}\mathbb{E}_{\rho_2}[f].$$

By the previous theorem, we must have a unique Gibbs state. $\square$

How does this apply to continuous symmetry breaking?

Consider $H(\theta) = -\sum_{x\sim y} J_{x\sim y}\,\vec{\theta}_x\cdot\vec{\theta}_y$ with $\vec{\theta}_x = (\theta_x^1, \dots, \theta_x^N)$, $\|\vec{\theta}_x\| = 1$

and let $R_\theta\vec{\theta} :=$ rotation of $\vec{\theta}$ by θ.

Suppose a rotationally-invariant Hamiltonian and a-priori measure

$$H(\theta) = H(R_\theta\theta) \quad \text{and} \quad \mu(d\theta) = \mu(R_\theta d\theta)$$

Rotation on configuration space induces rotation on observable space by

$$(R_\theta f)(\theta) := f(R_\theta\theta)$$

This, in turn, induces a rotation on the space of measures where

$R_\theta\rho$ is defined by $\mathbb{E}_{R_\theta\rho}[f] \equiv \mathbb{E}_\rho[R_\theta f] \quad \forall f$

Then, we have two dichotomous options:

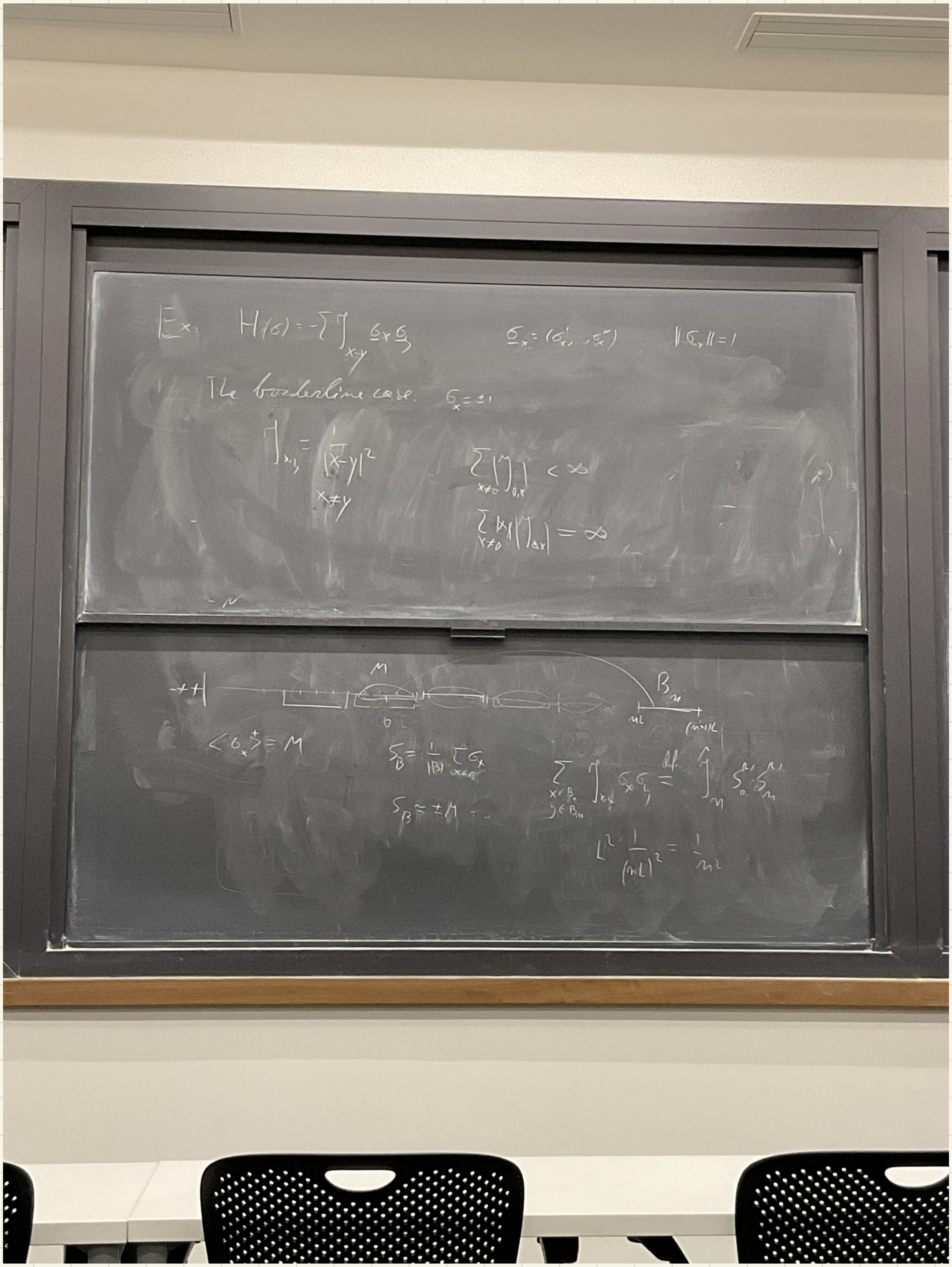
1. either we have a unique, rotationally-invariant Gibbs state
$\rho = R_\theta\rho \quad \forall\theta$

OR

2. $\exists$ an extremal Gibbs state ρ s.t. ρ and $R_\theta\rho$ are singular
- we can test for condition 2 and discover when the $\boxed{R_\theta\text{-symmetry of } \rho \text{ is broken}}$

Renormalization group huh?

the borderline $J \sim \frac{1}{|x-y|^2}$ model



Lecture 4/4 - Mermin-Wagner Theorem

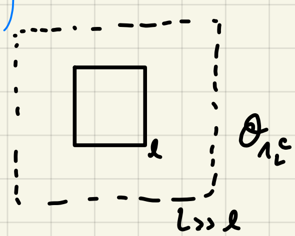
First we ought to verify that a rotated infinite-volume Gibbs state is still an infinite-volume Gibbs state: the DLR condition verifies this.

> **Theorem:** (Mermin-Wagner)
>
> For a two dimensional <u>finite-range</u> system of continuous spin variables with rotational symmetry, i.e.
> $$H(\sigma) = -\sum_{\substack{A \subset \Lambda \\ \mathrm{diam}(A) \le R}} J_A \, \phi_A(\sigma_A),$$
> if (i) ϕ_A is invariant under uniform rotations <u>and</u> (ii) ϕ_A varies smoothly under all rotations $\left(\text{ex: } H = -\sum_{x,y} J_{x-y} \vec{\sigma}_x \cdot \vec{\sigma}_y\right)$
>
> <u>then</u> any infinite-volume Gibbs state is invariant under uniform spin rotation; i.e. $\forall$ observables $f: \Omega \to \mathbb{R}$ and all $\mathcal{R}_\theta \in O(2)$,
> $$\int f \, d\mu = \int \mathcal{R}_\theta f \, d\mu \iff \mathbb{E}_\mu[f] = \mathbb{E}_\mu[\mathcal{R}_\theta f]$$

<u>Proof:</u> It suffices to show that $\forall$ extremal Gibbs states μ, $\exists c < \infty$ s.t. $\forall$ local $f: \Omega \to \mathbb{R}$ with $f \ge 0$,

$$\mathbb{E}_\mu[\mathcal{R}_\theta f] \le c \, \mathbb{E}_\mu[f]$$

(μ absolutely continuous to $\mathcal{R}_\theta \mu$ $\forall \mu$)



Fix a volume Λ_ℓ s.t. $f \in \mathcal{B}_{\Lambda_\ell}$. For any larger Λ_L, the tower rule and DLR condition give

$$\mathbb{E}_\mu[f] = \mathbb{E}_\mu[\mathbb{E}_\mu[f \mid \sigma_{\Lambda_L^c}]]$$

Consider a soft, nonuniform rotation of spins given by angle

$$\theta(x) = \begin{cases} \theta & |x| < \ell \\ \text{o.w.} & \\ 0 & |x| > L \end{cases}$$

and a rotation operator on functions $\tilde{\mathcal{R}} f(\sigma) := f(\{\mathcal{R}_{\theta(x)} \sigma_x\}_{x \in \Lambda})$

Then,

$$\mathbb{E}_\mu\left[\tilde{\mathcal{R}} f \,\middle|\, \sigma_{\Lambda_L^c}\right] \overset{\text{DLR}}{=} \int_{\Omega_{\Lambda_L}} \tilde{\mathcal{R}} f(\sigma) \frac{e^{-\beta H_{\Lambda_L}(\sigma_{\Lambda_L} \mid \sigma_{\Lambda_L^c})}}{Z_{\Lambda_L}(\sigma_{\Lambda_L^c})} \mu(d\sigma_{\Lambda_L})$$

$\tilde{\mathcal{R}}$ is bijective and μ is product measure, so $\mu(d\sigma) = \mu(d\tilde{\mathcal{R}}\sigma)$

$$= \int f(\tilde{\mathcal{R}}\sigma) \underbrace{e^{-\beta\left[H_{\Lambda_L}(\sigma_{\Lambda_L} \mid \sigma_{\Lambda_L^c}) - H_{\Lambda_L}(\tilde{\mathcal{R}}\sigma_{\Lambda_L} \mid \sigma_{\Lambda_L^c})\right]}}_{\text{weighting factor based on effect of } \tilde{\mathcal{R}} \text{ on the energy}} \frac{e^{-\beta H_{\Lambda_L}(\tilde{\mathcal{R}}\sigma_{\Lambda_L} \mid \sigma_{\Lambda_L^c})}}{Z_{\Lambda_L}(\sigma_{\Lambda_L^c})} \mu(d\tilde{\mathcal{R}}\sigma_{\Lambda_L})$$

We have

$$\underbrace{H_{\Lambda_L}(\tilde{\mathcal{R}}\sigma_{\Lambda_L}) - H_{\Lambda_L}(\sigma_{\Lambda_L})}_{\text{energy penalty}} \overset{\text{n.n., but works generally}}{=} -\sum_{|x-y|=1} J_{x-y}\left[\mathcal{R}_{\theta(x)}\vec{\sigma}_x \cdot \mathcal{R}_{\theta(y)}\vec{\sigma}_y - \vec{\sigma}_x \cdot \vec{\sigma}_y\right]$$

$$= -\sum_{|x-y|=1} J_{x-y}\left[\vec{\sigma}_x \cdot \left(\mathcal{R}_{\Delta\theta}\vec{\sigma}_y - \vec{\sigma}_y\right)\right]$$

We would like to bound $\vec{\theta}_x \cdot \left(R_{\Delta\theta}\vec{\theta}_y - \vec{\theta}_y\right)$ to show that the energy penalty is favorable.

For $\vec{\theta}_x \approx \vec{\theta}_y$, there is only a second-order term

$$\vec{\theta}_x \cdot \left(R_{\Delta\theta}\vec{\theta}_y - \vec{\theta}_y\right) \leq \frac{1}{2}|\Delta\theta|^2$$

Otherwise, we require a linear term that we must bound with trickery.

Corollary:

In the above setting, there can be no spontaneous magnetization. In other words, $\mathbb{E}_\Lambda[\vec{\theta}_x] = 0 \quad \forall x$.

Proof: By Mermin-Wagner, $\mathbb{E}_\Lambda[\vec{\theta}_x] = \mathbb{E}_\Lambda[-\vec{\theta}_x]$. The result follows.

$\square$

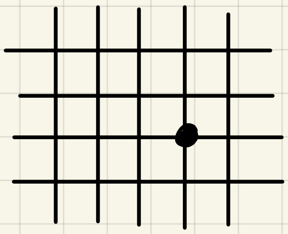
Schramm

Cycles in random permutations

Lecture 4/11 - Q-State Potts

Q-State Potts Model

Consider a setup on a graph $G=(V,E)$ with



$$\Omega_0 = \{1, \dots, Q\} \qquad \mu(\{\theta\}) = \frac{e^{-\beta H(\theta)}}{Z}$$

$$H = -\sum_{\{x,y\}\subset\Lambda} J_{x-y}\,\underbrace{(\delta_{\theta_x\theta_y} - 1)}_{\text{check this!}}$$

We call this the Q-state Potts model.

(When $Q=2$, we have Ising-like $\Omega_0 = \{1,2\} \Rightarrow \delta_{\theta_x\theta_y} = \frac{\theta_x\theta_y + 1}{2}$ in the Ising case. So, $Q=2$ is scaled + shifted Ising)

So,

Partition fn.

$$e^{\beta J_{x-y}(\delta_{\theta_x\theta_y}-1)} = \delta_{\theta_x\theta_y}\left(1 - e^{-\beta J_{x-y}}\right) + e^{-\beta J_{x-y}} = p_{x-y}\,\delta_{\theta_x\theta_y} + (1-p_{x-y})\cdot 1$$

$$\Rightarrow Z = \sum_\theta \prod_{e\in E}\left[p_{xy}\,\delta_{\theta_x\theta_y} + (1-p_{xy})\cdot 1\right] = \sum_{\theta: V\to\{1,\dots,Q\}} \sum_{n:E\to\{0,1\}} \prod_{e\in E} \underbrace{p_e^{n(e)}(1-p_e)^{1-n(e)}}_{\sim \text{Bernoulli}(p_e) \text{ if spins agree, } 0 \text{ otherwise}} \underbrace{\prod_{n_{xy}=1}\delta_{\theta_x\theta_y}}_{=\mathbb{1}[\theta \text{ is constant on cluster } n]}$$

($n:E\to\{0,1\}$: sum over random clusters where $n(e) = \mathbb{1}\{e \in \text{cluster}\}$)

$$\Rightarrow \boxed{Z = \sum_{n:E\to\{0,1\}} \prod_{\{x,y\}\in E} p_{xy}^{\,n(\{x,y\})}(1-p_{xy})^{1-n(\{x,y\})}\, Q^{N_c(n)}}$$

$N_c(n)$: # of connected clusters

where $p_{xy} = 1 - e^{-\beta J_{x-y}}$

Spin-spin correlation

Given a cluster n, we have

$$\mathbb{E}\left[\delta_{\theta_u\theta_v} \mid n\right] = \begin{cases} 1 & \text{if } n(\{u,v\}) = 1 \\ \frac{1}{Q} & \text{else} \end{cases}$$

depends only on if u,v are in same cluster!

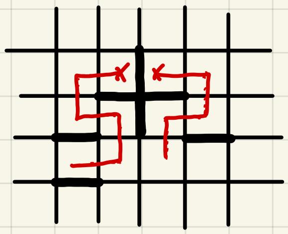
From here, we can see that the Q-state Potts model forms clusters with the same θ_u's that flip together. This makes it a perfect place to study percolation.

If $Q=2$, we have $\langle\theta_u\theta_v\rangle = n(\{u,v\})$ with θ as in Ising model.

Remarks:

- $Q=1$ is random percolation model
- $Q=2$ is scaled/shifted Ising
- If we take $Q \in \mathbb{R}^+$ instead, we get the F-K random cluster model
- As $Q \downarrow 0$, measure concentrates around clusters with low N_c; i.e. $\to$ minimum spanning tree

Kramers-Wannier Duality in 2D



We have the Q-state Potts model on a graph $G=(V,E)$ with a dual graph $G^*=(V^*,E^*)$.
Then, $P^*_{xy} = 1-P_{xy}$.

Note that percolation occurs in either G or G^*.
When one percolates, it chokes the dual, which decays exponentially.

When $G \equiv G^*$ and $P_{xy}=\frac{1}{2}$, the dual is identical and percolation occurs in both. ← Sunny says this might be wrong

Let $N_c^*(n)$ denote the # of connected clusters in the dual model.

Lemma:

not sure this is true

On any finite planar graph, $N_c(n) - N_c^*(n)$ is independent of n.

Proof: Consider adding any edge to n $\left(\text{i.e. } n'(e') := \begin{cases} 1 & e'=e \\ n(e') & \text{else} \end{cases}\right)$

If [edge e already inside a cluster] then $N_c(n') = N_c(n)$ (they were already in a cluster)
$N_c^*(n') = N_c^*(n)+1$ (breaks a dual cluster)

If [edge e joining two clusters] then $N_c(n') = N_c(n)-1$ (connects two clusters)
$N_c^*(n') = N_c^*(n)$ (doesn't touch dual)

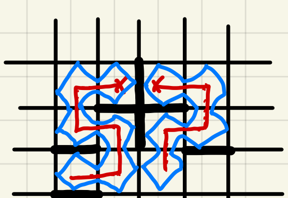
So, $\Delta(N_c(n) - N_c^*(n)) = -1$ independently of n. □

Note that we can write $N_c(n) = \frac{1}{2}\left[N_c(n)+N_c^*(n)\right] + \frac{1}{2}\underbrace{\left[N_c(n)-N_c^*(n)\right]}_{\text{constant}}$

$$\Longrightarrow Z^{(Q)} = \text{const}\cdot \sum_{n:E\to\{0,1\}} \left(\prod_{e:n(e)=1} P_{xy}\right)\cdot\left(\prod_{e:n(e)=0} 1-P_{xy}\right) \sqrt{Q}^{N_c(n)}\sqrt{Q}^{N_c^*(n)}$$ ← check this!!

As before, when $P_{xy}=\frac{1}{2}$, the model is dual to itself.

Another way to formulate this is to draw loops with midpoint-midpoint edges that don't cross edges in G or G^*. Each loop surrounds exactly 1 cluster in G or G^*.



Then, $Z^{(Q)} = \text{const}\cdot \sum_{n:E\to\{0,1\}} \left(\prod_{e:n(e)=1} P_{xy}\right)\cdot\left(\prod_{e:n(e)=0} 1-P_{xy}\right)\sqrt{Q}^{\#\text{ of loops}}$

This kind of formulation shows up often in quantum statistical physics.

When we consider the connection of an edge, conditioned on the rest of n (i.e. $\mathbb{P}\{n(e)=1 \mid \{n(e') : e' \neq e\}\}$),

$$\frac{\mathbb{P}\{n(e)=1 \mid v, u\}}{\mathbb{P}\{n(e)=0 \mid v, u\}} = \frac{p}{1-p} \qquad \frac{\mathbb{P}\{n(e)=1 \mid v, u\}}{\mathbb{P}\{n(e)=0 \mid v, u\}} = \frac{\frac{p}{p+(1-p)Q}}{\frac{(1-p)Q}{p+(1-p)Q}} = \frac{p}{(1-p)Q} \leq p$$

In the dual, 0 and 1 flip and the answers swap. So, the model is self-dual when

$$\frac{p}{1-p} = \frac{(1-p)Q}{p} \iff \frac{p}{1-p} = \sqrt{Q}$$

This is the Kramers-Wannier self-duality point in 2D.

FKG Monotonicity (Fortuin-Kostelyn-Ginibre)

The collection of possible clusters $n: E \to \{0,1\}$ (which we denote $\{0,1\}^E$) is <u>partially ordered</u>.

Def: (partial ordering)

An ordering $\succ$ is a partial ordering if

(i) $n' \succ n \iff n'_e \geq n_e \;\; \forall e$

(ii) $f: \{0,1\}^E \to \mathbb{R}$ is $\nearrow$ if $f(n') \geq f(n) \;\; \forall n' \succ n$

(iii) For prob. measures P_1, P_2 on $\{0,1\}^E$,

$$P_1 \succ P_2 \iff \forall f \nearrow, \;\; \int f(n) P_1(dn) \geq \int f(n) P_2(dn)$$

"P_1 dominates P_2"

not like $\mathbb{Z}^2$, but an algebra definition

Def: A partially-ordered set forms a "lattice" iff $\forall$ pairs (n', n), there exists $n \vee n'$, $n \wedge n'$ s.t. $n \vee n' \succ n, n'$ and $n \wedge n' \prec n, n'$

In this case, $(n \vee n')(e) = \max\{n(e), n'(e)\}$, $(n \wedge n')(e) = \min\{n(e), n'(e)\}$

Def: A probability measure μ is positively associated if $\forall f, g : \Omega \to \mathbb{R}$ s.t. $f, g \geq 0$ and $f, g \nearrow$, we have

$$\mathbb{E}_\mu[fg] \geq \mathbb{E}_\mu[f]\, \mathbb{E}_\mu[g]$$

"monotonic, nonnegative functions are positively associated"

check this definition

Theorem

For two measures P_1, P_2 on $\{0,1\}^E$,

$P_1 \succ P_2 \iff$ there exists a coupling $\mu(dn_1, dn_2)$ s.t.

(i) $\int g(n_j) \mu(dn_1, dn_2) = \int g(n) P_j(dn) \quad j = 1,2$ (correct marginals)

(ii) $n_1 \succ n_2 \;\; \mu$-a.s.

Note that for $f \nearrow$, the second condition implies

$$\mathbb{E}_{P_1}[f] - \mathbb{E}_{P_2}[f] = \int \underbrace{[f(n_1) - f(n_2)]}_{\geq 0} \mu(dn_1, dn_2) \geq 0$$

Theorem:

Let μ be a probability measure on a partially-ordered "lattice". A sufficient condition for μ to have positive association is that

$$\mathbb{E}_\mu[n \vee n'] \, \mathbb{E}_\mu[n \wedge n'] \geq \mathbb{E}_\mu[n] \, \mathbb{E}_\mu[n'] \quad \forall n, n'$$

Example (Ising)

$$\sigma' \succ \sigma \Rightarrow \sigma'_x \geq \sigma_x \quad \forall x \qquad \text{and} \qquad \mu(\sigma) = \frac{e^{-\beta \sum_{x,y} J_{x-y} \sigma_x \sigma_y}}{Z}$$

Consider

σ	+	−	+	−	−	−
σ'	−	−	−	+	+	−
$\sigma \vee \sigma'$	+	−	+	+	+	−
$\sigma \wedge \sigma'$	−	−	−	−	−	−

these states are more likely than the originals, since more spin agreement

(We can write the relation
$$(\sigma'_x \wedge \sigma_x)(\sigma'_y \wedge \sigma_y) + (\sigma'_x \vee \sigma_x)(\sigma'_y \vee \sigma_y) \geq \sigma'_x \sigma'_y + \sigma_x \sigma_y \text{)}$$

not sure what this has to do w/ anything

So, Ising spin model Gibbs measure μ satisfies the theorem, since $\sigma \wedge \sigma'$ and $\sigma \vee \sigma'$ are more likely than σ or σ'.

Example (FK random cluster model)

The relation $\mathbb{E}_\mu[n \vee n'] \, \mathbb{E}_\mu[n \wedge n'] \geq \mathbb{E}_\mu[n] \, \mathbb{E}_\mu[n']$ holds iff

$$\frac{\mu(n' \sqcup \{e\})}{\mu(n')} \geq \frac{\mu(n \sqcup \{e\})}{\mu(n)} \quad \forall n, n' \text{ s.t. } n'(e') \geq n(e') \quad \forall e' \neq e$$

We can verify this for FK random cluster model.

Example (Q-state Potts model)

Note that the Gibbs measure $\mu_{\beta,Q}(n) = \prod_{e \in E} p_{xy}^{n(e)} (1-p_{xy})^{1-n(e)} \, Q^{N_c(n)}$

has that

- $\mu_{\beta,Q}$ is decreasing in Q
- $\mu_{\beta,Q}$ is increasing in β
- $N_c(n)$ is decreasing in n
- $N_c(n) + |n|$ is increasing in n

Also, $\forall Q' \geq Q \geq 1$

$$\beta_c(Q') \geq \beta_c(Q) \geq \frac{Q}{Q'} \beta_c(Q')$$

This relates critical points of models for different Q's! So, critical behavior in one implies critical behavior in another!

Interpretation:

For any $\rho(dn)$ satisfying the FKG condition, $\forall f, g \geq 0$ with $f, g \nearrow$ positive associativity gives

$$\mathbb{E}_\rho[gf] \geq \mathbb{E}_\rho[g]\,\mathbb{E}_\rho[f] \Rightarrow \int g(n) f(n) \mu(dn) \geq \left(\int g(n) \mu(dn)\right)\left(\int f(n) \mu(dn)\right)$$

$$\Rightarrow \frac{\int g(n) f(n) \rho(dn)}{\int f(n) \rho(dn)} \geq \int g(n) \rho(dn)$$

So, letting $\mu(dn)$ be the tilted measure $\mu(dn) := f(n) \rho(dn)$, then

$$\mu \succ \rho$$

Holley's Theorem:

We have that $\rho \prec \rho'$ if and <u>only if</u> there exists a coupling $\mu(d\sigma, d\sigma')$ s.t. μ has marginal distributions agreeing with ρ and ρ' and μ is supported only on states $\sigma < \sigma'$.

The above theorem grants that if $\rho < \rho'$, then

$$\rho'(\sigma_x') - \rho(\sigma_x) = 2\mu(\sigma_x \neq \sigma_x')$$ $\Leftarrow$ something's wrong w/ this!

Add Lecture 4/18

Lecture 4/20 - Quantum Spin Chains

Suppose we have a Hamiltonian $\hat{H}$ that we haven't diagonalized.

We have the Q-thermal expectation value

$\{\Psi_m\}_m$ is any ONB

$$\hat{A} \longmapsto \langle \hat{A} \rangle_\beta = \frac{tr(\hat{A}e^{-\beta \hat{H}})}{tr(e^{-\beta \hat{H}})}, \quad \text{with} \quad Z := tr(e^{-\beta \hat{H}}) = \sum_{m=1}^{\infty} \langle \Psi_m | e^{-\beta \hat{H}} | \Psi_m \rangle$$

We can label Hamiltonians artificially by the time t_j at which we added it to the picture, granting a Dyson integral

$$e^{-\beta \hat{H}} = \sum_{m=0}^{\infty} \int \cdots \int_{0<t_1<\cdots<t_m<\beta} (-\hat{H})_{t_m} \cdots (-\hat{H})_{t_1} \, dt_1 \cdots dt_m, \qquad \text{with} \quad \hat{H}_t \equiv \hat{H}$$

(note that the Dyson integral came from e^{itH}: we apply analytic continuation from imaginary time to be in real time)

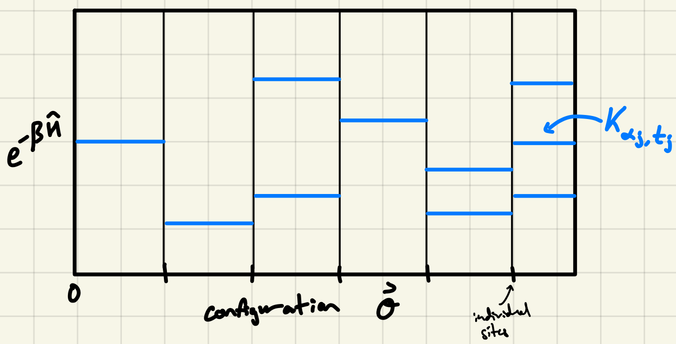
Writing $\quad -\hat{H} = \sum_{\alpha \in I} \hat{K}_\alpha$, (non-commuting) we get

$$e^{-\beta \hat{H}} = \sum_{m=0}^{\infty} \int \cdots \int_{0<t_1<\cdots<t_m<\beta} \sum_{\{\alpha_1, \ldots, \alpha_m\} \subseteq I} \left(\prod_{j=1}^{m} \hat{K}_{\alpha_j t_j} \right) dt_1 \cdots dt_m$$

$$= e^{\beta |I|} \int_\Omega \prod_{j=1}^{|\omega|} \hat{K}_{\alpha_j t_j} \, \rho(d\omega)$$

(time-ordered product)

In a sense, we have written out a random formulation for $\hat{H}$ with the following picture



In this picture, we visualize quantum evolution (in real time) as randomly applying operators $\hat{K}_{\alpha_j t_j}$ as we move through (inverse) time. We assign measure $\rho(d\omega)$ to the sequence of these transformations.

$I \times [0, \beta] \qquad \omega = \{\alpha_j, t_j\}_{j=1}^{|\omega|}$

$\rho(d\omega)$ is a Poisson process on $I \times [0, \beta]$

$\boxed{\text{figure out } I \text{ vs } E}$

Note that our integrals will look like the usual d-dimensional stuff + 1 extra dimension for the imaginary time! So, quantum stat mech in d-dim will look like classical stat mech in d+1-dim.

$$Z_{\Lambda, \beta} = e^{\beta |I|} \int_\Omega tr\left(\prod_{j=1}^{|\omega|} \hat{K}_{\alpha_n t_n} \cdots \hat{K}_{\alpha_1 t_1} \right) \rho(d\omega)$$

Q - Spin Operators

Hermitian (self-adjoint)

We have $\hat{\vec{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ satisfying $[\hat{S}_x, \hat{S}_y] = i\hat{S}_z$ with cyclic permutation.

Note that $|\hat{\vec{S}}|^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ commutes with each of $\hat{S}_x, \hat{S}_y, \hat{S}_z$.
This is because the magnitude of $\hat{\vec{S}}$ is invariant under rotations $e^{i\theta \hat{S}_j}$.
We can also derive $|\hat{\vec{S}}|^2 = s(s+1)\mathbb{1}$

Suppose our states live in a finite-dim Hilbert space $\mathcal{H}$.
If $\dim \mathcal{H} = N$, we get an ONB for $\mathcal{H}$ from eigenvectors of $\hat{S}_z$

$$\hat{S}_z |s,m\rangle = m|s,m\rangle, \qquad m \in \{-s, -s+1, \dots, s-1, s\}, \qquad s = \frac{N-1}{2}$$

s can be half-integer

(Fun story: we can prove the quantization for $\hat{S}_z$ by slicing the surface of a sphere into slices. In 3D, Archimedes proved equally-spaced slices have equal area, implying integer quantization)

Suppose we have two spins, modeled as

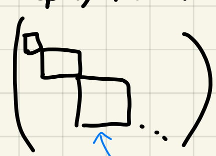
$$\mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2} = \text{span}\{|s_1,m_1\rangle \otimes |s_1,m_2\rangle\}_{m_1,m_2}$$
$$= \bigoplus_{|s_1-s_2|}^{s_1+s_2} \mathcal{H}_{s_{1,2}}$$

group representation of $\mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2}$

Let s_{12} be the possible magnitudes of the combined spin, i.e. the values

$$s_{12} \in \{|s_1-s_2|, \dots, s_1+s_2\} \quad \Rightarrow \quad \hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 =$$

$|s_1-s_2|$
$\vdots$
s_1+s_2



Each of these subspaces are the $\mathcal{H}_{s_{1,2}}$'s

Examples

① Q-bit $s=\frac{1}{2}$, $\dim \mathcal{H} = 2$

Pauli matrices

We can write $\hat{\vec{S}} = \frac{1}{2}(\sigma_x, \sigma_y, \sigma_z)$ with

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Writing the Heisenberg anti-ferromagnetic/ferromagnetic spin chain on $\mathbb{Z}$

$$H = \pm 2 \sum_m \vec{\sigma}_m \cdot \vec{\sigma}_{m+1} \Rightarrow \hat{H} = \pm \sum_m \hat{\vec{S}}_m \cdot \hat{\vec{S}}_{m+1}$$

+ for antiferromagnetic

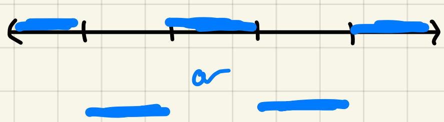
If they are all identical copies, we can write

$$\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \frac{1}{2}\left[|\hat{\vec{S}}_1 + \hat{\vec{S}}_2|^2 - \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_1 - \hat{\vec{S}}_2 \cdot \hat{\vec{S}}_2 \right] = \frac{1}{2}\left[S_{12}(S_{12}+1) - 2s(s+1)\mathbb{1} \right]$$

For $s=\frac{1}{2}$ spinors, $S_{12} \in \{0,1\}$. Let $|\psi\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$, and define $\hat{P}^{(0)}_{uv} := |\psi\rangle\langle\psi|$ (projection onto singlet)

Then, $(\hat{S}_{z,1} \hat{S}_{z,2})|\psi\rangle = -|\psi\rangle$, and $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \frac{1}{2} - 2\hat{P}^{(0)}_{uv}$ ← do the algebra

We can't have all links in their lowest states, and so we can assign an initial ground state



(dimerization)

Depending on evenness or oddness of the finite volume, one ground state will be preferable to another; translational symmetry breaking can occur if the presence of these two ground states remains in the infinite limit.

Lecture 4/25 - Quantum Spin Models

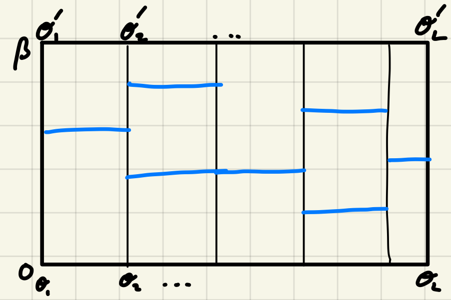
Recall the setup with

$$\hat{H} = -\sum_m \hat{\vec{S}}_m \cdot \hat{\vec{S}}_{m+1} =: \sum_{\alpha \in I} \hat{K}_\alpha$$

where $\hat{\vec{S}} = \frac{1}{2}(\sigma^x, \sigma^y, \sigma^z)$ are Pauli matrices and $|\sigma_j\rangle$ denote eigenstates of σ^z_j.

To compute expectations of the form $\langle \hat{A} \rangle_{\Lambda,\beta} = \dfrac{\mathrm{Tr}(e^{-\beta \hat{H}} \hat{A})}{\mathrm{Tr}(e^{-\beta \hat{H}})}$, we use the expansion

$$e^{\beta \sum_{\alpha \in I} \hat{K}_\alpha} = \sum_{n=0}^{\infty} \int \cdots \int_{0 < t_1 < \dots < t_n < \beta} \left(\prod_{j=1}^{n} K_{\alpha_j, t_j} \right) dt_1 \dots dt_n$$



$\omega = \{(\alpha_j, t_j)\}_j \subseteq I \times [0, \beta]$

poisson process in d+1-dim spacetime

To convert this to a probability measure, we normalize by dividing by $e^{\beta |I|}$. So, we get

$$e^{\beta \sum_{\alpha \in I} \hat{K}_\alpha} = e^{\beta |I|} \int_\Omega \left(\prod_{(\alpha,t) \in \omega} \hat{K}_{\alpha,t} \right) \rho_{\beta,\Lambda}(d\omega)$$

If we want matrix elements in $\mathcal{H} := \mathrm{span}\{|\sigma_1, \dots, \sigma_L\rangle\}$

$$\langle \sigma' | e^{\beta \sum_{\alpha \in I} \hat{K}_\alpha} | \sigma \rangle = e^{\beta |I|} \int_\Omega \langle \sigma' | \prod_{(\alpha,t) \in \omega} \hat{K}_{\alpha,t} | \sigma \rangle \rho_{\beta,\Lambda}(d\omega)$$

$$\Rightarrow Z_{\Lambda,\beta} = \mathrm{Tr}(e^{-\beta \hat{H}}) = \int_\Omega \sum_{\sigma} \langle \sigma' | \prod_{(\alpha,t) \in \omega} \hat{K}_{\alpha,t} | \sigma' \rangle \rho_{\beta,\Lambda}(d\omega)$$

over an ONB of $\mathcal{H}$

Returning to the Heisenberg Anti-Ferromagnet,

projection onto the singlet $|\psi\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$, $P^{(0)} = |\psi\rangle\langle\psi|$

$$\hat{H} = +\sum_m \hat{\vec{S}}_m \cdot \hat{\vec{S}}_{m+1} = \sum_m -4\,\hat{P}^{(0)}_{m,m+1} + \text{constant}$$

If we wished to extend to other spin values than spin-$\frac{1}{2}$, we can either leave the Hamiltonian as $\hat{\vec{S}}_m \cdot \hat{\vec{S}}_{m+1}$. Alternatively, we can write the singlet as

(with $s > \frac{1}{2}$, people define spin systems in order to get certain properties)

$$(s \in \tfrac{1}{2}\mathbb{Z}_+) \qquad |\psi\rangle := \frac{1}{\sqrt{2s+1}} \sum_{m=-s}^{s} (-1)^m |m, -m\rangle \Rightarrow \hat{P}^{(0)} = |\psi\rangle\langle\psi|$$

So, we consider

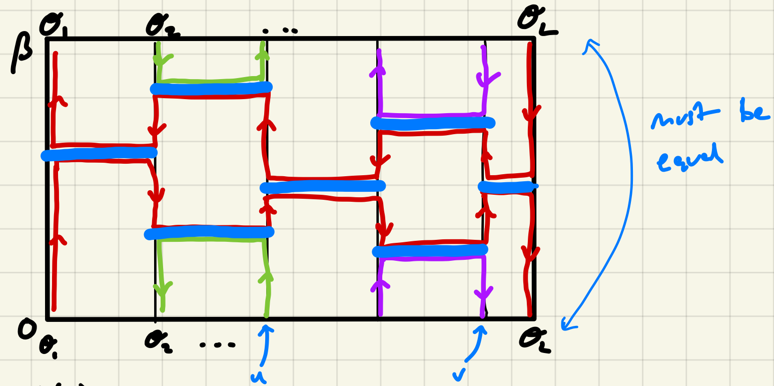
$$\hat{H} = -\sum_m \underbrace{(2s+1)\,\hat{P}^{(0)}_{m,m+1}}_{\hat{K}_m} \qquad (s \in \tfrac{1}{2}\mathbb{Z}_+)$$

This gives matrix elements

$$\langle \sigma'_m \sigma'_{m+1} | \hat{K}_m | \sigma_m \sigma_{m+1} \rangle = \sum_{m,m'=-s}^{s} (-1)^{m-m'} \mathbb{1}[\sigma_m = -\sigma_{m+1} = m]\, \mathbb{1}[\sigma'_m = -\sigma'_{m+1} = m']$$

We know that spins either align or are opposite. However, the form of this problem forces neighboring spins to be <u>opposite</u>.

Note that if $\sigma' = \sigma$ (as it does when computing traces), we can emphasize this constraint in the picture via:



Let N_ℓ be the number of loops (3 in the depicted picture). Then, we have a degree of freedom of choice for each loop, and for each choice we can select $(2s+1)$ options. So, in this case,

$$Z_{\Lambda,\beta} = e^{\beta L} \int_{\Omega} (2s+1)^{N_\ell(\omega)} \rho(d\omega)$$

Let's compute a spin-spin correlation.

$$\langle \hat{S}_u \cdot \hat{S}_v \rangle_{L,\beta} = 3 \langle \hat{S}^z_u \hat{S}^z_v \rangle_{L,\beta} = \frac{3\,\mathrm{Tr}(e^{-\beta \hat{H}} \hat{S}^z_u \hat{S}^z_v)}{\int_\Omega (2s+1)^{N_\ell(\omega)} \rho(d\omega)} = \frac{e^{\beta L} \int_\Omega \sum_\sigma \langle \sigma | \prod_{(x,t)\in\omega} \hat{K}_{(x,t)} \hat{S}^z_u \hat{S}^z_v | \sigma \rangle \, \rho_{\beta,\Lambda}(d\omega)}{\int_\Omega (2s+1)^{N_\ell(\omega)} \rho(d\omega)}$$

If u and v are not connected by ω, they will average to 0 as $\hat{S}^z_u, \hat{S}^z_v$ commute with the $\hat{K}$'s. If they are connected, their value will be determined by the loop, and will be

$$\underbrace{\left(\frac{1}{2s+1} \sum_{m=-s}^{s} m^2 \right)}_{:= M_s^2} \cdot (-1)^{u-v} \leftarrow \text{spins alternate}$$

$$\Longrightarrow \langle \hat{S}_u \cdot \hat{S}_v \rangle_{L,\beta} = 3 \langle \hat{S}^z_u \hat{S}^z_v \rangle_{L,\beta} = 3 (-1)^{u-v} M_s^2 \; \mathbb{P}\{u,v \text{ connected by } \omega\}$$

$\uparrow$ in same loop

Note the relation with the Q-State Potts Model, whose partition function and correlations are analogous as follows:

	quantum spin chain	Q-state Potts
dim	d	$d+1$
part. fn.	number of loops	number of clusters
correlations	u,v in same loop	u,v in same cluster

Infinite-width

If we take $L \to \infty$ (with fixed parity of L), one can use FKG to show that the infinite measure converges and is invariant under translations by even shifts.

For different choices of d and s, we can get different results for uniqueness of Gibbs states, correlation decay, etc.

A Dichotomy for 2D Loop Systems

In the infinite limit, either every point is contained in infinitely many loops or all points are in finitely many loops. In the finite case, the parity of the loops introduces dimerization, causing long range order and translational symmetry breaking.

Consider the following derivation: by translation invariance,

$$\sum_n n\left|\langle \hat{\vec{S}}_0 \cdot \hat{\vec{S}}_n \rangle\right| = \sum_{\substack{u>0\\ v<0}} \left|\langle \hat{\vec{S}}_u \cdot \hat{\vec{S}}_v \rangle\right| = M_s^2 \sum_{\substack{u>0\\ v<0}} \mathbb{P}\Big\{ \overbrace{v \quad 0 \quad u}^{\text{connected by } w} \Big\}$$

Since loops don't overlap in Heisenberg anti-ferromagnet, every loop containing the origin must add another connected u-v pair. So,

$$\sum_n n\left|\langle \hat{\vec{S}}_0 \cdot \hat{\vec{S}}_n \rangle\right| \geq M_s^2\, \mathbb{E}[\#\text{ loops encircling } 0]$$

If # of loops about 0 is infinite (Kolmogorov 0-1 gives $\mathbb{E}[\#] = \infty$), then the sum must also diverge. In particular, $\langle \hat{\vec{S}}_0 \cdot \hat{\vec{S}}_n \rangle$ decays <u>no quicker</u> than $\frac{1}{n^2}$.

So, we get that either

(i) dimerization + translational symmetry breaking + long range order
(multiple Gibbs measures, shifted by 1)

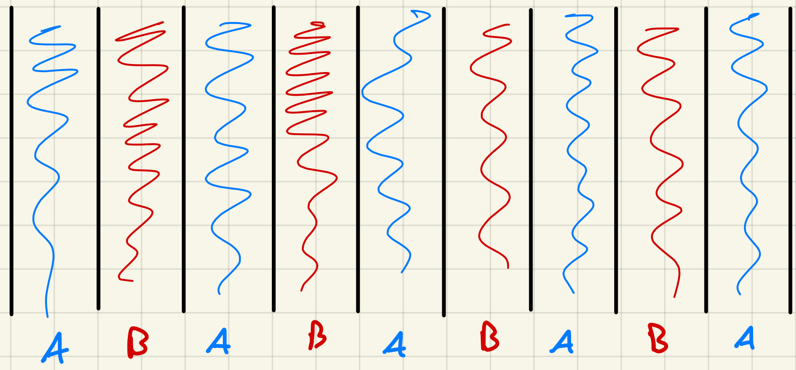
(ii) spin-spin correlation decays slower than $\frac{1}{|u-v|^2}$

This is a result of a general result: 2D loop dichotomy

either (i) long-range-order or (ii) slower correlation decay

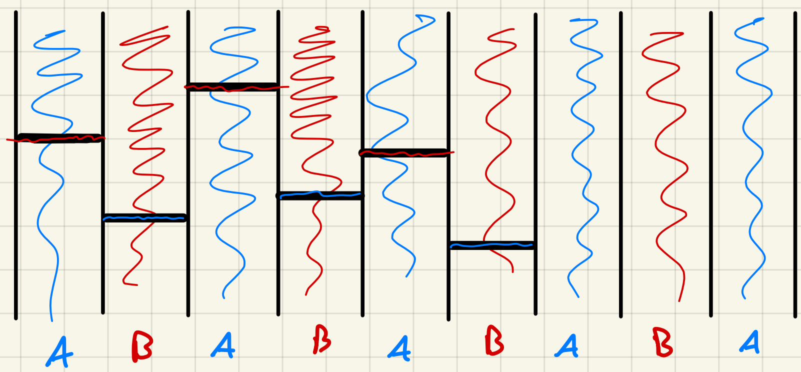
Lecture 4/27. Final Lecture

We will now discuss the A/B continuum percolation model. If the space is split into strips,



"consider the city of Venice"

Now consider randomly placed connections between strips

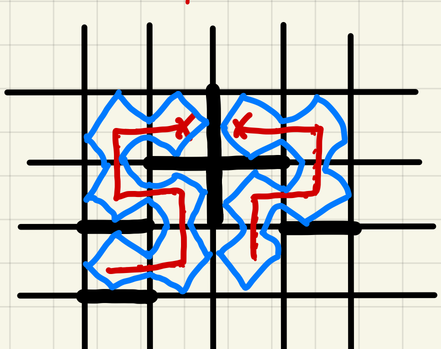


Each rung "connects" strips of the same color.

If we consider A as (+) and B as (−), we can place a total order on states via $\omega < \omega' \iff$ under ω', A is "more connected"

This arises naturally from investigation into quantum spin models.

However, recall FK-random cluster models (Q-state Potts w/ $Q \in \mathbb{R}^+$)

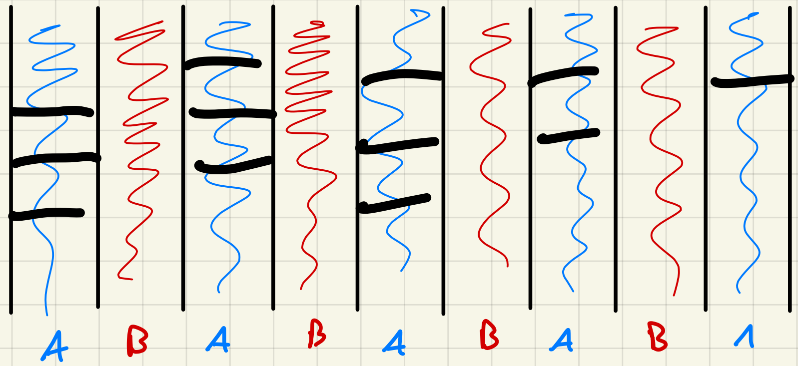


We can show that as we take the continuum limit of the lattice, we get a model equivalent to the A/B above, with the difference that the A/B measure is weighted by an extra factor

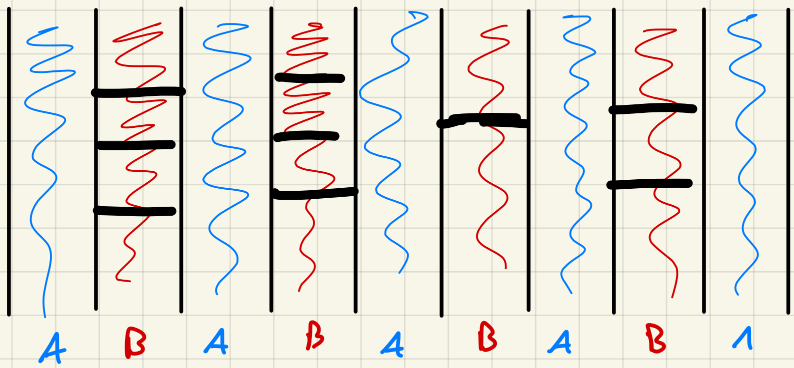
$$(2s+1)^{N_c(\omega)}$$

\# of clusters

Note that each time we add a rung to the same color, we add another cluster. In the extreme, we find two preferred types of states

(−) 

and

(+) 

In the infinite limit, we can find that both

$$\psi^- = \lim_{\substack{L\to\infty \\ L \text{ even}}} \frac{1}{L^d} \log\left(\mathrm{Tr}\, e^{-\beta H}\right) \quad \text{and} \quad \psi^+ = \lim_{\substack{L\to\infty \\ L \text{ odd}}} \frac{1}{L^d} \log\left(\mathrm{Tr}\, e^{-\beta H}\right)$$

When $s > \frac{1}{2}$, $\psi^- \neq \psi^+$, and there is symmetry breaking via dimerization.
When $s = \frac{1}{2}$, $\psi^- = \psi^+$, and there is slow decay of correlations.

Duality!