# ORF 543: Homework 1

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# Problem A

Suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$ is a Gaussian. Show that

$$\kappa_k(X) = \begin{cases} \mu, & k = 1 \\ \sigma^2, & k = 2 \\ 0, & k \geq 3 \end{cases}$$

*(5 points)*

## Solution

**Proof.** For Gaussian $X$, we can explicitly compute the expectation. Using the substitution $u := \frac{z-\mu}{\sigma\sqrt{2}}$,

$$\begin{aligned} \mathbb{E}\left[e^{itX}\right] &= \int_{-\infty}^{\infty} e^{itz} \frac{e^{-\frac{(z-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dz = \frac{\sigma\sqrt{2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{it(\sigma\sqrt{2}u+\mu)-u^2} du \\ &= \frac{e^{it\mu}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2+it\sigma\sqrt{2}u} du = \frac{e^{it\mu}}{\sqrt{\pi}} \left(\sqrt{\pi} e^{-\frac{t^2\sigma^2}{2}}\right) = e^{i\mu t - \frac{\sigma^2 t^2}{2}} \\ &\implies log\, \mathbb{E}\left[e^{itX}\right] = i\mu t - \frac{\sigma^2 t^2}{2} \end{aligned}$$

We can compute

$$\kappa_1(X) = \frac{1}{i} \frac{d}{dt} \left[ i\mu t - \frac{\sigma^2 t^2}{2} \right]_{t=0} = \frac{i\mu}{i} = \mu,$$

$$\kappa_2(X) = \frac{1}{-1} \frac{d^2}{dt^2} \left[ i\mu t - \frac{\sigma^2 t^2}{2} \right]_{t=0} = \frac{-\sigma^2}{-1} = \sigma^2,$$

and, since for $k \geq 3$ the derivatives vanish,

$$\kappa_k(X) = \frac{1}{i^k} \frac{d^k}{dt^k} \left[ i\mu t - \frac{\sigma^2 t^2}{2} \right]_{t=0} = 0,$$

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# Problem B

Suppose $X, Y$ are independent. Show that for all $k \geq 0$

$$\kappa_k(X+Y) = \kappa_k(X) + \kappa_k(Y)$$

*(5 points)*

## Solution

**Proof.** Since $X$ and $Y$ are independent, we know that $\mathbb{E}\left[e^{it(X+Y)}\right] = \mathbb{E}\left[e^{itX}e^{itY}\right] = \mathbb{E}\left[e^{itX}\right]\mathbb{E}\left[e^{itY}\right]$. So,

$$\begin{aligned}
\kappa_k(X+Y) &= \frac{1}{i^k}\left.\frac{d^k}{dt^k}\right|_{t=0} log\,\mathbb{E}\left[e^{it(X+Y)}\right] = \frac{1}{i^k}\left.\frac{d^k}{dt^k}\right|_{t=0} log\,\left(\mathbb{E}\left[e^{itX}\right]\mathbb{E}\left[e^{itY}\right]\right) \\
&= \frac{1}{i^k}\left.\frac{d^k}{dt^k}\right|_{t=0} \left(log\,\mathbb{E}\left[e^{itX}\right] + log\,\mathbb{E}\left[e^{itY}\right]\right) \\
&= \frac{1}{i^k}\left.\frac{d^k}{dt^k}\right|_{t=0} log\,\mathbb{E}\left[e^{itX}\right] + \frac{1}{i^k}\left.\frac{d^k}{dt^k}\right|_{t=0} log\,\mathbb{E}\left[e^{itY}\right] \\
&= \kappa_k(X) + \kappa_k(Y)
\end{aligned}$$

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# Problem C

Suppose $X_1, ..., X_n$ are i.i.d random variables with the same distribution as a random variable $X$. Compute the cumulants of

$$S_n := \frac{1}{n} \sum_{i=1}^{n} X_i$$

in terms of the cumulants of $X$ and of $n$. *(10 points)*

## Solution

**Proof.** We want to first investigate how the cumulant changes when we scale a random variable. Let $Y$ be any random variable and let $a \in \mathbb{R}$ be arbitrary. Define the cumulant generating function of $Y$ as

$$K_Y(t) = log\ \mathbb{E}\left[e^{itY}\right]$$

Then,

$$K_{aY}(t) = log\ \mathbb{E}\left[e^{it(aY)}\right] = log\ \mathbb{E}\left[e^{i(at)Y}\right] = K_Y(at)$$

So, we find that the $k^{th}$ cumulant of a scaled variable can be found to be

$$\kappa_k(aY) = \frac{1}{i^k} \left.\frac{d^k}{dt^k}\right|_{t=0} K_{aY}(t) = \frac{1}{i^k} \left.\frac{d^k}{dt^k}\right|_{t=0} K_Y(at) = a^k \cdot \frac{1}{i^k} \left.\frac{d^k}{dt^k}\right|_{t=0} K_Y(t) = a^k \kappa_k(Y),$$

where the third equality comes from the chain rule and the fact that $at$ and $t$ look the same when evaluated at $t = 0$ (we can think of this by imagining performing $k$ derivatives on $K_Y(at)$ and $k$ derivatives on $K_Y(t)$; when we evaluate at $t = 0$, they will only differ by a factor of $a^k$). With this in mind, we see that scaling a random variable by a constant scales the $k^{th}$ cumulant by the $k^{th}$ power of that constant. This, along with the result from part **(b)** and the fact that all the $X_i$ have the same cumulants as $X$, is enough to see that for all $k \geq 0$:

$$\kappa_k(S_n) = \frac{1}{n^k} \kappa_k\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^k} \sum_{i=1}^{n} \kappa_k(X_i) = \frac{1}{n^k} \cdot n \cdot \kappa_k(X) = \frac{\kappa_k(X)}{n^{k-1}}$$

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# Problem D

Fix $\vec{x} \in \mathbb{R}^{n_0}$. Prove that

$$\kappa_k(z^{(2)}(\vec{x})) = \begin{cases} 0, & k \text{ odd} \\ O\left(n_1^{1-\frac{k}{2}}\right), & k \text{ even} \end{cases}$$

*(10 points)*

## Solution

**Proof.** We can write the output of the final layer of the network as

$$z^{(2)}(\vec{x}) = b^{(2)} + \sum_{j=1}^{n_1} W_j^{(2)} \sigma\left(z_j^{(1)}\right),$$

where $W_j^{(2)} \sim \mathcal{N}(0, \frac{C_W}{n_1})$, $W_{ij}^{(1)} \sim \mathcal{N}(0, \frac{C_W}{n_0})$, and $b_j^{(1)}, b^{(2)} \sim \mathcal{N}(0, C_b)$ are independent, and $z_j^{(1)} = W_j^{(1)}\vec{x} + b_j^{(1)}$. (A helpful thing to note here is that this means all the $z_j^{(1)}$'s are independent since they are functions of $W_j^{(1)}$'s and $b_j^{(1)}$'s, which are independent for different $j$'s; this is abused repeatedly here and in later parts). Now, we start small by observing that since $W_j^{(2)} \sim \mathcal{N}\left(0, \frac{C_W}{n_1}\right)$

$$\mathbb{E}\left[e^{itW_j^{(2)}\sigma\left(z_j^{(1)}\right)}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{C_W}{n_1}}} e^{itws} e^{-\frac{w^2}{2\frac{C_W}{n_1}}} dw \cdot PDF(S),$$

where $S$ denotes the random variable $\sigma(z_j^{(1)})$ and $PDF(S)$ is the differential probability density of the continuous variable $s$ over all the values that $S$ can take. Then, we can use the substitution $w' = \frac{w}{\sqrt{2\frac{C_W}{n_1}}}$ and some usual Gaussian integral magic to see

$$\begin{aligned}\mathbb{E}\left[e^{itW_j^{(2)}\sigma\left(z_j^{(1)}\right)}\right] &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it\sqrt{2\frac{C_W}{n_1}}sw'} e^{-w'^2} dw' \cdot PDF(s) \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \sqrt{\pi} e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}s^2} PDF(s) = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}s^2} PDF(s) \\ &= \mathbb{E}\left[e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}\cdot\sigma^2(z_j^{(1)})}\right]\end{aligned}$$

where the last step comes from the definition of expectation. Therefore, we see that

$$\kappa_k\left(W_j^{(2)}\sigma\left(z_j^{(1)}\right)\right) = \frac{1}{i^k}\left.\frac{d^k}{dt^k}\right|_{t=0} log\ \mathbb{E}\left[e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}\cdot\sigma^2(z_j^{(1)})}\right]$$

Note, however, that the function $log\ \mathbb{E}\left[e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}\cdot\sigma^2(z_j^{(1)})}\right]$ is an even function of $t$, and therefore the $k^{th}$ derivative of this must vanish for odd $k$. This means that for odd $k$, $\kappa_k\left(W_j^{(2)}\sigma\left(z_j^{(1)}\right)\right) = 0$, and so we find that by independence, for odd $k$ it holds that

$$\kappa_k\left(z^{(2)}\right) = \kappa_k\left(b^{(2)}\right) + \sum_{j=1}^{n_1}\kappa_k\left(W_j^{(2)}\sigma\left(z_j^{(1)}\right)\right) = \kappa_k\left(b^{(2)}\right) = 0,$$

where we know that $\kappa_k\left(b^{(2)}\right) = 0$ because the odd cumulants of a zero mean Gaussian are 0. Suppose now that $k$ is even. Define $W = \sqrt{n_1}\cdot W_j^{(2)}$ to be a rescaled version of the random variable $W_j^{(2)}$. Then, $W \sim \mathcal{N}(0, C_W)$. So, we can say that

$$\kappa_k\left(W\sigma\left(z_j^{(1)}\right)\right) = n_1^{\frac{k}{2}}\kappa_k\left(W_j^{(2)}\sigma\left(z_j^{(1)}\right)\right)$$

The cumulant in the left hand side of this equation is written in terms of variables that have nothing to do with $n_1$; so we can say that $\kappa_k\left(W_j^{(2)}\sigma\left(z_j^{(1)}\right)\right) = O\left(n_1^{-\frac{k}{2}}\right)$. This yields that for even $k$,

$$\kappa_k\left(z^{(2)}\right) = \kappa_k\left(b^{(2)}\right) + \sum_{j=1}^{n}\kappa_k\left(W_j^{(2)}\sigma\left(z_j^{(1)}\right)\right) = \kappa_k\left(b^{(2)}\right) + \sum_{j=1}^{n_1} O\left(n_1^{-\frac{k}{2}}\right) = O\left(n_1^{1-\frac{k}{2}}\right),$$

where we note that $\kappa_k\left(b^{(2)}\right) = \begin{cases} C_b = O(1) = O\left(n_1^{1-\frac{k}{2}}\right), & k = 2 \\ 0, & k \neq 2 \end{cases}$. We have arrived at the result that

$$\kappa_k\left(z^{(2)}\right) = \begin{cases} 0, & k \text{ odd} \\ O\left(n_1^{1-\frac{k}{2}}\right), & k \text{ even} \end{cases}$$

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# Problem E

Fix $\vec{x} \in \mathbb{R}^{n_0}$ and define

$$\Sigma^{(2)} := C_b + \frac{C_W}{n_1} \sum_{i=1}^{n_1} \left( \sigma(z_i^{(1)}(\vec{x})) \right)^2$$

Show that

$$\mathbb{E}\left[\Sigma^{(2)}\right] = K^{(2)}$$

Use this to show that for all $t \in \mathbb{R}$

$$log\ \mathbb{E}\left[e^{itz^{(2)}(\vec{x})}\right] = -\frac{t^2}{2} K^{(2)} + log\ \mathbb{E}\left[e^{-\frac{t^2}{2}\Delta^{(2)}}\right]$$

where

$$\Delta^{(2)} := \Sigma^{(2)} - \mathbb{E}\left[\Sigma^{(2)}\right]$$

*(15 points)*

## Solution

**Proof.** We have

$$\mathbb{E}\left[\Sigma^{(2)}\right] = C_b + \frac{C_W}{n_1} \sum_{i=1}^{n_1} \mathbb{E}\left[ \left( \sigma(z_i^{(1)}(\vec{x})) \right)^2 \right]$$

Much of this proof will feel like repeating steps made in part **(d)**; that is because they are. Note that $z_i^{(1)}$ is a random variable distributed as $\mathcal{N}\left(0, C_b + \frac{C_W}{n_0}||\vec{x}||^2\right) = \mathcal{N}\left(0, K^{(1)}\right)$. So, we see that

$$\mathbb{E}\left[\left(\sigma(z_i^{(1)})\right)^2\right] = \left\langle \sigma^2 \right\rangle_{K^{(1)}}$$

$$\implies \mathbb{E}\left[\Sigma^{(2)}\right] = C_b + \frac{C_W}{n_1} \cdot n_1 \cdot \left\langle \sigma^2 \right\rangle_{K^{(1)}} = C_b + C_W \cdot \left\langle \sigma^2 \right\rangle_{K^{(1)}} = K^{(2)}$$

Now, we start small by observing that since $W_j^{(2)} \sim \mathcal{N}\left(0, \frac{C_W}{n_1}\right)$

$$\mathbb{E}\left[e^{itW_j^{(2)}\sigma\left(z_j^{(1)}\right)}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{C_W}{n_1}}} e^{itws} e^{-\frac{w^2}{2\frac{C_W}{n_1}}}\, dw \cdot PDF(S),$$

where $S$ denotes the random variable $\sigma(z_j^{(1)})$ and $PDF(S)$ is the differential probability density of the continuous variable $s$ over all the values that $S$ can take. Then, we can use the substitution $w' = \frac{w}{\sqrt{2\frac{C_W}{n_1}}}$ and some usual Gaussian integral magic to see

$$\begin{aligned}
\mathbb{E}\left[e^{itW_j^{(2)}\sigma\left(z_j^{(1)}\right)}\right] &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it\sqrt{2\frac{C_W}{n_1}}sw'} e^{-w'^2} dw' \cdot PDF(s) \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sqrt{\pi} e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}s^2} PDF(s) = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}s^2} PDF(s) \\
&= \mathbb{E}\left[e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}\cdot\sigma^2(z_j^{(1)})}\right]
\end{aligned}$$

where the last step comes from the definition of expectation. So, we see that since all our random variables are independent and $b^{(2)}$ being Gaussian $\implies \mathbb{E}\left[e^{itb^{(2)}}\right] = e^{-\frac{t^2}{2}C_b}$,

$$\mathbb{E}\left[e^{itW_j^{(2)}\sigma\left(z_j^{(1)}\right)}\right] = \mathbb{E}\left[e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}\cdot\sigma^2(z_j^{(1)})}\right]$$

Problem E continued on next page. . . 

$$\implies \mathbb{E}\left[e^{itb^{(2)}}\right]\prod_{j=1}^{n_1}\mathbb{E}\left[e^{itW_j^{(2)}\sigma\left(z_j^{(1)}\right)}\right] = e^{-\frac{t^2}{2}C_b}\prod_{j=1}^{n_1}\mathbb{E}\left[e^{-\frac{t^2}{2}\cdot\frac{C_W}{n_1}\cdot\sigma^2(z_j^{(1)})}\right]$$

$$\implies \mathbb{E}\left[exp\left\{it\left(b^{(2)}+\sum_{j=1}^{n_1}W_j^{(2)}\sigma\left(z_j^{(1)}\right)\right)\right\}\right] = \mathbb{E}\left[exp\left\{-\frac{t^2}{2}\left(C_b+\frac{C_W}{n_1}\sum_{j=1}^{n_1}\sigma^2\left(z_j^{(1)}\right)\right)\right\}\right]$$

$$\implies \mathbb{E}\left[e^{itz^{(2)}}\right] = \mathbb{E}\left[e^{-\frac{t^2}{2}\Sigma^{(2)}}\right]$$

Since we can write $\Sigma^{(2)} = \Delta^{(2)} + \mathbb{E}\left[\Sigma^{(2)}\right] = \Delta^{(2)} + K^{(2)}$, we find that

$$\mathbb{E}\left[e^{itz^{(2)}}\right] = \mathbb{E}\left[e^{-\frac{t^2}{2}K^{(2)}}\cdot e^{-\frac{t^2}{2}\Delta^{(2)}}\right] = e^{-\frac{t^2}{2}K^{(2)}}\cdot\mathbb{E}\left[e^{-\frac{t^2}{2}\Delta^{(2)}}\right]$$

$$\implies log\,\mathbb{E}\left[e^{itz^{(2)}}\right] = -\frac{t^2}{2}K^{(2)} + log\,\mathbb{E}\left[e^{-\frac{t^2}{2}\Delta^{(2)}}\right]$$

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# Problem F

Show that

$$\mathbb{E}\left[\Delta^{(2)}\right] = 0$$

and that for all $q \geq 2$ we have

$$\mathbb{E}\left[\left(\Delta^{(2)}\right)^q\right] = O\left(n_1^{-\lceil \frac{q}{2} \rceil}\right)$$

*(15 points)*

## Solution

**Proof.** Clearly, we have that since $\mathbb{E}\left[\Sigma^{(2)}\right] = K^{(2)}$ is nonrandom,

$$\mathbb{E}\left[\Delta^{(2)}\right] = \mathbb{E}\left[\Sigma^{(2)} - K^{(2)}\right] = \mathbb{E}\left[\Sigma^{(2)}\right] - K^{(2)} = 0$$

We will first show that we can define a moment of a random variable $X$ in terms of its lower moments and cumulants. Observe that if we let the moment generating function of $X$ be

$$M_X(t) = \mathbb{E}\left[e^{itX}\right] = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \mathbb{E}\left[X^j\right],$$

we can then relate the cumulant generating function to this with

$$K_X(t) = log\, \mathbb{E}\left[e^{itX}\right] = log\, M_X(t) \implies M_X(t) = e^{K_X(t)}$$

We can take $k$ derivatives of $M_X(t)$ using the Leibniz Rule to find that

$$\begin{aligned}
\frac{d^k}{dt^k} M_X(t) = \frac{d^k}{dt^k} e^{K_X(t)} &= \frac{d^{k-1}}{dt^{k-1}}\left[\left(\frac{d}{dt}K_X(t)\right) M_X(t)\right] \\
&= \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{d^{k-1-j}}{dt^{k-1-j}}\left[\frac{d}{dt}K_X(t)\right] \frac{d^j}{dt^j}\left[M_X(t)\right] = \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{d^{k-j}}{dt^{k-j}} K_X(t) \frac{d^j}{dt^j} M_X(t)
\end{aligned}$$

We can evaluate this derivative at $t = 0$. Let $\mu_j(X) = \frac{1}{i^j} \frac{d^j}{dt^j}\Big|_{t=0} M_X(t)$ be the $j^{th}$ moment and $\kappa_j(X) = \frac{1}{i^j} \frac{d^j}{dt^j}\Big|_{t=0} K_X(t)$ be the $j^{th}$ cumulant. Then, we get that

$$i^k \mu_k(X) = \sum_{j=0}^{k-1} \binom{k-1}{j} i^{k-j} \kappa_{k-j}(X) \cdot i^j \mu_j(X) \implies \mu_k(X) = \sum_{j=0}^{k-1} \binom{k-1}{j} \kappa_{k-j}(X) \mu_j(X)$$

We are interested in investigating the order of growth of $\mathbb{E}\left[\left(\Delta^{(2)}\right)^q\right] = \mu_q\left(\Delta^{(2)}\right)$. However, it will be necessary to first understand the order of growth of the cumulants $\kappa_q\left(\Delta^{(2)}\right)$. We know from the definition of the cumulant generating function as the power series of cumulants that

$$\sum_{j=0}^{\infty} \frac{\kappa_j\left(\Delta^{(2)}\right)}{j!} (iu)^j = log\, \mathbb{E}\left[e^{iu\Delta^{(2)}}\right]$$

With the substitution of $u = i\frac{t^2}{2}$, we find that

$$\sum_{j=0}^{\infty} \frac{\kappa_j\left(\Delta^{(2)}\right)}{(-2)^j j!} (t^2)^j = log\, \mathbb{E}\left[e^{-\frac{t^2}{2}\Delta^{(2)}}\right]$$

With the result from part **(e)**, we know that

$$log\, \mathbb{E}\left[e^{-\frac{t^2}{2}\Delta^{(2)}}\right] = log\, \mathbb{E}\left[e^{itz^{(2)}}\right] + \frac{t^2}{2}K^{(2)},$$

yielding that for all $k > 0$,

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{\kappa_j\left(\Delta^{(2)}\right)}{(-2)^j j!}t^{2j} &= log\, \mathbb{E}\left[e^{itz^{(2)}}\right] + \frac{t^2}{2}K^{(2)} \\
\implies \frac{d^{2k}}{dt^{2k}}\Big|_{t=0}\left[\sum_{j=0}^{\infty} \frac{\kappa_j\left(\Delta^{(2)}\right)}{(-2)^j j!}t^{2j}\right] &= \frac{d^{2k}}{dt^{2k}}\Big|_{t=0}\left[log\, \mathbb{E}\left[e^{itz^{(2)}}\right] + \frac{t^2}{2}K^{(2)}\right] \\
\implies \left[\sum_{j=k}^{\infty} \frac{\kappa_j\left(\Delta^{(2)}\right)}{(-2)^j j!}\frac{(2j)!}{(2j-2k)!}t^{2j-2k}\right]_{t=0} &= (-1)^k \kappa_{2k}\left(z^{(2)}\right) + K^{(2)} \cdot \mathbb{1}_{k=1} \\
\implies \frac{(2k)!}{(-2)^k k!}\kappa_k\left(\Delta^{(2)}\right) &= (-1)^k \kappa_{2k}\left(z^{(2)}\right) + K^{(2)} \cdot \mathbb{1}_{k=1} \\
\implies \kappa_k\left(\Delta^{(2)}\right) &= \frac{2^k k!}{(2k)!}\kappa_{2k}\left(z^{(2)}\right) - K^{(2)} \cdot \mathbb{1}_{k=1},
\end{aligned}$$

where $\mathbb{1}_{k=1}$ is an indicator that is 1 when $k = 1$ and 0 otherwise. We can plug in the result from part **(d)** to find that $\kappa_{2k}\left(z^{(2)}\right) = O\left(n_1^{1-k}\right)$, and so

$$\kappa_k\left(\Delta^{(2)}\right) = O\left(n_1^{1-k}\right)$$

We now have all that we need. We will show the claim by strong induction on $q$. Clearly, the base case holds for $q = 1$, since $\mathbb{E}\left[\left(\Delta^{(2)}\right)^1\right] = 0 = O(1)$. Suppose now that the claim holds for all moments $k < q$; we want to show that the claim holds for the $q^{th}$ moment. We can write out our recursive relation

$$\mu_q\left(\Delta^{(2)}\right) = \sum_{k=0}^{q-1}\binom{q-1}{k}\kappa_{q-k}\left(\Delta^{(2)}\right)\mu_k\left(\Delta^{(2)}\right) = \sum_{k=0}^{q-2} O(1)\cdot O\left(n_1^{1-(q-k)}\right)\cdot O\left(n_1^{-\lceil \frac{k}{2}\rceil}\right),$$

where the last equality comes from our previous result and the application of our inductive hypothesis. Note here that the final index of the sum decreased to $q-2$; this is because element of the sum with $k = q-1$ evaluates to 0 since $\kappa_1\left(\Delta^{(2)}\right) = \mathbb{E}\left[\Delta^{(2)}\right] = 0$. Using the identity that $\lceil \frac{k}{2}\rceil + \lfloor \frac{k}{2}\rfloor = k$ for all integers $k$, we can write

$$\mu_q\left(\Delta^{(2)}\right) = \sum_{k=0}^{q-2} O\left(n_1^{1-q+\lfloor \frac{k}{2}\rfloor}\right)$$

Observe that the largest exponent occurs for the largest possible value of $k$, which is $k = q-2$. So, we find that the entire sum has an order of growth equal to $O\left(n_1^{1-q+\lfloor \frac{q-2}{2}\rfloor}\right)$. We can simplify this exponent: note that for all integers $q$ we have $\lfloor \frac{q-2}{2}\rfloor = \lfloor \frac{q}{2}\rfloor - 1 = q - \lceil \frac{q}{2}\rceil - 1$. Applying this,

$$\mu_q\left(\Delta^{(2)}\right) = O\left(n_1^{1-q+q-\lceil \frac{q}{2}\rceil - 1}\right) = O\left(n_1^{-\lceil \frac{q}{2}\rceil}\right)$$

This is exactly our inductive claim for the $q^{th}$ moment. So, we can say by induction that for all $q > 1$,

$$\mathbb{E}\left[\left(\Delta^{(2)}\right)^q\right] = O\left(n_1^{-\lceil \frac{q}{2}\rceil}\right)$$

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# Problem G

Use **(e)** and **(f)** to prove that

$$log\ \mathbb{E}\left[e^{itz^{(2)}(\vec{x})}\right] = -\frac{t^2}{2}K^{(2)} + \frac{t^4}{8}\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] + O(n_1^{-2})$$

Conclude that

$$\kappa_4\left(z^{(2)}(\vec{x})\right) = \frac{3C_W^2}{n_1}\left(\left\langle\sigma^4(z)\right\rangle_{K^{(1)}} - \left\langle\sigma^2(z)\right\rangle_{K^{(1)}}^2\right)$$

*(15 points)*

## Solution

**Proof.** We start by examining

$$log\ \mathbb{E}\left[e^{-\frac{t^2}{2}\Delta^{(2)}}\right] = log\ \mathbb{E}\left[\sum_{k=0}^{\infty}\frac{\left(\frac{-t^2}{2}\right)^k}{k!}\left(\Delta^{(2)}\right)^k\right] = log\ \sum_{k=0}^{\infty}\frac{\left(\frac{-t^2}{2}\right)^k}{k!}\mathbb{E}\left[\left(\Delta^{(2)}\right)^k\right]$$

We have seen from part **(f)** that the expectation vanishes when $k = 1$ and has order $O\left(n_1^{-\lceil\frac{k}{2}\rceil}\right)$ when $k \geq 3$. So, we can break what we have into

$$log\ \left(1 + 0 + \frac{t^4}{8}\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] + \sum_{k=3}^{\infty}\frac{\left(\frac{-t^2}{2}\right)^k}{k!}\mathbb{E}\left[\left(\Delta^{(2)}\right)^k\right]\right) = log\ \left(1 + \frac{t^4}{8}\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] + \beta\right),$$

where $\beta = \sum_{k=3}^{\infty}\frac{\left(\frac{-t^2}{2}\right)^k}{k!}\mathbb{E}\left[\left(\Delta^{(2)}\right)^k\right] = O\left(n_1^{-2}\right)$. We can expand the function $log(1+x)$ about $\beta$, and again about 0, to see that since $\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] = O(n_1^{-1})$

$$log\ \left(1 + \frac{t^4}{8}\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] + \beta\right) = log\left(1 + \frac{t^4}{8}\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right]\right) + O\left(n_1^{-2}\right) = \frac{t^4}{8}\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] + O\left(n_1^{-2}\right)$$

We can plug this into the result from part **(e)** to see

$$log\ \mathbb{E}\left[e^{itz^{(2)}(\vec{x})}\right] = -\frac{t^2}{2}K^{(2)} + \frac{t^4}{8}\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] + O(n_1^{-2})$$

So, we can find the fourth cumulant to be

$$\kappa_4\left(z^{(2)}\right) = \frac{1}{i^4}\left.\frac{d^4}{dt^4}\right|_{t=0}\left[-\frac{t^2}{2}K^{(2)} + \frac{t^4}{8}\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] + O(n_1^{-2})\right] = 3\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right]$$

We can observe the cuteness of this expectation via

$$\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] = \mathbb{E}\left[\left(\Sigma^{(2)}\right)^2 - 2\mathbb{E}\left[\Sigma^{(2)}\right]\Sigma^{(2)} + \mathbb{E}\left[\Sigma^{(2)}\right]^2\right] = \mathbb{E}\left[\left(\Sigma^{(2)}\right)^2\right] - \mathbb{E}\left[\Sigma^{(2)}\right]^2 = Var\left[\Sigma^{(2)}\right]$$

Returning to the definition of $\Sigma^{(2)}$, since all the $z_i^{(1)}$'s are independent

$$\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] = Var\left[\Sigma^{(2)}\right] = Var\left[C_b + \frac{C_W}{n_1}\sum_{i=1}^{n_1}\left(\sigma(z_i^{(1)})\right)^2\right] = \frac{C_W^2}{n_1^2}\sum_{i=1}^{n_1}Var\left[\left(\sigma(z_i^{(1)})\right)^2\right]$$

$$= \frac{C_W^2}{n_1^2}\sum_{i=1}^{n_1}\left(\left\langle\sigma^4(z)\right\rangle_{K^{(1)}} - \left\langle\sigma^2(z)\right\rangle_{K^{(1)}}^2\right) = \frac{C_W^2}{n_1}\left(\left\langle\sigma^4(z)\right\rangle_{K^{(1)}} - \left\langle\sigma^2(z)\right\rangle_{K^{(1)}}^2\right)$$

Problem G continued on next page. . . 

We then evaluate the fourth cumulant to be

$$\kappa_4\left(z^{(2)}\right) = 3\mathbb{E}\left[\left(\Delta^{(2)}\right)^2\right] = \frac{3C_W^2}{n_1}\left(\left\langle\sigma^4(z)\right\rangle_{K^{(1)}} - \left\langle\sigma^2(z)\right\rangle_{K^{(1)}}^2\right)$$

■

# Problem H

Assume that $f : \mathbb{R} \to \mathbb{R}$ is a smooth function that grows no faster than a polynomial at infinity. Show the following width refinement

$$\mathbb{E}\left[f(z^{(2)}(\vec{x}))\right] = \langle f(z)\rangle_{K^{(2)}} + \frac{C_W^2}{8n_1}\left(\langle\sigma^4(z)\rangle_{K^{(1)}} - \langle\sigma^2(z)\rangle_{K^{(1)}}^2\right)\langle D^4 f(z)\rangle_{K^{(2)}} + O(n_1^{-2}),$$

where $D^4$ stands for the fourth derivative. *(25 points)*

## Solution

**Proof.** We can say from the first result in part **(g)** that the cumulant generating function takes the form

$$log\ \mathbb{E}\left[e^{itz^{(2)}}\right] = -\frac{t^2}{2!}K^{(2)} + \frac{t^4}{4!}\kappa_4(z^{(2)}) + O\left(n_1^{-2}\right)$$

Exponentiating,

$$\mathbb{E}\left[e^{itz^{(2)}}\right] = e^{-\frac{t^2}{2!}K^{(2)}} \cdot e^{\frac{t^4}{4!}\kappa_4(z^{(2)})+O\left(n_1^{-2}\right)}$$

Now, consider the Fourier Transform $F(\cdot)$ of our nice and smooth arbitrary function $f$ such that $f(z) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega z}d\omega$. We can write

$$\mathbb{E}\left[f(z^{(2)})\right] = \mathbb{E}\left[\int_{-\infty}^{\infty} F(\omega)e^{i\omega z^{(2)}}d\omega\right] = \int_{-\infty}^{\infty} F(\omega)\mathbb{E}\left[e^{i\omega z^{(2)}}\right]d\omega,$$

where the expectation and integral play nice because $f$ is smooth and poly-bounded, and we can take the $F(\omega)$ out of the expectation because the expectation is over the random variable $z^{(2)}$. We can recognize the form of this expectation and substitute in our earlier approximation for it to get that

$$\int_{-\infty}^{\infty} F(\omega)\mathbb{E}\left[e^{i\omega z^{(2)}}\right]d\omega = \int_{-\infty}^{\infty} F(\omega)\cdot e^{-\frac{\omega^2}{2!}K^{(2)}} \cdot e^{\frac{\omega^4}{4!}\kappa_4(z^{(2)})+O\left(n_1^{-2}\right)}$$

Recognizing from part **(a)** that $e^{-\frac{\omega^2}{2}K^{(2)}}$ is the expectation $\mathbb{E}\left[e^{i\omega x}\right]$ over a random variable $x \sim \mathcal{N}\left(0, K^{(2)}\right)$,

$$= \int_{-\infty}^{\infty} F(\omega)\cdot\left\langle e^{i\omega z}\right\rangle_{K^{(2)}} \cdot e^{\frac{\omega^4}{4!}\kappa_4(z^{(2)})+O\left(n_1^{-2}\right)}$$

We can Taylor expand the exponential to see that

$$e^{\frac{\omega^4}{4!}\kappa_4(z^{(2)})+O\left(n_1^{-2}\right)} = 1 + \left(\frac{\omega^4}{4!}\kappa_4(z^{(2)}) + O\left(n_1^{-2}\right)\right) + \frac{1}{2!}\left(\frac{\omega^4}{4!}\kappa_4(z^{(2)}) + O\left(n_1^{-2}\right)\right)^2 + \ldots$$

Now, we know that $\kappa_4\left(z^{(2)}\right)$ is of order $O\left(n_1^{-1}\right)$, and so any power $q > 1$ will have $\left(\frac{\omega^4}{4!}\kappa_4(z^{(2)}) + O\left(n_1^{-2}\right)\right)^q$ on the order of $O\left(n_1^{-2}\right)$. Therefore, the exponential reduces to $1 + \frac{\omega^4}{4!}\kappa_4(z^{(2)}) + O\left(n_1^{-2}\right)$, and so

$$\begin{aligned}
\mathbb{E}\left[f(z^{(2)})\right] &= \int_{-\infty}^{\infty} F(\omega)\cdot\left\langle e^{i\omega z}\right\rangle_{K^{(2)}}\cdot\left(1 + \frac{\omega^4}{4!}\kappa_4(z^{(2)}) + O\left(n_1^{-2}\right)\right)d\omega \\
&= \int_{-\infty}^{\infty} F(\omega)\cdot\left\langle e^{i\omega z}\right\rangle_{K^{(2)}}d\omega + \int_{-\infty}^{\infty} F(\omega)\cdot\omega^4\cdot\left\langle e^{i\omega z}\right\rangle_{K^{(2)}}\cdot\frac{\kappa_4\left(z^{(2)}\right)}{4!}d\omega + O\left(n_1^{-2}\right) \\
&= \int_{-\infty}^{\infty}\left\langle F(\omega)e^{i\omega z}\right\rangle_{K^{(2)}}d\omega + \frac{\kappa_4\left(z^{(2)}\right)}{4!}\int_{-\infty}^{\infty}\left\langle \omega^4 F(\omega)e^{i\omega z}\right\rangle_{K^{(2)}}d\omega + O\left(n_1^{-2}\right)
\end{aligned}$$

Problem H continued on next page. . . 

We use the laws of Fourier land to see that the Fourier Transform of the fourth derivative $D^4 f(z) = f^{(4)}(z)$ is precisely equal to $i^4\omega^4 F(\omega) = \omega^4 F(\omega)$, where $F(\omega)$ is the Fourier Transform of $f$. This allows us swap the integrals and expectations once again and undo the Fourier Transforms, yielding:

$$\mathbb{E}\left[f(z^{(2)})\right] = \langle f(z)\rangle_{K^{(2)}} + \frac{\kappa_4\left(z^{(2)}\right)}{4!}\left\langle D^4 f(z)\right\rangle_{K^{(2)}} + O\left(n_1^{-2}\right)$$

Substituting the second result from part **(g)**, we get our final refinement

$$\mathbb{E}\left[f(z^{(2)})\right] = \langle f(z)\rangle_{K^{(2)}} + \frac{C_W^2}{8n_1}\left(\left\langle\sigma^4(z)\right\rangle_{K^{(1)}} - \left\langle\sigma^2(z)\right\rangle_{K^{(1)}}^2\right)\left\langle D^4 f(z)\right\rangle_{K^{(2)}} + O\left(n_1^{-2}\right)$$

■