1/30 -

1.1 - Locality

Def:

The operator $A \in \mathcal{B}(\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N)$ is called local iff

$$\inf_{x,y \in \mathbb{Z}^d} \frac{-1}{\|x-y\|} \log(\|A_{xy}\|) > 0$$

abs. val if $N=1$,
any matrix norm for $N>1$

This happens iff $\exists\, C, \mu \in (0,\infty)$ s.t. $\|A_{xy}\| \le C e^{-\mu\|x-y\|}$

Example: (discrete Laplacian)

We define the discrete Laplacian via $(-\Delta\psi)_x := \sum_{y \sim x} \psi_x - \psi_y \quad (x \in \mathbb{Z}^d)$
Letting $\{R_j\}_{j=1}^d$ be the right shift operators on $\ell^2(\mathbb{Z}^d)$,

$$-\Delta = 2d\,\mathbb{1} - \sum_{j=1}^d (R_j + R_j^*)$$

With this normalization, $\sigma(-\Delta) = \sigma_{a.c.}(-\Delta) = [0, 4d]$.

Note that $-\Delta$ is local since $(-\Delta)_{xy} = 0$ for $\|x-y\| > 1$.

1.2 - Bloch Decomposition & Fourier Series

Def:

$[-\pi,\pi)^d$, the d-torus

The Fourier transform is a map $\mathcal{F}: \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$ via

$$(\mathcal{F}\psi)(k) = \sum_{x \in \mathbb{Z}^d} e^{-ik\cdot x}\psi_x \quad (\psi \in \ell^1 \cap \ell^2,\ k \in \mathbb{T}^d)$$

and extended to all of ℓ^2 via BLT. It has inverse

$$(\mathcal{F}^{-1}\hat{\varphi})_x = \frac{1}{(2\pi)^d}\int_{k \in \mathbb{T}^d} e^{ik\cdot x}\hat{\varphi}(k)\,dk$$

With this, $\mathcal{F}$ is unitary (Parseval's thm). The value is that $\mathcal{F}$ diagonalizes periodic operators!

Def:

$A \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ is periodic iff $A_{xy} = A_{x+z,\, y+z} \quad (x,y,z \in \mathbb{Z}^d)$

Def:

For $a : \mathbb{T}^d \to \mathbb{C}$ bdd. a.e., we have the multiplication operator $M_a \in \mathcal{B}(L^2(\mathbb{T}^d))$ via

$$(M_a \hat{\psi})(k) = a(k)\hat{\psi}(k) \qquad (\hat{\psi} \in L^2(\mathbb{T}^d),\ k \in \mathbb{T}^d)$$

Lemma:

If $A \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ is periodic, then $\exists\, a : \mathbb{T}^d \to \mathbb{C}$ s.t.

① $FAF^* = M_a$

② $a(k) = \sum_{x \in \mathbb{Z}^d} e^{-ik\cdot x} A_{0,x}$

③ $\sigma(A) = \sigma_{a.c.}(A) = \mathrm{im}(a)$

We call a the symbol associated to A.

Proof: see notes :)

$\square$

Example

- The right shift operators $\{R_j\}_{j=1}^d$ defined by

$$(R_j \psi)_y = \psi_{y-e_j} \qquad (y \in \mathbb{Z}^d,\ \psi \in \ell^2(\mathbb{Z}^d))$$

is periodic with symbol

$$r_j(k) = \sum_{x \in \mathbb{Z}^d} e^{-ik\cdot x}(R_j)_{0,x} = \sum_x e^{-ik\cdot x}\langle \delta_0, \delta_{x-e_j}\rangle = e^{-ik\cdot e_j} = e^{-ik_j}$$

- The discrete Laplacian $-\Delta$ is periodic with symbol

$$\varepsilon(k) = 2d - 2\sum_{j=1}^d \cos(k_j)$$

- The position operators $\{X_j\}_{j=1}^d$ defined by $(X_j \psi)_y = y_j \psi_y$ gets mapped to

$$\mathcal{F} X_j \mathcal{F}^* = i \partial_{k_j}$$

- If A is periodic with symbol a then $[X_j, A]$ gets mapped to multiplication by the derivative

$$\mathcal{F} [X_j, A] \mathcal{F}^* = i M_{\partial_j a}$$

- If M_v is a multiplication operator on real space by $v: \mathbb{Z}^d \to \mathbb{R}$, then A is mapped to the convolution operator

$$\mathcal{F} M_v \mathcal{F}^* = C_{\mathcal{F} v}$$

<u>Theorem:</u> (Riemann-Lebesgue)

It holds that

A is local and periodic with symbol a $\iff$ $a: \mathbb{T}^d \to \mathbb{C}$ is analytic in an annulus

More generally,

A is polynomially-local w/ degree p and periodic w/ symbol a $\iff$ $a: \mathbb{T}^d \to \mathbb{C}$ is C^p in an annulus

2/1 -

Recall the generic Hilbert space

$$\mathcal{H} := \ell^2(\mathbb{Z}^d \to \mathbb{C}) \otimes \mathbb{C}^N \quad \text{with standard basis} \quad \{\delta_x \otimes e_j\}_{\substack{x \in \mathbb{Z}^d \\ j \in 1,\dots,N}}$$

and bounded, S.A. Hamiltonian $H = H^* \in \mathcal{B}(\mathcal{H})$.

Recall that H is local iff $\exists\, C, \mu > 0$ s.t.

$$\|H_{xy}\| \le C e^{-\mu\|x-y\|} \quad \leftarrow \text{local integral kernel}$$

1.3: Consequences of Locality

* Lieb-Robinson

Note that for the continuum Laplacian, $\sigma(-\Delta) = [0,\infty)$ is unbounded with dispersion $E(k) = \|k\|^2$.

Compare with the lattice Laplacian, $\sigma(-\Delta) = [0, 4d]$ is bounded with $E(k) = \sum_{j=1}^{d} 4\sin^2(\tfrac{1}{2}k_j) \le 4d$

So, locality + boundedness is necessary. Here's what it gets us:

Theorem (Lieb-Robinson, 1 particle)

has to be on the lattice

"max" velocity

Let $H = H^* \in \mathcal{B}(\mathcal{H})$ be local + bdd. Then, $\exists\, v_H > 0$ s.t. $\exists\, D > 0$ s.t.

$$\mathbb{P}\left\{\begin{array}{l}\text{particle starting at origin exits} \\ B_{tv}(0_{\mathbb{Z}^d}) \text{ after time } t\end{array}\right\} \le D e^{-\frac{1}{2}\mu_H (v - v_H)t} \quad (t \ge 0,\ v \ge v_H)$$

Proof: We start in state $\delta_0 \otimes \varphi$ for some $\varphi \in \mathbb{C}^N$. Time-evolving the system, at time t we have $e^{-itH}(\delta_0 \otimes \varphi)$. The probability of us ending in some state $\delta_x \otimes \psi \in \mathcal{H}$ is $|\langle \delta_x \otimes \psi, e^{-itH}\delta_0 \otimes \varphi\rangle|^2$ for $x \in B_{tv}(0)$, $\psi \in \mathbb{C}^N$. Thus,

take sup over φ, ψ

$$\mathbb{P}\left\{\begin{array}{l}\text{particle starting at origin exits} \\ B_{tv}(0_{\mathbb{Z}^d}) \text{ after time } t\end{array}\right\} \le \sum_{x \notin B_{tv}(0)} \|\langle \delta_x, e^{-itH}\delta_0\rangle\|^2$$

We may bound powers of H via

$$\|(H^n)_{xy}\| \overset{\text{insert } \mathbb{1}}{=} \Big\| \sum_{z_1,\dots,z_{n-1}\in\mathbb{Z}^d} H_{xz_1} H_{z_1z_2} \cdots H_{z_{n-1}y} \Big\| \overset{\text{locality}}{\le} \sum_{z_1,\dots,z_{n-1}} C_H^n e^{-\mu_H(\|x-z_1\|+\dots+\|z_{n-1}-y\|)}$$

$$\overset{\text{translation invariance \& triangle ineq.}}{\le} C_H^n e^{-\frac{1}{2}\mu_H\|x-y\|} \underbrace{\Big(\sum_{z\in\mathbb{Z}^d} e^{-\frac{1}{2}\mu_H\|z\|}\Big)}_{=:D\,\le\,\left(\sum_{z\in\mathbb{Z}} e^{-\frac{1}{2\sqrt{d}}\mu_H|z|}\right)^d}{}^{n-1} \le C_H^n D^{n-1} e^{-\frac{1}{2}\mu_H\|x-y\|}$$

So, we may estimate the propagator

$$\|e^{-itH}(0,x)\| = \|\langle\delta_x, e^{-itH}\delta_0\rangle\| \le \sum_{n=0}^{\infty}\frac{t^n}{n!}\|(H^n)_{x,0}\|$$

$$\le \sum_{n=0}^{\infty}\frac{t^n}{n!} C_H^n D^{n-1} e^{-\frac{1}{2}\mu_H\|x\|} = \frac{1}{D} e^{(C_H D t - \frac{1}{2}\mu_H\|x\|)}$$

So, $\mathbb{P}\left\{\begin{matrix}\text{particle starting at origin exits} \\ B_{tv}(0_{\mathbb{Z}^d}) \text{ after time } t\end{matrix}\right\} \le \sum_{x\notin B_{tv}(0)} \|e^{-itH}(0,x)\|^2$

$$\le \frac{1}{D^2}\sum_{x\notin B_{tv}(0)} e^{(2C_H D t - \mu_H\|x\|)}$$

split into $e^{-\frac{1}{2}\mu_H\|x\|}\cdot \underbrace{e^{-\frac{1}{2}\mu_H\|x\|}}_{\le e^{-\frac{1}{2}\mu_H vt}}$ since $\|x\|\ge vt$

$$\le \frac{1}{D^2} e^{(2C_H D t - \frac{1}{2}\mu_H v t)} \underbrace{\sum_{x\notin B_{tv}(0)} e^{-\frac{1}{2}\mu_H\|x\|}}_{\le \sum_{x\in\mathbb{Z}^d}\dots = D}$$

$$\le \frac{1}{D} e^{(2C_H D t - \frac{1}{2}\mu_H v t)}$$

$$= \frac{1}{D} e^{-\frac{1}{2}\mu_H\Big(v - \underbrace{\frac{4C_H D}{\mu_H}}_{=:v_H}\Big)t}$$

$\square$

We see that locality $\Rightarrow$ stuck inside a ball $\Leftrightarrow$ $\exists$ max. velocity. Next, we will see that the holomorphic functional calculus <u>preserves locality</u>.

<u>Lemma:</u> (Holmgren's Bound)

Let $A\in\mathcal{B}(\mathcal{H})$ and $\{\varphi_j\}_j$ an ONB for $\mathcal{H}$. Then,

$$\|A\| \le \max_{i\leftrightarrow j}\sup_i \sum_j |\langle\varphi_i, A\varphi_j\rangle|$$

<u>Proof:</u> see notes.

$\square$

Theorem (Combes-Thomas Estimate): ← works in continuum case too

Let $H = H^* \in \mathcal{B}(\mathcal{H})$ be local. Then,

$$\| R(z)_{xy} \| \le \frac{2}{\delta} e^{-\tilde{\mu}_H \delta \|x-y\|} \qquad (x,y \in \mathbb{Z}^d,\ z \in \rho(H))$$

for some $\tilde{\mu}_H > 0$, where $\delta := \operatorname{dist}(z, \sigma(H))$.

Proof: Let $f : \mathbb{Z}^d \to \mathbb{R}$ be bdd. and L-Lipschitz for some L TBD.
← deformation

Define

$$H_f := e^{f(X)} H e^{-f(X)} \quad \text{(not S.A.!)} \Rightarrow (H_f)_{xy} = e^{f(x)} H_{xy} e^{-f(y)} = e^{f(x)-f(y)} H_{xy}$$

$$\Rightarrow R_f(z)_{xy} = (H_f - z\mathbb{1})^{-1}_{xy} = \left[e^{-f(X)} (H - z\mathbb{1})^{-1} e^{f(X)}\right]_{xy} = e^{f(x)-f(y)} R(z)_{xy}$$

So,

$$\|R(z)_{xy}\| = \left|e^{f(y)-f(x)}\right| \, \|R_f(z)_{xy}\| \le \left|e^{f(x)-f(y)}\right| \, \|R_f(z)\|$$

We may bound op. norm of $R_f(z)$ via

$$\|(H_f - z\mathbb{1})\psi\| = \left\| \left[(H - z\mathbb{1}) + (H_f - H)\right]\psi \right\| \overset{\text{reverse triangle ineq.}}{\ge} \underbrace{\|(H - z\mathbb{1})\psi\|}_{\ge \delta \|\psi\| \text{ by trivial bound}} - \|(H_f - H)\psi\|$$

By Holmgren's Bound,

$$\begin{aligned}
\|H_f - H\| &\le \max_{x \leftrightarrow y} \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \|(H_f - H)_{xy}\| \\
&= \max_{x \leftrightarrow y} \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \underbrace{\left|e^{f(x)-f(y)} - 1\right|}_{\le e^{|f(x)-f(y)|} - 1} \|H_{xy}\| \\
&\le \max_{x \leftrightarrow y} \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \left(e^{L\|x-y\|} - 1\right) C_H e^{-\mu_H \|x-y\|} \\
&\overset{\text{translation invariance of } \|y-x\|}{=} \sum_{y \in \mathbb{Z}^d} C_H e^{-\mu_H \|y\|} \left(e^{L\|y\|} - 1\right) \\
&\overset{e^{a\|x\|} - 1 \le 2a e^{2a\|x\|}}{\le} 2 L C_H \sum_{y \in \mathbb{Z}^d} e^{-(\mu_H - 2L)}
\end{aligned}$$

Letting $L \le \min\left\{ \frac{1}{4}\mu_H, \frac{\delta}{4 C_H D} \right\}$, we see $\|H_f - H\| \le \frac{\delta}{2}$.

Thus,

$$\|(H_f - z\mathbb{1})\psi\| \ge \delta\|\psi\| - \frac{\delta}{2}\|\psi\| \Rightarrow \|H_f - z\mathbb{1}\| \ge \frac{\delta}{2} \Rightarrow \|R_f(z)\| \overset{\text{trivial bound}}{\le} \frac{2}{\delta}.$$

So,

$$\|R(z)_{xy}\| \le \frac{2}{\delta} e^{-|f(x)-f(y)|}.$$

$\forall k \in \mathbb{N}$, define $f_k(\cdot) := L \min\{k, \|\cdot - y\|\} \Rightarrow$ no clue how this helps

So,

$$\|R(z)_{xy}\| \le \frac{2}{\delta} e^{-\tilde{\mu}_H \delta \|x-y\|} \quad \text{as desired.}$$

$\square$

Corollary:

Let $H = H^* \in \mathcal{B}(\mathcal{H})$ be local. Then, the holomorphic functional calculus <u>preserves locality</u> in the sense that $f : \mathbb{R} \to \mathbb{C}$ real-analytic implies $f(H) = \frac{i}{2\pi} \oint_\Gamma R(z) f(z)\, dz$ is local.

Remark: Holo f'al calc. preserves locality. Often, the question of whether H defines a metal or an insulator boils down to whether the measurable f'al calc. on H preserves locality.

1.4 - Types of Motion

We would like to separate motion $\to$ ballistic] superconductor; diffusive] conductor; localized] insulator

Def.

"Second moment of position operator"

We define the transport coefficient of H by

$$M_{ij}(t) := \langle e^{-itH}\delta_0, X_i X_j e^{-itH}\delta_0 \rangle_{\ell^2(\mathbb{Z}^d \to \mathbb{C})}$$

We are interested in the large-time asymptotics.

$M_{ij}(t) \sim t^2 \iff$ ballistic motion $\qquad \sqrt{M_{ij}(t)} \overset{t\to\infty}{\approx} X(t)$

$M_{ij}(t) \sim t \iff$ diffusive motion $\leftarrow$ this is the scaling of Brownian motion

$M_{ij}(t) \sim O(1) \iff$ localized motion

Prop:

Periodic Hamiltonians have ballistic motion.

Proof: In momentum space (i.e. $\mathcal{F}$ diagonalizes H), $\mathcal{F}\delta_0 = (k \mapsto 1)$. By periodicity

$$\mathcal{F} e^{-itH} \mathcal{F}^* = e^{-it\mathcal{F}H\mathcal{F}^*} = e^{-itM_h}$$

where $h: \mathbb{T}^d \to \mathrm{Herm}_{N\times N}(\mathbb{C})$ is H's symbol. Thus,

$$M_{ij}(t) \overset{\mathcal{F}\text{ unitary}}{=} \langle \mathcal{F}\delta_0, \mathcal{F}e^{itH}\mathcal{F}^*\mathcal{F}X_i\mathcal{F}^*\mathcal{F}X_j\mathcal{F}^*\mathcal{F}e^{-itH}\mathcal{F}^*\mathcal{F}\delta_0 \rangle$$

$$= \int_{k\in\mathbb{T}^d} dk\; e^{ith(k)}\, i\partial_i i\partial_j\, e^{-ith(k)} - e^{-ith(k)}\left(-(\partial_i h)(\partial_j h)t^2 - it(\partial_i\partial_j h)\right)$$

$$= t^2\Big(\underbrace{\int (\partial_i h)(\partial_j h)}_{v^2}\Big) + it\Big(\underbrace{\int \partial_i \partial_j h}_{=0}\Big)$$

"Bloch electrons don't see the lattice"

$\square$

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Note that if our Hamiltonian is reflection-symmetric $\left(\begin{smallmatrix} \text{isotropic,} \\ x \mapsto e^{-itH}(0,x) \\ \text{is even} \end{smallmatrix} \right)$ then

$$M_{ij}(t) = \sum_{x \in \mathbb{Z}^d} x_i x_j \left| e^{-itH}(0,x) \right|^2 \overset{i \neq j}{=} 0$$

This is true for isotropic Hamiltonians such as $-\Delta$.

So, perhaps the interesting quantity is

$$M(t) := \sum_{x \in \mathbb{Z}^d} \|x\|^2 \left| e^{-itH}(0,x) \right|^2$$

Example (trivial localization):

If H is diagonal w.r.t position, $H_{xy} = H_x \delta_{xy}$ (i.e. potential, no kinetic), then $M_{ij}(t) = 0$ $\forall i,j$. We are interested in what settings reduce to this, even in the presence of kinetic energy.

Diffusion:

Why is $M(t) \sim t$ called diffusion? For intuition, consider the continuum and let $n(x,t)$ be particle density at position x, time t. The diffusion/heat eq reads

$$\partial_t n(x,t) = -D \Delta_x n(x,t) \qquad (x \in \mathbb{R}^d,\ t > 0) \qquad D > 0 \text{ diffusion const}$$

Then,

$$\partial_t \sum_{x \in \mathbb{Z}^d} x_i x_j n(x,t) = \sum_x x_i x_j \partial_t n(x,t) \overset{\text{diffusion}}{=} \sum_x x_i x_j (-D \Delta_x n)(x,t)$$

$$\underset{\substack{\text{I.B.P., even} \\ \text{on the lattice} \\ \text{it holds}}}{=} D \sum_{x \in \mathbb{Z}^d} (-\Delta_x x_i x_j) n(x,t) = 2D\delta_{ij} \sum_{x \in \mathbb{Z}^d} n(x,t)$$

So, since $\langle x_i, x_j \rangle_n = \dfrac{\sum_x x_i x_j n(x,t)}{\sum_x n(x,t)} \Rightarrow \partial_t \langle x_i, x_j \rangle_n = 2D\delta_{ij}$

Thus,

$$\langle x_i, x_j \rangle_n \sim 2tD\delta_{ij} + C \quad \Rightarrow \quad \lim_{t \to \infty} \frac{\langle x_i, x_j \rangle_n}{t} = 2D\delta_{ij}.$$

From this, we define

$$\boxed{D := \frac{1}{2} \lim_{t \to \infty} \frac{M(t)}{t}}$$

Spectral Types & Dynamics (from Teschl)

We can decompose $\sigma(H)$ into parts corresponding to the types of motion. Indeed, we may have different behavior for different initial states. So, we define

$$D(\psi) = \frac{1}{2} \lim_{t\to\infty} \frac{1}{t} \langle \psi, e^{itH} X^2 e^{-itH} \psi \rangle \quad \text{for initial state } \psi$$

and

$$D(E) = D(\text{wave packet about } E) \quad \text{for } E \in \mathbb{R}$$

The big open problem is to find reasonable systems for which $D \in (0, \infty)$. It's tough/unknown how to do this directly via second moments. We can get a bit of mileage through functional analysis.

Recall the spectral measure associated with H, ψ:

$$\mu_{H,\psi} = \langle \psi, \chi_{\cdot}(H) \psi \rangle$$

By Lebesgue decomposition (w.r.t. Lebesgue on $\mathbb{R}$) splits into 3 parts:

1. pure point (eigenvalues, has mass)
2. abs cont. (has density w.r.t. Lebesgue)
3. singular cont. (continuous but no mass or density w.r.t. Lebesgue)

Thus,

$$f(H) = \underbrace{\sum_i^{\text{ONB of e'vectors}} f(\lambda_i)\, \psi_i \otimes \psi_j^*}_{\text{p.p.}} + \underbrace{\int_{\lambda\in\mathbb{R}} f(\lambda)\, P_H^{a.c.}(\lambda)\, d\lambda}_{\text{a.c.}} + \underbrace{\int_{\lambda\in\mathbb{R}} f(\lambda)\, dP^{s.c.}(\lambda)}_{\text{s.c.}}$$

> We will see the following connections
>
> 1. pure point $\longleftrightarrow$ localization $\longleftrightarrow$ bound states
> 2. abs. cont. $\longleftrightarrow$ delocalization $\longleftrightarrow$ scattering states
> 3. sing. cont. $\longleftrightarrow$??

So, looking at spectral type can answer "is it localized", but not "is it diffusive".

Let's look closer at the above connections. If we look only at p.p.,

Suppose $\psi \in \ell^2$ is s.t. $H\psi = \lambda\psi \Rightarrow e^{-itH}\psi = e^{-it\lambda}\psi$

$$\Rightarrow |\langle \varphi, e^{-itH}\psi\rangle|^2 = |\langle \varphi, \psi\rangle|^2 \Rightarrow \text{const. in time!}$$

__Theorem:__ (Wiener)

Let μ be a finite, complex Borel measure on $\mathbb{R}$ with Fourier transform

$$\hat{\mu}(t) := \int_{E\in\mathbb{R}} e^{-itE} d\mu(E)$$

__Then__, the Césaro avg. of $\hat{\mu}$ obeys

$$\lim_{T\to\infty} \frac{1}{T}\int_{t=0}^{T} |\hat{\mu}(t)|^2 dt = \sum_{E\in\mathbb{R}} |\mu(\{E\})|^2 < \infty$$

i.e. it only picks up the p.p. part.

__Proof:__ $$\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2 dt = \frac{1}{T}\int_0^T\left(\overline{\int_E e^{-itE} d\mu(E)} \int_{\tilde{E}} e^{-it\tilde{E}} d\mu(\tilde{E})\right) dt$$

$$\overset{\text{Fubini}}{=} \int_{E,\tilde{E}} \overline{d\mu(E)}\, d\mu(\tilde{E})\, \underbrace{\frac{1}{T}\int_0^T e^{-it(E-\tilde{E})} dt}_{\to\, \chi_{\{0\}}(E-\tilde{E}),\ \text{check this}}$$

$$\xrightarrow{\text{D.C.T.}} \int_E \overline{\mu(\{E\})}\, d\mu(E) = \sum_{E\in\mathbb{R}} |\mu(\{E\})|^2$$

$\square$

Recall the decomposition of $\mathcal{H}$ into

$$\mathcal{H} = \mathcal{H}_{pp}(H) \oplus \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{sc}(H), \quad \text{with} \quad \mathcal{H}_{\#}(H) := \{\psi\in\mathcal{H} : \mu_{H,\psi} \text{ is } \#\}$$

We have $\mu_{H,\psi}$ is $\# \iff \psi\in \operatorname{ran}(P_{\#}(H))$ and $[H, P_{\#}(H)] = 0$.
Thus by polarization, the complex measures $\mu_{H,\varphi,\psi}$ are $\#$ for all φ provided that $\mu_{H,\psi}$ is $\#$.

So far, the above gives

1. If $\psi\in\mathcal{H}_c(H) = \mathcal{H}_{ac}(H)\oplus\mathcal{H}_{sc}(H)$ __then__ $\lim_{T\to\infty}\frac{1}{T}\int_0^T |\hat{\mu}_{H,\psi}(t)|^2 dt = 0$
2. If $\psi\in\mathcal{H}_{ac}(H)$, __then__ $\lim_{t\to\infty}|\hat{\mu}_{H,\psi}(t)| = 0$ by Riemann-Lebesgue

The same holds for the off-diagonal complex measures. Since

$$|\hat{\mu}_{H,\varphi,\psi}(t)| = |\langle\varphi, e^{-itH}\psi\rangle|$$

we see that a.c. vectors get __more orthogonal__ to themselves over time.
This is why a.c. $\longleftrightarrow$ scattering.

Theorem:

there is a version of this for unbounded A, K

Let A be SA. and bdd. Let $K \in \mathcal{B}(\mathcal{H})$ be bounded and compact. Then, $\forall \psi \in \mathcal{H}$,

$$① \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T \| K e^{-itA} P_c(A)\psi\|^2 \, dt = 0$$

$$② \quad \lim_{t\to\infty} \| K e^{-itA} P_{a.c.}(A)\psi\| = 0$$

Proof: Let $\psi \in \mathcal{H}_{\#}(H)$, $\# \in \{c, a.c.\}$ to avoid writing projections. By compactness, $K = \lim_{n\to\infty} F_n$ in norm, with $F_n = \sum_{i=1}^{m} \varphi_i \otimes \varphi_i^*$ with $\{\varphi_i\}_{i=1}^m$ ONB of $\operatorname{im}(F_n)$.
So,

$$\|F_n e^{-itA}\psi\|^2 \underset{\text{ONB}}{=} \sum_{i=1}^{m} \underbrace{|\langle \varphi_i, e^{-itA}\psi\rangle|^2}_{|\hat{\mu}_{A,\varphi_i,\psi}(t)|^2}$$

Taking $\|K - F_n\| \le \frac{1}{n}$, $\quad \|K e^{-itA}\psi\|^2 \le 2\|F_n e^{-itA}\psi\|^2 + \frac{2}{n^2}\|\psi\|^2$.
Take $t\to\infty$ via Wiener/Riemann-Lebesgue, then take $n\to\infty$.

$\square$

Theorem: (RAGE ((Ruelle, Amrein, Georgescu, Enss)))

$\exists$ version for unbounded

Let $H = H^* \in \mathcal{B}(\mathcal{H})$ and $\{K_n\}_n$ a sequence of compact operators s.t. $\operatorname{s-lim}_{n\to\infty} K_n = \mathbb{1}$. Then,

$$\mathcal{H}_c(H) = \left\{ \psi \in \mathcal{H} : \lim_{n\to\infty}\lim_{T\to\infty} \frac{1}{T}\int_0^T dt \, \|K_n e^{-itH}\psi\|^2 = 0 \right\}$$

eventually you leave the box - delocalized

$$\mathcal{H}_{pp}(H) = \left\{ \psi \in \mathcal{H} : \lim_{n\to\infty} \sup_{t\ge 0} \|(\mathbb{1} - K_n) e^{-itH}\psi\| = 0 \right\}$$

even at large times, you don't leave the box - localized!

Remark: For example, take $K_n = \chi_{B_n(0)}(X) = \sum_{x\in B_n(0)} \delta_x \otimes \delta_x^* \xrightarrow{s} \mathbb{1}$ to be a box of side length n.

Conditions for a.c. spectrum that we will cover:

① Limiting absorption principle: $|\langle \psi, (H - z\mathbb{1})^{-1}\psi\rangle|$ (Herglotz!)

② Mourre theory $\quad i[H, B] \ge 0$

③ Index theory: $\quad \operatorname{index}(\Lambda U \Lambda + \Lambda^\perp) \ne 0$ for some proj. Λ
$\Rightarrow \sigma(U) = \sigma_{a.c.}(U) = \mathbb{S}^1$

2/8 - When do we have a.c. spectrum?

Stability

Defn:

For $A \in \mathcal{B}(\mathcal{H})$, we define the essential spectrum

Fredholm

$$\sigma_{ess}(A) = \{ z \in \mathbb{C} : (A - z\mathbb{1}) \in \mathcal{F}(\mathcal{H}) \}$$

Theorem:

$$\sigma_{ess}(A) = \sigma_{ess}(A+K) \quad \text{for all } K \text{ compact.}$$

Proof: $z \notin \sigma_{ess}(A) \iff A - z\mathbb{1} \in \mathcal{F}(\mathcal{H}) \iff A + K - z\mathbb{1} \in \mathcal{F}(\mathcal{H}) \iff z \notin \sigma_{ess}(A+K)$ $\square$

Theorem:

1-trace class, $tr(|T|) < \infty$

Let $A = A^* \in \mathcal{B}(\mathcal{H})$ and $T = T^* \in \mathcal{J}_1(\mathcal{H})$. Then,

$$\sigma_{ac}(A) = \sigma_{ac}(A+T)$$

We see that σ_{ess} is stable under $\mathcal{K}(\mathcal{H})$ and σ_{ac} is stable under $\mathcal{J}_1(\mathcal{H})$.

This makes sense since $\sigma_{ac} \subseteq \sigma_{ess}$ and $\mathcal{J}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$.

Limiting Absorption Principle (Jaksic 2006, "What is ac spectrum?")

Lemma:

Let μ be a finite Borel measure. Define its Borel transform via $f(z) := \int_{E \in \mathbb{R}} \frac{1}{E - z} \, d\mu(E)$. Then,

① $\lim_{\epsilon \to 0^+} \frac{1}{\pi} \operatorname{Im}\{f(E + i\epsilon)\}$ exists for Lebesgue-a.e. $E \in \mathbb{R}$.

② $\{E \in \mathbb{R} : \operatorname{Im}\{f(E + i0^+)\} = \infty\} = spt(\mu_{sing})$

$\{E \in \mathbb{R} : \operatorname{Im}\{f(E + i0^+)\} \in (0, \infty)\} = spt(\mu_{ac})$

$\{E \in \mathbb{R} : \lim_{\epsilon \downarrow 0} \epsilon \operatorname{Im}\{f(E + i\epsilon)\} > 0\} = spt(\mu_{sing})$

Proof. Jaksic :) $\square$

Prop. (Yafaev) all these things also work for unbounded

Let $H = H^* \in \mathcal{B}(\mathcal{H})$. Assume that $\mathcal{D} \subseteq \mathcal{H}$ dense s.t.

$$\sup_{\substack{E\in[a,b]\\ \varepsilon\in(0,1)}} \left|\langle \varphi, \underbrace{R(E+i\varepsilon)}_{(H-(E+i\varepsilon)\mathbb{1})^{-1}} \varphi\rangle\right| < \infty \qquad (\varphi\in\mathcal{D}) \qquad \circledast$$

Then, $\sigma(H)\cap[a,b] = \sigma_{ac}(H)\cap[a,b]$ purely.

Proof: From $\circledast$, we know $\left[\sup_{\varepsilon\in(0,1)} \int_a^b \frac{1}{\pi} \operatorname{Im}\{\overbrace{\langle\varphi, R(E+i\varepsilon)\varphi\rangle}^{\text{Herglotz fn.}}\}^p dE\right] < \infty$ for some $p > 1$.

For any $\tilde{a} < \tilde{b}$, Stone's formula gives

$$\frac{1}{2}\left(\langle\varphi, \chi_{[\tilde a,\tilde b]}(H)\varphi\rangle + \langle\varphi, \chi_{(\tilde a,\tilde b)}(H)\varphi\rangle\right)$$

$$= \lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_{\tilde a}^{\tilde b} \operatorname{Im}\{\langle\varphi, R(E+i\varepsilon)\varphi\rangle\} dE$$

$$\Rightarrow \langle\varphi, \chi_I(H)\varphi\rangle \le \lim_{\varepsilon\downarrow 0} \int_I \frac{1}{\pi}\operatorname{Im}\{\langle\varphi, R(E+i\varepsilon)\varphi\rangle\} dE \cdot 1$$

$$\overset{\text{Hölder}}{\le} \sup_{\varepsilon\in(0,1)} \left(\int (\ldots)^p\right)^{\frac{1}{p}} \quad |I|^{1-\frac{1}{p}}$$

$$\Rightarrow \langle\varphi, \chi_\cdot(H)\varphi\rangle \llcorner [a,b] \ll \text{Lebesgue}$$

$\square$

Defn:

We say H has the limiting absorption principle (LAP) at $E\in\mathbb{R}$ iff $\forall \varphi\in L^2$ sufficiently nice (for $\ell^2(\mathbb{Z}^d)$, take compact spt, otherwise Sobolev?) and $\forall \sigma > 0$, $\exists C(\sigma)\in(0,\infty)$ s.t.

$$\sup_{\varepsilon\neq 0} \|R(E+i\varepsilon)\varphi\|_{H^{-\frac{1}{2}-\sigma}} \le C(\sigma)\frac{1}{\sqrt{E}}\|\varphi\|_{H^{\frac{1}{2}+\sigma}}$$

where $\|\psi\|_{H^s} := \|\underbrace{\langle x\rangle^s}_{\text{see final calc.}}\psi\|_{L^2}$ with $\langle x\rangle := \sqrt{1+\|x\|^2}$

Claim: Any H obeying LAP $\forall E\in[a,b]$ has pure a.c. spectrum on $[a,b]$.

Scattering Theory & Wave Operators (R&S III)

Defn:

For $A, B \in \mathcal{B}(\mathcal{H})$ S.A., define (when they exist) the wave operators

$$\Omega^{\pm}(A,B) := \operatorname*{s\text{-}lim}_{t \to \pm\infty} e^{-itA} e^{itB} P_{ac}(B), \qquad \mathcal{H}^{\pm} := \operatorname{im}(\Omega^{\pm})$$

Remark: ① If φ is an eigenvector of B w/ eigenvalue λ,

$$e^{-itA} e^{itB} \varphi = e^{-itA} e^{it\lambda} \varphi$$

does not converge unless φ is evector of A too.
So, we needed to project onto the continuous part of B.

② Often, A is the operator of interest and B is known $(-\Delta$, free theory, $\ldots)$

Prop:

If $\Omega^{\pm}(A,B)$ exist, then:

① $\Omega^{\pm}(A,B)$ are partial isometries with initial space $P_{ac}(B)\mathcal{H}$ and final space $\mathcal{H}^{\pm}$

$\ker(\Omega^{\pm}(A,B))^{\perp}$

② $\mathcal{H}^{\pm}$ are invariant spaces for A:

$$\Omega^{\pm}(A,B)\, \mathcal{D}(B) \subseteq \mathcal{D}(A) \quad \text{and}$$
$$A\Omega^{\pm}(A,B) = \Omega^{\pm}(A,B) = B$$

③ $\mathcal{H}^{\pm} \subseteq \operatorname{im}(P_{ac}(A))$

Proof: ① Clearly, $(P_{ac}(B)\mathcal{H})^{\perp} = \ker(\Omega^{\pm}(A,B))$. Conversely, if $\varphi \in P_{ac}(B)\mathcal{H}$, then

$$\|\Omega^{\pm}(A,B)\varphi\| = \lim \|e^{-itA} e^{itB} \varphi\| \overset{\text{unitary}}{=} \|\varphi\|$$

② Note that since $[e^{isB}, P_{ac}(B)] = 0$, for any fixed s we see

$$\Omega^{\pm} = e^{-isA}\Omega^{\pm} e^{isB} \Rightarrow e^{isA}\Omega^{\pm} = \Omega^{\pm} e^{isB} \xrightarrow{\text{take a derivative}} A\Omega^{\pm} = \Omega^{\pm} B$$

To see that $\mathcal{H}^{\pm}$ is invariant for A: $\varphi \in \mathcal{H}^{\pm} \Rightarrow \exists \psi : \varphi = \Omega^{\pm}\psi$

$$\Rightarrow A\varphi = A\Omega^{\pm}\psi = \Omega^{\pm} B\psi \in \mathcal{H}^{\pm}$$

③ $A|_{\mathcal{H}^{\pm}}$ is unitarily equivalent to $B|_{P_{ac}(B)\mathcal{H}}$ via $\Omega^{\pm}$.

□

Theorem:

Let $A, B \in \mathcal{B}(\mathcal{H})$ be S.A. and assume $\underset{t\to\infty}{\text{slim}}\, e^{-itA} e^{itB}$ exists.
Then,
$$\sigma_{ac}(B) \subseteq \sigma_{ac}(A).$$

Proof: Comes from ③ in above prop.

$\square$

Claim: (Chain Rule)

If A, B, C are SA. and $\Omega^{\pm}(A,C)$, $\Omega^{\pm}(C,B)$ exist, **then**
$$\Omega^{\pm}(A,B) = \Omega^{\pm}(A,C)\,\Omega^{\pm}(C,B)$$

Defn: (Completeness)

We say A, B are complete iff
$$\Omega^{\pm}(A,B) \text{ exist and } \mathcal{H}^{+} = \mathcal{H}^{-} = P_{ac}(A)\mathcal{H}$$
If, in addition, $\sigma_{sing}(A) = \emptyset$ (or equivalently $\mathcal{H}^{+} = \mathcal{H}^{-} = P_{pp}(A)^{\perp}\mathcal{H}$), they have asymptotic completeness.

Prop:

$\Omega^{\pm}(A,B)$ and $\Omega^{\pm}(B,A)$ exist $\iff$ A, B are complete

Proof: $(\Rightarrow)$
$$P_{ac}(A) = \Omega^{\pm}(A,A) \overset{\text{chain rule}}{=} \Omega^{\pm}(A,B)\,\Omega^{\pm}(B,A)$$
$$\Rightarrow P_{ac}(A)\mathcal{H} \subseteq \mathcal{H}^{\pm}.$$

Reverse inclusion was seen earlier.

$(\Leftarrow)$?

$\square$

How do we know when $\Omega^{\pm}$ exist? Cook's method!

Theorem: (Cook's method)

Let A, B S.A.. Assume

① $\exists\, D \subseteq \mathcal{D}(B) \cap \operatorname{im}(P_{ac}(B))$ s.t. D is dense in $\operatorname{im}(P_{ac}(A))$

② $\exists\, T > 0$ s.t. $\forall\, |t| > T$, $\forall\, \varphi \in D$,

⊛ $e^{-itB}\varphi \in \mathcal{D}(A)$

⊛ $\int_T^\infty dt \left(\|(B-A)e^{-itB}\varphi\| + \|(B-A)e^{itB}\varphi\| \right) < \infty$

Then, $\Omega^\pm(A,B)$ exist.

Proof: Define $\zeta(t) := e^{itA} e^{-itB} \varphi$ for fixed $\varphi \in D$.
$\forall\, t > T$, $e^{-itB}\varphi \in \mathcal{D}(A) \cap \mathcal{D}(B)$. Also, $\zeta'(t) = -i e^{itA}(B-A)e^{-itB}\varphi$

By FTOC, $\zeta(t) - \zeta(s) = \int_s^t \zeta' \Rightarrow \|\zeta(t) - \zeta(s)\| \leq \int_s^t \|(B-A)e^{-iuB}\varphi\|\, du$
By our integrability assumption, $\{\zeta(t)\}_t$ is Cauchy $\forall\, \varphi \in D$.
By a density argument, Ω^+ exists. Repeat for Ω^-.

$\square$

Examples:

therefore, equality

⊛ If $B - A \in \mathcal{T}_1(\mathcal{H})$, **then** $\sigma_{ac}(B) \subseteq \sigma_{ac}(A)$.
This is stability of σ_{ac}

⊛ $B = -\Delta$ (discrete) Laplacian and $A = -\Delta + V(x)$
Then, $\sigma_{ac}(A) \supseteq \sigma_{ac}(B)$ if V is "sufficiently nice"

① V has cpt. spt.
OR
② V has fast enough decay at ∞
OR
③ V has "sparse" spt : $|\operatorname{spt}(V) \cap B_R(x)| \lesssim R^{d-1}$
(Krishna '92)

2/13 -

We will study the formula for DC conductivity, as this will lead to the integer quantum Hall effect and formulae for the diagonal elements of the conductivity matrix σ.

Perturbation Theory

Consider Ohm's Law $V = IR = I/\sigma$. So, for some perturbation V to the Hamiltonian, we are interested in the coefficient of I's linear response.

Recall Rayleigh-Schrodinger perturbation theory (i.e. analytic perturb. theory) from Griffiths, where we write $H' = H + \varepsilon V$ and compute

$$\Delta E_j' = \varepsilon \langle \varphi_j, V \varphi_j \rangle \quad , \quad \Delta \varphi_j' = \dots$$

This stuff only works for discrete and finitely-degenerate spectrum of H. So, we must do something else - the Kubo linear response theory.

Def: (Mixed states & density matrices)

Recall pure states $\varphi \in \mathcal{H}$, where the expectation of an observable $A = A^* \in \mathcal{B}(\mathcal{H})$ is given by $\langle \varphi, A\varphi \rangle \underset{\|\varphi\|=1}{=} \operatorname{tr}(\varphi \otimes \varphi^* A) =: \operatorname{tr}(P_\varphi A)$

If we have some <u>distribution over pure states</u> $\{\varphi_i\}_{i=1}^N$ (normalized) w.p. $\{p_i\}_{i=1}^N \subseteq [0,1]$ s.t. $\sum_i p_i = 1$, we may define $\rho := \sum_i p_i \varphi_i \otimes \varphi_i^*$ and confirm

$$① \ \langle \varphi, \rho\varphi \rangle = \sum_i p_i |\langle \varphi_i, \varphi \rangle|^2 \geq 0 \implies \rho \geq 0$$

$$② \ \operatorname{tr}(\rho) = \sum_i p_i \|\varphi_i\|^2 = 1 \implies \operatorname{tr}(\rho) = 1$$

So, we define a density matrix as any $\rho \in \mathcal{B}(\mathcal{H})$ s.t.

$$① \ \rho \geq 0 \qquad ② \ \rho \in \mathcal{T}_1 \qquad ③ \ \operatorname{tr}(\rho) = 1$$

With this, expectations are now $\operatorname{tr}(\rho A)$.

Many-Body QM Intuition

For m distinguishable particles, with $\mathcal{H}$ as the single-particle Hilbert space, the total state space is $\bigotimes_{j=1}^{m} \mathcal{H}$

Note that $L^2(E)^{\otimes m} = L^2(\underbrace{E \times \dots \times E}_{m})$, and so we may view wavefunctions φ based on their symmetries under swapping arguments to φ (i.e. $\varphi(a_1, a_2) = \pm \varphi(a_2, a_1)$)

For m indistinguishable particles $<$ fermions (anti-symmetric) $\bigwedge_{j=1}^{m} \mathcal{H}$ (wedge); bosons (symmetric)

tensor products of subspaces of (anti-)symmetric fns, or equivalently $\mathcal{H}^{\otimes m}$ with symmetries quotiented out

⊛ We may lift operators on $\mathcal{H}$ to ones on $\mathcal{H}^{\wedge m}$ via the 2nd quantized lift of $H \in \mathcal{B}(\mathcal{H})$ via

$$d\Gamma(H) = \sum_{j=1}^{m} \underbrace{\mathbb{1} \wedge \dots \wedge \mathbb{1}}_{j-1 \text{ times}} \wedge H \wedge \underbrace{\mathbb{1} \wedge \dots \wedge \mathbb{1}}_{m-j-1}$$

⊛ If $\{e_n\}_n$ is ONB of $\mathcal{H}$, then $\{e_{n_1} \wedge \dots \wedge e_{n_m}\}_{n_1, \dots, n_m}$ ONB of $\mathcal{H}^{\wedge m}$.

⊛ If $\Psi \in \mathcal{H}^{\wedge m}$ then expectation value is $\langle \Psi, d\Gamma(A) \Psi \rangle$ for single-particle observable $A = A^* \in \mathcal{B}(\mathcal{H})$.
If $\Psi = \varphi_1 \wedge \dots \wedge \varphi_m$ then $\langle \Psi, d\Gamma(A) \Psi \rangle = \dots = \sum_{j=1}^{m} \langle \varphi_j, B \varphi_j \rangle$

At zero temp, m Fermions will occupy the m lowest ground states, and so (if H is discrete w/ $\{\varphi_j\}_j$ ONB with energies λ_j) then the many-body ground state is $\varphi_1 \wedge \dots \wedge \varphi_m$ (Slater determinant) with energy $\lambda_1 + \dots + \lambda_m$.

We have expectations

$$\langle \varphi_1 \wedge \dots \wedge \varphi_m, d\Gamma(A) \varphi_1 \wedge \dots \wedge \varphi_m \rangle = tr\Big(\big(\underbrace{\sum_{j=1}^{m} \varphi_j \otimes \varphi_j^*}\big) A\Big)$$

density matrix $A = \chi_{\{\lambda_1, \dots, \lambda_m\}}(A) = \chi_{(-\infty, \lambda_m]}(A)$

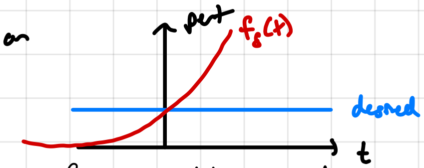
All this goes to show that we may handle many-body zero-temp ground states via density matrix $\rho = \chi_{(-\infty, E_F)}$ with E_F (Fermi energy) the cutoff for filled energies. In total,

> Many-body zero-temp ground state expectation of a single-particle observable $A = A^* \in \mathcal{B}(\mathcal{H})$ is
>
> $\langle A \rangle = tr(\rho_F A)$ with $\rho_F = \chi_{(-\infty, E_F)}(A)$

Kubo Formula

The Kubo formula is a perturbation theory for $tr(\rho B)$ for density matrices ρ.

⊛ We need a regularization: we gradually turn on the perturbation via some fn $f_\delta(t)$ s.t. $f_\delta(t) \to 1$ as $\delta \downarrow 0$ (i.e. $f(t) = e^{\delta t}$).



So, we start evolving from $t=-\infty$ and are perturbed when $t=0$

⊛ So, we have the perturbed Hamiltonian

$$H'(t) = H + \varepsilon f(t) A$$

equilibrium, won't time evolve

⊛ We are also given an initial state ρ_0 : $[H, \rho_0] = 0$

Note that if $\rho_0 = \rho_F$ then $tr(\rho_0) = \infty$. So, we must restrict ourselves to $B = B^* \in \mathcal{B}(\mathcal{H})$ s.t. $\rho_0 B \in \mathcal{J}_1$.

⊛ We let $\rho'(t)$ be the time-evolved state (w/ $H'(t)$) and seek

$$tr(\rho'(0) B) = tr(\rho_0 B) + \varepsilon \chi_{BA} + O(\varepsilon^2)$$

↖ first order in ε after $\delta \downarrow 0$ is taken

Theorem (Kubo):

$$\chi_{BA} = -i \int_{-\infty}^{0} tr\left(e^{-itH} B e^{itH} [\overset{\text{pert.}}{A}, \overset{\text{initial state}}{\rho_0}]\right) dt$$

Proof. The correct time evolution for density matrices is $\dot{\rho} = -i[H, \rho]$

(since $\rho = \varphi \otimes \varphi^*$ and $\partial_t \varphi = -iH\varphi \overset{\text{Leibniz rule}}{\Rightarrow} i\dot{\rho} = i\dot{\varphi} \otimes \varphi^* - i\varphi \otimes \dot{\varphi}^* = H\varphi \otimes \varphi^* - \varphi \otimes \varphi^* H = [H, \rho]$)

So, ρ' must satisfy the ODE with b.c. $\rho'(-\infty) = \rho_0$:

$$i\dot{\rho}'(t) = [H'(t), \rho'(t)] = [H + \varepsilon f(t) A, \rho'(t)] = [H + \varepsilon f(t) A, \rho_0 + \varepsilon \rho_1(t)]$$

$$= \varepsilon [H, \rho_1(t)] + \varepsilon f(t) [A, \rho_0] + O(\varepsilon^2)$$

So, looking at $\rho_1 = \frac{\rho' - \rho_0}{\varepsilon}$, we see

$$i\dot{\rho}_1(t) = [H, \rho_1(t)] + f(t)[A, \rho_0] \qquad \text{with b.c. } \rho_1(-\infty) = 0$$

Define the superoperator $H^\times \equiv [H, \cdot]$ on $\mathcal{B}(\mathcal{H})$, yielding ODE

$$i\dot{\rho}_1(t) = H^\times \rho_1(t) + f(t) A^\times \rho_0 \qquad \text{w.b.c. } \rho_1(-\infty) = 0$$

We make the ansatz $\rho_1(t) \overset{?}{=} -i \int_{t'=-\infty}^{t} f(t') \underbrace{e^{-i(t-t')H^\times}}_{\text{f'al calc. on B.A. } \mathcal{B}(\mathcal{H})} A^\times \rho_0 \, dt' =: Ans(t)$

Clearly, $Ans(-\infty) = 0$ ✓.

We confirm

$$\dot{Ans}(t) = -i\partial_t \int_{-\infty}^{t} \cdots \overset{\text{Leibniz on superoperators}}{=} -i f(t) A^\times \rho_0 - i \int_{-\infty}^{t} dt' f(t') \underbrace{e^{-i(t-t')H^\times} H^\times}_{\text{these commute by f'al calc.}} A^\times \rho_0$$

$$= -i f(t) A^\times \rho_0 - i H^\times Ans(t)$$

$$\Rightarrow i\dot{Ans}(t) = f(t)[A, \rho_0] + [H, Ans(t)] \;✓$$

Good ansatz! Now, plugging in $t=0$ and taking $\delta \downarrow 0$,

$$\rho'(0) = \rho_0 + \varepsilon\left(-i \int_{-\infty}^{0} e^{itH^\times} [A, \rho_0] \, dt\right)$$

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Zero-Temp DC Conductivity

We will produce formulas under two different assumptions

① Time-reversal invariance (TRI) $\Rightarrow$ $\sigma_{ij}(E_F) = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \| G(x,0; E_F + i\varepsilon)\|^2$

② No TRI but $\exists$ spectral gap $\Rightarrow$ $\sigma_{ij}(E_F) = \operatorname{tr}\left(P\left[[\Lambda_1, P], [\Lambda_2, P]\right]\right)$

For interpretation of DC bias, we define a velocity op. in the j^{th} direction as the current

- $V_j := i[H, X_j]$ since $\partial_t \langle X_j \rangle_\varphi = \langle i[H, X_j]\rangle_\varphi$
- $V_j(t) = e^{itH} V_j e^{-itH}$ for notation

Taking a perturbation $A = -E_0 X_j$, we would have by Kubo that

$$\times \quad \sigma_{ij}(E_F) = \chi_{BA} = -i \lim_{\delta \downarrow 0} \int_{-\infty}^{0} dt\, e^{\delta t} \operatorname{tr}\left(V_i(-t)[X_j, P]\right)$$

This is no good, since $V_i(-t)[X_j, P]$ isn't generally trace class!
The first workaround will be to replace tr with trace per unit volume

Def.

We define the trace per unit volume of A via

$$\widetilde{\operatorname{tr}}(A) := \lim_{L \to \infty} \frac{1}{(2L+1)^d} \sum_{\substack{x \in \mathbb{Z}^d \\ \|x\|_1 \le L}} \langle \delta_x, A \delta_x \rangle$$

we won't justify why this limit exists, here

> **Theorem (ish):**
>
> If H has TRI (i.e. $H_{xy} = \overline{H_{xy}}$), then
>
> $$\sigma_{ij}(E_F) = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \, \mathbb{E}_\omega\left[|G(x,0; E_F + i\varepsilon)|^2 \right]$$
>
> where $\langle \delta_0, R(z) \delta_x \rangle =: G(0,x; z)$

Proof (ish):

Writing $e^{\delta t} = \partial_t\left(\frac{e^{\delta t} - 1}{\delta}\right)$, integration by parts on the Kubo formula would read

$$\sigma_{ij}(E_F) = \lim_{\delta \downarrow 0} i \int_{-\infty}^{0} dt\, \frac{e^{\delta t} - 1}{\delta} \partial_t \underbrace{\widetilde{\operatorname{tr}}\left(V_i(-t)[X_j, P]\right)}_{= \widetilde{\operatorname{tr}}\left(V_i e^{itH}[X_j, P] e^{-itH}\right) \overset{[P,H]=0}{=} \widetilde{\operatorname{tr}}\left(V_i [X_j(t), P]\right)}$$

$$= \lim_{\delta \downarrow 0} i \int_{-\infty}^{0} dt\, \frac{e^{\delta t} - 1}{\delta} \widetilde{\operatorname{tr}}\left(V_i [V_j(t), P]\right)$$

dQ is PVM of H $\quad = \lim_{\delta\downarrow 0} i \int_{-\infty}^{0} dt \, \frac{e^{\delta t}-1}{\delta} \int_{\lambda_1,\lambda_2\in\mathbb{R}} d\lambda_1 d\lambda_2 \, e^{it(\lambda_1-\lambda_2)} \tilde{\mathrm{tr}}\left(V_i \, dQ(\lambda_1)[V_j, \Delta] \, dQ(\lambda_2)\right)$

Letting $\Delta = \chi_{(-\infty,E_F)}(H)$ be the Fermi proj., since dQ is H's PVM we have

$$= \lim_{\delta\downarrow 0} i \int_{-\infty}^{0} dt \, \frac{e^{\delta t}-1}{\delta} \int_{\lambda_1,\lambda_2\in\mathbb{R}} d\lambda_1 d\lambda_2 \, e^{it(\lambda_1-\lambda_2)} (f(\lambda_2)-f(\lambda_1)) \underbrace{\tilde{\mathrm{tr}}\left(V_i \, dQ(\lambda_1) V_j \, dQ(\lambda_2)\right)}_{=: dM_{ij}(\lambda_1,\lambda_2)}, \quad f := \chi_{(-\infty,E_F)}$$

Since $e^{\delta t}-1 = t\int_{\zeta=0}^{\delta} e^{\zeta t} d\zeta \Rightarrow \int_{t=-\infty}^{0} dt \, t \, e^{it(\lambda_1-\lambda_2-i\zeta)} = \frac{1}{(\lambda_1-\lambda_2-i\zeta)^2}$, we swap integrals to get

$$\theta_{ij}(E_F) = \lim_{\delta\downarrow 0} i \int_{\lambda_1,\lambda_2\in\mathbb{R}} \frac{1}{\delta}\int_{\zeta=0}^{\delta} \frac{1}{(\lambda_1-\lambda_2-i\zeta)^2} (f(\lambda_2)-f(\lambda_1)) \, dM_{ij}(\lambda_1,\lambda_2)$$

For reasonable $g:\mathbb{R}\to\mathbb{R}$ we know $\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_{\zeta=0}^{\delta} g(\zeta)\,d\zeta = \lim_{\delta\downarrow 0} g(\delta)$, and so

$$\theta_{ij}(E_F) = \lim_{\delta\downarrow 0} i \int_{\lambda_1,\lambda_2\in\mathbb{R}} \frac{1}{(\lambda_1-\lambda_2-i\delta)^2} (f(\lambda_2)-f(\lambda_1)) \, dM_{ij}(\lambda_1,\lambda_2)$$

We apply the Kramers-Kronig relation between distributions

$$\lim_{\epsilon\downarrow 0} \frac{1}{x\pm i\epsilon} \overset{\mathcal{D}}{=} \mp i\pi\delta(x) + \mathcal{P}\left(\frac{1}{x}\right), \quad \text{where } \mathcal{P}\left(\tfrac{1}{x}\right) f \equiv \lim_{\gamma\downarrow 0} \int_{(-\infty,-\gamma)\cup(\gamma,\infty)} dx \, \frac{f(x)}{x}$$

$\overset{\mathcal{D}}{=}$: as distributions, integrating against test fns. Cauchy principal value

$$\overset{\text{diff wrt } x}{\Longrightarrow} \lim_{\epsilon\downarrow 0} \frac{-1}{(x\pm i\epsilon)^2} \overset{\mathcal{D}}{=} \mp i\pi\delta'(x) + \mathcal{P}'\left(\frac{1}{x}\right)$$

$\leftarrow$ even fn of x since LHS and δ' are

TRI gives that $dM_{ij}(\lambda_1,\lambda_2) = dM_{ij}(\lambda_2,\lambda_1)$. Integrating this even fn over odd measure will zero it out, and so we can ignore the $\mathcal{P}'(\frac{1}{x})$ part.

$$\Rightarrow \theta_{ij}(E_F) = i\pi \int_{\lambda_1,\lambda_2\in\mathbb{R}} \delta'(\lambda_1-\lambda_2)(f(\lambda_2)-f(\lambda_1)) \, dM_{ij}(\lambda_1,\lambda_2)$$

$\delta'(x) = \lim_{\epsilon\downarrow 0}\frac{\delta(x+\epsilon)-\delta(x)}{\epsilon}$
$$= i\pi \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_{\lambda_1\in\mathbb{R}} (f(\lambda_1+\epsilon)-f(\lambda_1)) \, dM_{ij}(\lambda_1,\lambda_1+\epsilon)$$
— term that $=0$ because of $f(\lambda_2)-f(\lambda_1)$ part

dM_{ij} is continuous
$$= i\pi \int_{\lambda_1\in\mathbb{R}} f'(\lambda_1) \, dM_{ij}(\lambda_1,\lambda_1) = i\pi \int_{\lambda_1\in\mathbb{R}} \delta(\lambda_1-E_F) \, dM(\lambda_1,\lambda_1)$$

$$= i\pi \int_{\lambda_1,\lambda_2} \delta(\lambda_1-E_F)\,\delta(\lambda_2-E_F) \, dM_{ij}(\lambda_1,\lambda_2)$$

$\delta_\epsilon(\lambda) := \frac{1}{\pi}\mathrm{Im}\left\{\frac{1}{\lambda+i\epsilon}\right\}$
$$= i\pi \lim_{\epsilon\downarrow 0}\int_{\lambda_1,\lambda_2} \delta_\epsilon(\lambda_1-E_F)\,\delta_\epsilon(\lambda_2-E_F) \, dM_{ij}(\lambda_1,\lambda_2)$$

undo the dQ's
$$= i \lim_{\epsilon\downarrow 0} \tilde{\mathrm{tr}}\left(V_i \, \delta_\epsilon(H-E_F) \, V_j \, \delta_\epsilon(H-E_F)\right)$$

Since $\mathrm{Im}\{R(z)\} = \frac{1}{2i}(R(z)-R(z)^*) \overset{H \text{ S.A.}}{=} \frac{1}{2i}(R(z)-R(\bar{z})) \overset{\text{resolvent identity}}{=} \mathrm{Im}\{z\} R(z) R(\bar{z})$

$$\Rightarrow \theta_{ij}(E_F) = \lim_{\epsilon\downarrow 0} \frac{\epsilon^2}{\pi} \tilde{\mathrm{tr}}\left(R(E_F+i\epsilon)R(E_F-i\epsilon) V_i R(E_F-i\epsilon)R(E_F+i\epsilon) V_j\right)$$

$R(z)[H,A]R(\bar{z}) = \ldots = -[R(z),A]$
$$= \lim_{\epsilon\downarrow 0} \frac{\epsilon^2}{\pi} \tilde{\mathrm{tr}}\left([R(E_F-i\epsilon), X_i][R(E_F+i\epsilon), X_j]\right)$$

We will now look at ergodic, random operators $\Omega \ni \omega \mapsto A_\omega \in \mathcal{B}(\mathcal{H})$, for which we may use Birkhoff's ergodic theorem relating space avg. with avg. over randomness:

$$\widetilde{tr}(A) = \lim_{L\to\infty} \frac{1}{(2L+1)^d} \sum_{\substack{x\in\mathbb{Z}^d \\ \|x\|_\infty \le L}} \langle \delta_x, A_\omega \delta_x \rangle \overset{a.s.}{=} \mathbb{E}_\omega\left[\langle \delta_0, A_\omega \delta_0 \rangle\right]$$

can be evaluated anywhere

$$\Rightarrow \Theta_{ij}(E_F) = \lim_{\varepsilon\downarrow 0} \frac{\varepsilon^2}{\pi} \mathbb{E}_\omega\left[\left\langle \delta_0, \left[R_\omega(E_F+i\varepsilon), X_i\right]\left[R_\omega(E_F-i\varepsilon) X_j\right] \delta_0 \right\rangle\right]$$

ergodic random Hamiltonian

$X_i\delta_0 = 0$
$$= \lim_{\varepsilon\downarrow 0} \frac{\varepsilon^2}{\pi} \mathbb{E}_\omega\left[\left\langle \delta_0, R_\omega(E_F+i\varepsilon) X_i X_j R_\omega(E_F-i\varepsilon) \delta_0 \right\rangle\right]$$

insert $\mathbb{1} = \sum_{x\in\mathbb{Z}^d} \delta_x \otimes \delta_x^*$
$$= \lim_{\varepsilon\downarrow 0} \frac{\varepsilon^2}{\pi} \sum_{x\in\mathbb{Z}^d} x_i x_j \, \mathbb{E}_\omega\left[\left|G(x,0;E_F+i\varepsilon)\right|^2\right]$$

$\square$

2/20-

Recall last time: for $T=0$ and an electric field in j-direction, we measure current $\vec{j} = \sigma \vec{E}$ in the i-direction to get

$$\sigma_{ij}(E_F) = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \, \mathbb{E}\Big[\underbrace{|G(x,0;E_F + i\varepsilon)|}_{G(x,y;z) \equiv \langle \delta_x, (H - z\mathbb{1})^{-1} \delta_y \rangle}{}^2 \Big]$$

We are interested in when $\sigma \equiv 0$, since this would prove its an insulator.

<u>Stupid example:</u> (H diagonal w.r.t. position)

$H = V(X)$ (i.e. $\nexists$ kinetic energy) $\Rightarrow$ G diagonal $\Rightarrow$ $\sigma \equiv 0$.

<u>Counter-example:</u> (periodic ops)

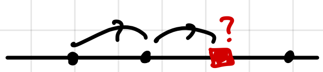
$H_{xy} = H_{x+a, y+a}$ $\quad \forall x, y, a \in \mathbb{Z}^d \Rightarrow \sigma_{ij}(E_F) = \infty$ if $E_F \in \sigma(H)$.

won Nobel prize for this, did it two doors down from Jadwin 343

Anderson Model & Random Operators

look at Aizenman-Warzel textbook!

We start from assuming that real materials have impurities. So, translation invariance ($\Leftrightarrow$ periodicity) is <u>not</u> a reasonable assumption.



- (~1950s) Wigner used random matrices to study atomic/molecular levels

- (~1960s) Anderson:

$$H = \underbrace{-\Delta}_{\substack{\text{discrete Laplacian} \\ \sigma(-\Delta) = [-2d, 2d]}} + \lambda V_\omega(X)$$

$\lambda > 0$ coupling strength

$V_\omega(x) \equiv \omega_x$ random sequence

$\omega: \mathbb{Z}^d \to \{\text{random variables}\}$ stochastic process

think of this as time index

This is a random Schrodinger operator. Anderson worked initially under the i.i.d. assumption ($\omega_x, \omega_y \sim \mu$ independently)

- (~1970s) Made Anderson model rigorous, proved (Kunz-Souillard) p.p. spectrum of iid model

- (1982) Frolich-Spencer performed multiscale analysis (KAM in math) (super hard perturbative) to show

$$|G(x,y;z)| \le C e^{-\mu|x-y|} \quad \text{w.h.p.}$$

- (1983) Aizenman-Molchanov invented fractional moment method to show

$$\mathbb{E}\big[|G(x,y;z)|^s\big] \le C e^{-\mu|x-y|}$$ for large λ, small enough s

Random Operators

We work in prob. space $(\Omega, \mathcal{F}, \mathbb{P})$.

Def:

- A map $T: \Omega \to \Omega$ is measure-preserving if
$$\mathbb{P}[S] = \mathbb{P}[T^{-1}(S)] \qquad \forall S \in \mathcal{F}$$
- For a group G time-evolution action on $(\Omega, \mathcal{F}, \mathbb{P})$, and a group morphism $T: G \to Aut(\Omega)$ (i.e. $T_{gh} = T_g \circ T_h$), we call $(\Omega, \mathcal{F}, \mathbb{P}, T)$ a measure-preserving G-dynamical system.
- A RV $X: \Omega \to \mathbb{R}$ is invariant if $X \circ T_g = X \quad \forall g \in G$.
- A G-dynamical system is ergodic if all invariant RV's are $\mathbb{P}$-a.s. constant: $\exists c_X \in \mathbb{R}$ s.t. $\mathbb{P}(\{X = c_X\}) = 1$.

Def:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be prob. space, $\mathcal{H}$ a separable Hilbert space. The SA-operator-valued map
$$A: \Omega \to \{B = B^* \in \mathcal{B}(\mathcal{H})\}$$
is a random operator if $\forall f: \mathbb{R} \to \mathbb{C}$ measurable, $\forall \varphi, \psi \in \mathcal{H}$, the map
$$\Omega \ni \omega \mapsto \langle \varphi, f(A_\omega) \psi \rangle \quad \text{is } \mathcal{F}\text{-meas.}$$
We say $\omega \mapsto A(\omega)$ is weakly measurable.

Def:

The random op. $\omega \mapsto A_\omega$ is ergodic random op iff A_ω and $A_{T_g(\omega)}$ are unitarily equivalent $\forall g \in G, \omega \in \Omega$.

depending on ω, g

Theorem (Birkhoff): space avg = randomness avg

Let $(\Omega, \mathcal{F}, \mathbb{P}, T)$ be an ergodic $\mathbb{Z}^d$-dyn. sys. and $X \in L^1(\Omega, \mathbb{P})$ be a random variable, then

$$\lim_{L \to \infty} \frac{1}{(2L+1)^d} \sum_{\substack{x \in \mathbb{Z}^d \\ |x|_\infty \le L}} X(T_x \omega) \overset{\mathbb{P}\text{-a.s.}}{=} \mathbb{E}_\mathbb{P}[X]$$

Theorem (Pastur 1980s):

Let $(\Omega, \mathcal{F}, \mathbb{P}, T)$ be an ergodic $\mathbb{Z}^d$-dyn. sys. and $H_\cdot = H_\cdot^*$ be an ergodic random op. Then, $\exists$ deterministic sets (a.s. spectra) $\Sigma, \Sigma_{pp}, \Sigma_{ac}, \Sigma_{sc} \subseteq \mathbb{R}$ s.t.

$$\sigma_{\#}(H_\omega) = \Sigma_{\#} \quad \mathbb{P}\text{-a.s.}$$

Proof sketch: Recall $\sigma(H_\omega) = \{\lambda \in \mathbb{R} : tr(\chi_{(a,b)}(H_\omega)) > 0 \ \forall a < \lambda < b\}$

Define $X_{ab} : \Omega \to [0, \infty)$ to be

$$\omega \mapsto tr(\chi_{(a,b)}(H_\omega))$$

Note that

$$X_{ab}(T_x \omega) = tr(\chi_{(a,b)}(H_{T_x\omega})) \overset{H \text{ erg.}}{=} tr(\chi_{(a,b)}(U^* H_\omega U))$$
$$= tr(U^* \chi_{(a,b)}(H_\omega) U) = X_{ab}(\omega)$$

and so X_{ab} is ergodic.

So, X_{ab} is $\mathbb{P}$-a.s. constant; call it α_{ab}. Then,

$$\Sigma := \{\lambda \in \mathbb{R} : \forall a, b \in \mathbb{Q} \text{ s.t. } \lambda \in (a,b), \ \alpha_{ab} > 0\}$$

does the job.

$\square$

Anderson Model

$\sigma(-\Delta) = [-2d, 2d]$

can be thought of as "density of impurities"

Let $\quad H_\omega := -\Delta + \lambda V_\omega(X) \qquad V_\omega(x) := \omega_x$

We work in the prob. space $\Omega := \mathbb{R}^{\mathbb{Z}^d}$, $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$ the product measure (iid).
Say $f : \Omega \to \mathbb{C}$ measurable if it depends on finitely many vars in $\Lambda \subseteq \mathbb{Z}^d$

$$\Rightarrow \mathbb{P}(f) \equiv \mathbb{E}_{\mathbb{P}}[f] = \int_{\omega \in \Omega} f(\omega)\, d\mathbb{P}(\omega) = \prod_{x \in \Lambda} \int_{\omega_x \in \mathbb{R}} d\mu(\omega_x) f(\omega)$$

We assume the single-site measure μ is "nice":

Def: μ is τ-Holder continuous if $\exists \tau \in (0,1]$ s.t. $\mu(I) \leq C|I|^\tau \ \forall I \subseteq \mathbb{R}$ interval.

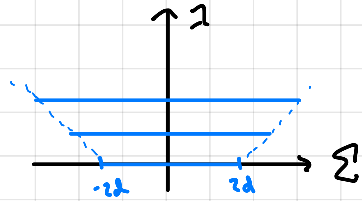
We use $G = \mathbb{Z}^d$ to be lattice translation $T_x \omega = \omega(\cdot - x)$

Theorem:

Anderson model H_ω is ergodic random operator!

Theorem (Kunz - Souillard):

"Spectrum expands as $\lambda \uparrow$"



For $H_\omega = -\Delta + \lambda V_\omega(X)$ ergodic,

$$\Sigma = [-2d, 2d] + \lambda \,\mathrm{supp}(\mu) = \{\lambda \in \mathbb{R} : \lambda = a+b,\ a \in \sigma(-\Delta),\ b \in \sigma(\lambda V_\omega(X))\}$$
$$\big(\sigma(-\Delta) + \sigma(\lambda V_\omega(X))\big)$$

Proof sketch: $\boxed{\subseteq}$ always holds. Just to show it again, suppose

$E \notin [-2d, 2d] + \lambda \,\mathrm{supp}(\mu) \iff \mathrm{dist}(E, \lambda \,\mathrm{supp}(\mu)) > 2d$

$\Rightarrow -\Delta + \lambda V_\omega(X) - E\mathbb{1} = (\lambda V_\omega(X) - E\mathbb{1})(\mathbb{1} - (\lambda V_\omega(X) - E\mathbb{1})^{-1}\Delta)$

$\Rightarrow \|(\lambda V_\omega(X) - E\mathbb{1})^{-1}\Delta\| \le \underbrace{\|-\Delta\|}_{\le 2d}\ \underbrace{\|(\lambda V_\omega(X) - E\mathbb{1})^{-1}\|}_{< \frac{1}{2d}} < 1$

$\Rightarrow$ invertible! $E \notin \sigma(H_\omega)$

$\boxed{\supseteq}$ Weyl criterion gives

$$E \in \sigma(-\Delta) \iff \forall \varepsilon > 0,\ \exists \varphi_\varepsilon \in \mathcal{H} \text{ s.t. } \|\varphi_\varepsilon\| = 1,\ \|(-\Delta - E\mathbb{1})\varphi_\varepsilon\| < \varepsilon$$

So, let $E \in [-2d, 2d]$ and $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be such a seq. By locality, we may assume $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is uniformly compactly supported. in the box Λ. For all $\tilde{E} \in \lambda \,\mathrm{supp}(\mu)$

$$\mathbb{P}\left\{\omega \in \Omega : \sup_{x \in \Lambda} |\lambda \omega_x - \tilde{E}| < \varepsilon\right\} = \prod_{x \in \Lambda} \mu(B_\varepsilon(\tilde{E})) > 0$$

For such ω's, we have $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is Weyl for $E + \tilde{E}$:

$$\|(H_\omega - (E + \tilde{E})\mathbb{1})\varphi_\varepsilon\| \le \underbrace{\|(-\Delta - E\mathbb{1})\varphi_\varepsilon\|}_{\text{Weyl}} + \underbrace{\|(V_\omega(X) - \tilde{E}\mathbb{1})\varphi_\varepsilon\|}_{\text{selection of } \omega} \le 2\varepsilon$$

So,

$$\mathbb{P}\{E + \tilde{E} \in \sigma(H_\omega)\} > 0$$

Ergodicity and Pastur's theorem gives the result. $\square$

2/22 -

Recall that in the Anderson model (iid) $H_\omega := -\Delta + \lambda V_\omega(X)$ we know H_ω is $\mathbb{Z}^d$ translation-ergodic:

$$H_{T_x\omega} = U_x^* H_\omega U_x \quad \text{with} \quad U_x \in \mathcal{B}(\ell^2(\mathbb{Z}^d)) \text{ unitary translation } (U_x\psi)(y) = \psi(x+y)$$

and

$$(U_x^* V_\omega(X) U_x)(\psi) = V_\omega(X - x\mathbb{1})(\psi) = y \longmapsto \omega_{y-x}\psi(y)$$

Facts & Conjectures about Anderson

(Fact) • $d=1 \Rightarrow$ Anderson model is localized $\forall \lambda>0$, at all energies.

(Conjecture) • $d=2 \Rightarrow$ — — — — — — ("Scaling theory of loc." - Anderson et. al)

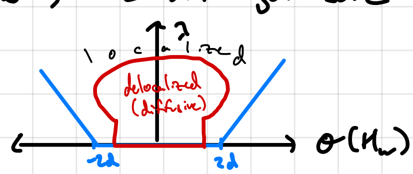
We also expect this to be true due to connection with supersymmetry (Efetov) and 2D O(3) model, for which $\nexists$ phase transition.

(Fact) • $d>2 \Rightarrow \exists \lambda_c(d)$ s.t. $\forall \lambda > \lambda_c$, localized at all energies.

(Fact) • $d>2 \Rightarrow \forall \lambda>0$, if E is "close to $\partial\Sigma$" then localized.

(Conjecture) • $d>2$, λ suff. small, $d(E,\partial\Sigma)$ suff large, then <u>de</u>localized.
(Extended states conjecture)

We will prove the third bullet above today. Before, we will get some more intuition for localization.



$(d>2)$

Criteria for localization at E_F

(i) Zero DC cond.: (@ zero temp) $\sigma_{ij}(E_F) = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\pi} \sum_{x\in\mathbb{Z}^d} x_i x_j \, \mathbb{E}\left[|\langle \delta_0, (H_\omega - (E_F + i\varepsilon)\mathbb{1})^{-1}\delta_x\rangle|^2\right] = 0$

(ii) dynamical criterion: $M_{ij}(t) = \mathbb{E}\left[|\langle \delta_0, \chi_{B_\varepsilon(E_F)}(H)\, e^{itH} X_i X_j\, e^{-itH} \chi_{B_\varepsilon(E_F)}(H)\,\delta_0\rangle|\right]$ bounded as $t\to\infty$

e'vectors have exponential decay

(iii) pure point spectrum around E_F: $\exists \varepsilon>0$ s.t. $\sigma(H) \cap B_\varepsilon(E_F) = \sigma_{pp}(H) \cap B_\varepsilon(E_F)$ can't be inf. degen. e'vals

$\boxed{\text{(iv) fractional moment criterion:}}$ $\exists C,\mu \in (0,\infty)$ s.t. $\forall x,y$, $\sup_{\varepsilon>0} \mathbb{E}\left[|G(x,y;E_F+i\varepsilon)|^s\right] \le Ce^{-\mu\|x-y\|}$ and $s\in(0,1)$

Aizenman proved this! it implies most others

(v) 2^{nd} moment criterion: $\exists C,\mu \in (0,\infty)$ s.t. $\forall x,y \in \mathbb{Z}^d$, $\sup_{\varepsilon>0} \varepsilon\, \mathbb{E}\left[|G(x,y;E_F+i\varepsilon)|^2\right] \le Ce^{-\mu\|x-y\|}$

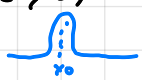
(vi) dynamic localization: $\exists \varepsilon>0$ s.t. $\sup_{t>0} \mathbb{E}\left[|\langle \delta_x, e^{-itH}\chi_{B_\varepsilon(E_F)}(H)\,\delta_y\rangle|\right] \le Ce^{-\mu\|x-y\|}$

(vii) QM many ground state $\Leftrightarrow$ Fermi proj. $P = \chi_{(-\infty, E_F)}(H)$ has $\mathbb{E}[\|P_{xy}\|] \leq C e^{-\mu\|x-y\|}$

(viii) functional analysis criterion : $\sup_{\substack{f \text{ bdd, meas,} \\ f|_{B_\varepsilon(E_F)^c} \text{ const.}}} \mathbb{E}[\|f(H)_{xy}\|] \leq C e^{-\mu\|x-y\|}$

Recall that H local
$\Rightarrow$ $f(H)$ local for analytic f.
If it also holds for bdd. meas. f,
we are localized.

Criteria for delocalization (i.e. diffusive)

- fully continuous spectrum
- $\sigma_{ij}(E_F) \in (0,\infty)$
- $\mu_{ij}(E_F) \sim t$
- inverse participation ratio : $\sum_{\substack{x \in \mathbb{Z}^d \\ |x|_\infty \leq N}} |\psi_x|^p \overset{\text{loc.}}{=} 1$  loc

$\overset{\text{deloc.}}{=} \left(\frac{1}{\sqrt{N}}\right)^p$  deloc

A-priori bound & Fractional moments

In a vibes sense, the Green's fn for a compact (only pp.) $H - z\mathbb{1}$ is

$$G(x,y;z) \equiv (H - z\mathbb{1})^{-1}_{xy} = \sum_{j=1}^{\infty} \frac{\varphi_j(x)\overline{\varphi_j(y)}}{E_j - z} \Rightarrow \mathbb{E}[|G(x,y;z)|] \sim \int \frac{1}{|E-z|} d\nu(E)$$

We expect this to scale as $\sim \int_{-1}^{1} \frac{1}{|x|} = \infty$. Uh oh!

However, the ingenuity is that $\mathbb{E}[|G(x,y;z)|^s] \sim \int_{-1}^{1} \frac{1}{|x|^s} dx = \frac{2}{1-s}$ $\quad (s \in (0,1))$

Lemma: (Schur Complement)

Suppose $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, let $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $A: \mathcal{H}_1 \to \mathcal{H}_1$, $C: \mathcal{H}_1 \to \mathcal{H}_2$, $B: \mathcal{H}_2 \to \mathcal{H}_1$, $D: \mathcal{H}_2 \to \mathcal{H}_2$

Assume D is invertible and $S := A - BD^{-1}C \in \mathcal{B}(\mathcal{H}_1)$ is invertible.

Then, $L^{-1} = \begin{bmatrix} S^{-1} & -S^{-1}BD^{-1} \\ -D^{-1}CS^{-1} & D^{-1} + D^{-1}CS^{-1}BD^{-1} \end{bmatrix}$

We will now start proving the fractional moments stuff!

Theorem (Graf '94):

$\forall s \in (0, \overset{\mu\text{'s Hölder constant}}{\tau})$, we have $\sup_{\varepsilon > 0} \mathbb{E}\left[|G(x,y; E+i\varepsilon)|^s \right] < \infty \qquad (\forall x,y \in \mathbb{Z}^d)$

Proof: We will use finite-rank perturbation theory? $H' = H + F$, F finite rank.
We will only prove the diagonal case $y = x$.

Decompose $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 = \text{range}(P_x) := \text{range}(\delta_x \otimes \delta_x^*)$ (1-dim)
$\mathcal{H}_2 = \mathcal{H}_1^\perp$ (∞-dim)

Then,

$$H - z\mathbb{1} = \begin{bmatrix} (\lambda \omega_x - z)\mathbb{1}_{\mathcal{H}_1} & P_x(-\Delta)P_x^\perp \\ P_x^\perp(-\Delta)P_x & \tilde{H} - z\mathbb{1}_{\mathcal{H}_2} \end{bmatrix} \quad \text{where } \tilde{H} := P_x^\perp H P_x^\perp \text{ indep of } \omega_x!$$

Since $\tilde{H}$ is still S.A. and $\text{Im}\{z\} > 0 \Rightarrow \tilde{H} - z\mathbb{1}$ invertible ✓
and $\underbrace{S}_{\in \mathbb{C} \text{ since } \dim(\mathcal{H}_1)=1} := \lambda\omega_x - z - P_x(-\Delta)\underbrace{P_x^\perp(\tilde{H} - zP_x^\perp)^{-1}}_{\substack{\text{Herglotz fn. of } z \\ \Rightarrow \text{Im}\{\dots\} > 0}} \underbrace{P_x^\perp(-\Delta)P_x}_{\text{invertible}}$ = real + ($\text{Im}\{\dots\} > 0$) $\Rightarrow$ invertible! ✓

we may apply Schur complement to find $(H - z\mathbb{1})^{-1}$. Since we are only concerned with the $(H - z\mathbb{1})^{-1}_{xx}$ element, we get

$$G(x,x;z) = \frac{1}{\lambda\omega_x - \beta} \quad \text{for some } \beta \in \mathbb{C} \text{ is indep. of } \omega_x!$$

Lemma: $\forall s \in (0, \underset{\text{depends on regularity of } \mu}{\tau})$, $\quad \sup_{\beta \in \mathbb{C}} \int_{v \in \mathbb{R}} |\lambda v - \beta|^{-s} \, d\mu(v) \le \frac{\tau}{\tau - s} (C_\mu)^{s/\tau} \left(\frac{2}{\lambda}\right)^s < \infty$

Proof of lemma: For all $D > 0$,

$$\int_{v \in \mathbb{R}} |\lambda v - \beta|^{-s} d\mu(v) = \underbrace{\int_{\{|\lambda v - \beta|^{-s} \le D\}} |\lambda v - \beta|^{-s} d\mu(v)}_{\le D \text{ since } \mu(\mathbb{R}) = 1} + \int_{\{|\lambda v - \beta|^{-s} > D\}} |\lambda v - \beta|^{-s} d\mu(v)$$

Using the layer-cake representation $\int_{\{f > t\}} f \, d\mu = \int_{t'=t}^\infty \mu(\{f > t'\}) \, dt'$, we get

$$\int_{v \in \mathbb{R}} |\lambda v - \beta|^{-s} d\mu(v) \le D + \int_{t=D}^\infty \mu(\{|\lambda v - \beta|^{-s} > t\}) \, dt \qquad (\forall D > 0)$$

The condition $|\lambda v - \beta|^{-s} > t \Leftrightarrow |\lambda v - \beta|^2 < t^{-2/s} \Leftrightarrow (\lambda v - \beta_R)^2 + \beta_I^2 < t^{-2/s} \Rightarrow |\lambda v - \beta_R|^{-s} > t$
and so $\mu(\{|\lambda v - \beta|^{-s} > t\}) \le \mu\left(\left\{ \left|v - \frac{\beta_R}{\lambda}\right| < \frac{t^{-1/s}}{\lambda} \right\}\right) \underset{\tau\text{-Hölder regularity}}{\le} C_\mu \left(2 \frac{t^{-1/s}}{\lambda}\right)^\tau$ independent of β!

Thus, $\int_{v \in \mathbb{R}} |\lambda v - \beta|^{-s} d\mu(v) \le D + \left(\frac{2}{\lambda}\right)^\tau C_\mu \int_{t=D}^\infty t^{-\tau/s} \, dt = D + \left(\frac{2}{\lambda}\right)^\tau C_\mu \cdot \frac{D^{1 - \tau/s}}{\frac{\tau}{s} - 1}$

converges if $\tau/s > 1$

Optimizing over D, $\int_{v \in \mathbb{R}} |\lambda v - \beta|^{-s} d\mu(v) \le \frac{\tau}{\tau - s} C_\mu^{s/\tau} \left(\frac{2}{\lambda}\right)^s$. □

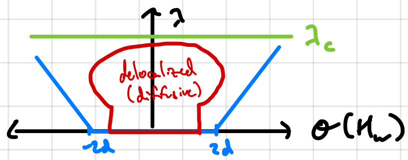
Use the lemma to integrate over ω_x. Then, since the bound is indep. of $\omega_{\{x\}^c}$, we are done. Off-diagonal proof is the same, since Schur gives

$$G(x,y;z) = \frac{\beta}{\lambda\omega_x - \alpha} \quad \text{for } \alpha, \beta \in \mathbb{C} \text{ indep. of } \omega_x.$$

□

2/27 - Loc. @ high λ, all E

Recall the picture



We will derive what happens at the green line (i.e. high $\lambda \Rightarrow$ loc. $\forall E, d$).

Lemma (Decoupling):

$$\int_{v\in\mathbb{R}} \frac{|\lambda v-\alpha|^s}{|\lambda v-\beta|^s}\, d\mu(v) \geq \lambda^s \underset{\text{indep. of } \alpha,\beta}{M} \int_{v\in\mathbb{R}} |\lambda v-\beta|^{-s}\, d\mu(v)$$

Proof: For simplicity, let $v \leftarrow \lambda v$, and so we WTS

$$\int_{v\in\mathbb{R}} \frac{|v-\alpha|^s}{|v-\beta|^s}\, d\mu(v) \geq M \int_{v\in\mathbb{R}} |v-\beta|^{-s}\, d\mu(v)$$

Note that $\forall u,v,\beta \in \mathbb{C}$, supposing WLOG $|u-\beta| \geq |v-\beta|$,

$$\Rightarrow |u-\beta|^s = |v-\beta-v+u|^s \leq (|v-\beta|+|v|+|u|)^s \underset{s\in(0,1)}{\leq} |v-\beta|^s+|v|^s+|u|^s$$
$$\Rightarrow |u|^s+|v|^s \geq |u-\beta|^s - |v-\beta|^s$$

So,

$$\begin{aligned}
&(|v|^s|u|^{-s}-1)\,|u-\beta|^s + (|u|^s|v|^{-s}-1)\,|v-\beta|^s + |u|^s + |v|^s \\
&\geq (|v|^s|u|^{-s}-1)\,|u-\beta|^s + (|u|^s|v|^{-s}-1)\,|v-\beta|^s + |u-\beta|^s - |v-\beta|^s \\
&\underset{|u-\beta|\geq|v-\beta|}{\geq} (\underbrace{|v|^s|u^{-s}| + |u|^s|v|^{-s}}_{t+\frac{1}{t}\geq 2 \text{ for } t\geq 0} - 2)\,|v-\beta|^s \geq 0
\end{aligned}$$

Dividing by $|v-\beta|^s|u-\beta|^s$,

$$|v-\beta|^{-s} + |u-\beta|^{-s} \leq \frac{|v|^s}{|v-\beta|^s}\left(|u|^{-s}+|u-\beta|^{-s}\right) + \frac{|u|^s}{|u-\beta|^s}\left(|v|^{-s}+|v-\beta|^{-s}\right) \quad (\forall u,v,\beta\in\mathbb{C})$$

Take $v = v-\alpha$, $u = u-\alpha$, $\beta = \beta-\alpha$ to get

$$|v-\beta|^{-s} + |u-\beta|^{-s} \leq \frac{|v-\alpha|^s}{|v-\beta|^s}\left(|u-\alpha|^{-s}+|u-\beta|^{-s}\right) + \frac{|u-\alpha|^s}{|u-\beta|^s}\left(|v-\alpha|^{-s}+|v-\beta|^{-s}\right)$$

Integrating by $\iint \cdot\, d\mu(u)\, d\mu(v)$,

$$\begin{aligned}
\int d\mu(v)\, |v-\beta|^{-s} &\leq \left(\int d\mu(v)\, \frac{|v-\alpha|^s}{|v-\beta|^s}\right)\left(\int d\mu(u)\, \left(|u-\alpha|^{-s}+|u-\beta|^{-s}\right)\right) \\
&\underset{\text{earlier lemma}}{\leq} \underbrace{\left(2\frac{\tau}{\tau-s}\, C_\mu^{s/\tau}\, 2^s\right)}_{=:\, M^{-1}} \int d\mu(v)\, \frac{|v-\alpha|^s}{|v-\beta|^s}.
\end{aligned}$$

$\square$

Theorem:

There is a $\lambda_c > 0$ s.t. $\forall \lambda \geq \lambda_c$, $\forall E \in \mathbb{R}$,

$\exists s \in (0,1)$ and $C, \mu \in (0,\infty)$ s.t. $\displaystyle\sup_{\varepsilon > 0} \mathbb{E}\left[|G(x,y; \underbrace{E+i\varepsilon}_{=:z})|^s \right] \leq C e^{-\mu \|x-y\|}$

Proof: We begin with the Schrödinger equation

$$(H - z\mathbb{1}) R(z) = \mathbb{1} \xrightarrow{H = -\Delta + V(x)} -\Delta R(z) = \mathbb{1} + z R(z) - \lambda V_\omega(x) R(z)$$

Letting $-\Delta = 2d\mathbb{1} - A$, where $(A\psi)_x = \sum_{y \sim x} \psi_y$ is the adjacency matrix,

$$\Rightarrow (z - 2d) R(z) - \lambda V_\omega(x) R(z) = -A R(z) - \mathbb{1}$$

Taking the x,y matrix elements,

$$(2d - z + \lambda \omega_x) G(x,y;z) = \delta_{xy} + \sum_{\tilde{x} \sim x} G(\tilde{x}, y; z)$$

$$\Rightarrow |2d - z + \lambda\omega_x|^s \, |G(x,y;z)|^s = \left| \delta_{xy} + \sum_{\tilde{x}\sim x} G(\tilde{x},y;z) \right|^s \qquad (\forall s \in (0,1))$$

Since $(a+b)^s \leq a^s + b^s$ for $a,b \geq 0$, $s \in (0,1)$,

$$|2d - z + \lambda\omega_x|^s \, |G(x,y;z)|^s \leq \delta_{xy} + \sum_{\tilde{x}\sim x} |G(\tilde{x},y;z)|^s$$

$$\Rightarrow \mathbb{E}\left[|2d - z + \lambda\omega_x|^s \, |G(x,y;z)|^s \right] \leq \delta_{xy} + \sum_{\tilde{x}\sim x} \mathbb{E}\left[|G(\tilde{x},y;z)|^s \right]$$

lets us pull out the constant

By decoupling,

$$\mathbb{E}\left[|2d - z + \lambda\omega_x|^s \, |G(x,y;z)|^s \right] \geq \lambda^s M \, \mathbb{E}\left[|G(x,y;z)|^s \right]$$

$$\Rightarrow \mathbb{E}\left[|G(x,y;z)|^s \right] \leq \frac{1}{\lambda^s M} \delta_{xy} + \frac{1}{\lambda^s M} \sum_{\tilde{x}\sim x} \mathbb{E}\left[|G(\tilde{x},y;z)|^s \right]$$

For $x, y \in \mathbb{Z}^d$, denote $f(x,y) := \mathbb{E}\left[|G(x,y;z)|^s \right]$. Then, for x far from y (which is what we care about),

$$\boxed{f(x,y) \leq \frac{1}{\lambda^s M} \sum_{\tilde{x}\sim x} f(\tilde{x}, y)}$$

Subharmonicity in space

Lemma (Subharmonicity):

Consider a kernel $B: \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty)$ obeying
$B_{xy} \leq \gamma \sum_{u\sim x} B_{uy}$ $\forall x,y \in \mathbb{Z}^d$ and some $\gamma < \frac{1}{2d}$,

Then, $B_{xy} \leq \frac{2}{m} \exp\left(-\frac{1}{2} m \|x-y\|\right)$ with $m = \frac{1}{\gamma} - 2d$

Proof of lemma: Let $(A\psi)_x = \sum_{u\sim x} \psi_u$ be the adjacency operator, and so

$$B_{xy} \leq \gamma (AB)_{xy} \Rightarrow ((\underbrace{\mathbb{1} - \gamma A}_{= -\Delta + m\mathbb{1}}) B)_{xy} \leq 0 \leq \delta_{xy}$$

$$\Rightarrow ((-\Delta + m\mathbb{1})B)_{xy} \leq \delta_{xy} \Longrightarrow B_{\tilde{x}y} \leq (-\Delta + m\mathbb{1})^{-1}_{\tilde{x}y} \Rightarrow B_{\tilde{x}y} \leq \frac{2}{m} \exp\left(-\frac{1}{2} m \|x-y\|\right)$$

use heat kernel, Bessel fns to show $(-\Delta + m\mathbb{1})^{-1}_{\tilde{x}y} \geq 0$

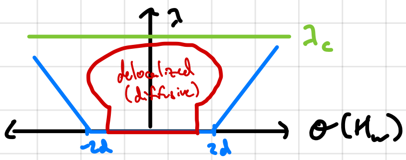
Combes-Thomas if $m > 0$

$\square$

This lemma completes the proof. $\square$

2/20 -  Loc @ all λ, extreme E

Recall the picture



We saw above that we have $\lambda_c = \left(\frac{2d}{\mu}\right)^{1/s}$ so that $m>0$.
from decoupling

Now, let's look at localization below the green line.

We have $H_\omega \equiv -\Delta + \lambda V_\omega(X)$, $H_0 \equiv -\Delta$, and so the resolvent identity yields

$$R_\omega(z) = R_0(z) + R_0(z)\underbrace{(H_0 - H_\omega)}_{-\lambda V_\omega(X)} R_\omega(z)$$

$\mathbb{E}[|\cdot|^s]$

$$\Longrightarrow \mathbb{E}[G_\omega(x,y;z)] \le \underbrace{\mathbb{E}[|G_0(x,y;z)|^s]}_{\text{nonrandom}} + \sum_{\tilde{x}\sim x} \lambda^s |G_0(x,\tilde{x};z)|^s \, \mathbb{E}[|\omega_{\tilde{x}}|^s |G_\omega(\tilde{x},y;z)|^s]$$

Note that we are in a different regime (λ is on the right, decoupling must go in other direction). Specifically, we are gonna need

① need $z \notin \sigma(-\Delta)$ to use Combes-Thomas on G_0.
② need λ^s small
③ need another decoupling lemma in the other direction

Lemma (Decoupling 2):

Suppose that $\overbrace{\int |v|^{2s} d\mu(v)}^{=: B_{2s}} < \infty$ (finite s-moment) and μ is τ-Hölder regular. Then, $\exists\, D(B_{2s},\tau) \in (0,\infty)$ s.t.

$$\int_{v\in\mathbb{R}} |v|^s |\lambda v - \beta|^{-s}\, d\mu(v) \le D \int_{v\in\mathbb{R}} |\lambda v - \beta|^{-s}\, d\mu(v)$$

Proof: Using our earlier lemma,

$$\int_{v\in\mathbb{R}} |v|^s |\lambda v - \beta|^{-s}\, d\mu \overset{\text{C.S.}}{\le} \sqrt{B_{2s}} \sqrt{\frac{\tau}{\tau - 2s}\, C_\mu^{2s/\tau} \left(\frac{2}{\lambda}\right)^{2s}}$$

Also, $\forall Q \in \mathbb{R}^+$, $\int_{v\in\mathbb{R}} |\lambda v - \beta|^{-s} d\mu \ge \int_{\{|\lambda v| \le Q\}} |\lambda v - \beta|^{-s} d\mu$

We know $|\lambda v - \beta| \le |\lambda v| + |\beta| \le Q + |\beta|$, and so

$$\int_{\{|\lambda v|\le Q\}} |\lambda v - \beta|^{-s}\, d\mu \ge (Q+|\beta|)^{-s}\left(1 - \mu\{|\lambda v| > Q\}\right)$$

Markov's ineq. states $\forall f(\cdot)$ monotone inc.

$$\mu\{|x|>c\} \le \frac{\int f(|x|)\,d\mu}{f(c)} \Rightarrow \mu\{|\lambda v|>Q\} \le \frac{B_{2s}}{\left(\frac{Q}{\lambda}\right)^{2s}}$$

Choosing Q s.t. $B_{2s}/\left(Q/\lambda\right)^{2s} = \frac{1}{2}$ (i.e. $Q = \left(2\lambda^{-2s}B_{2s}\right)^{1/2s}$)

$$\int_{v\in\mathbb{R}} |\lambda v-\beta|^{-s}\,d\mu \ge \frac{1}{2}\left(\left(2\lambda^{-2s}B_{2s}\right)^{1/2s} + |\beta|\right)^{-1}$$

Case 1: $|\beta| \le \left(\lambda s^{-2s}B_{2s}\right)^{1/2s}$

This follows clearly, and we get D in that regime.

Case 2: $|\beta| > (\dots)^{1/2s}$

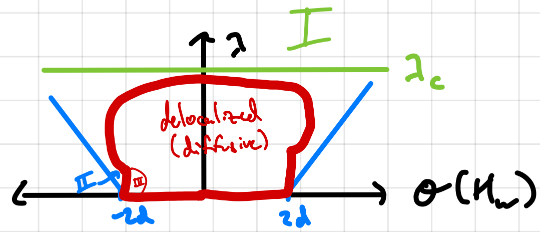
Here,
$$\int_{v\in\mathbb{R}} \frac{|v|^s}{|\lambda v-\beta|^s}\,d\mu \le \int_{|v|\le\frac{|\beta|}{2}}\cdots + \int_{|v|>\frac{|\beta|}{2}}\cdots$$
$$\le \left(\frac{2}{|\beta|}\right)^s B_s + \left(\frac{2}{|\beta|}\right)^s \int_{|v|>\frac{|\beta|}{2}} \frac{|v|^{2s}}{|\lambda v-\beta|^s}\,d\mu$$
a-priori bound
$$\le \left(\frac{2}{|\beta|}\right)^s \left(B_s + B_{2s}M\right) \le D\,\frac{1}{2}\left(\left(2\lambda^{-2s}B_{2s}\right)^{1/2s} + |\beta|\right)^{-1}$$

for large enough D.

$\square$

So, ① – ③ above yield localization.

3/5 -



There are the following mechanisms for localization:

① large λ $\Rightarrow$ complete loc. via subharmonicity

② $E \notin \sigma(-\Delta)$ and λ suff. small $\Rightarrow$ loc. via subharmonicity

③ low density of states ("Lifschitz tails")

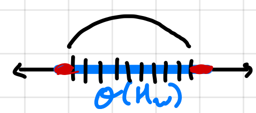
④ Complete loc. in 1D

We already proved I and II. We will tackle III and IV today.

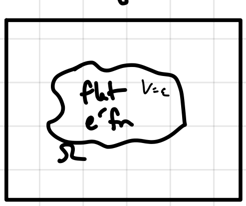
III - Low Density of States

We have $H_\omega = -\Delta + \lambda V_\omega(x)$. Truncate to $\Lambda_L = [-L,L]^d \cap \mathbb{Z}^d$ to get a $N := (2L+1)^d \times N$ matrix H_L acting on $\mathbb{C}^{\Lambda_L}$ (boundary conditions don't matter).

As $N \to \infty$, the N eigenvalues of H_N fill out $\sigma(H_\omega)$



We are interested in the states corresponding to the red regions. For $E \approx -2d + \varepsilon$ (or $2d - \varepsilon$), we expect the eigenfunctions of the Laplacian to be approximately constant. So, the probability of such a state $\sim e^{-|\Omega|}$.



This exponential decay of probability of eigenstates near the fringes (in contrast to the semicircle law) is called Lifschitz tails.

Through black magic, we'd be able to get quantitative bounds

① $$\mathbb{P}\{\omega : \text{dist}(\sigma(H_L(\omega)), E) \leq C L^{-\beta}\} \leq \tilde{C} L^{-\alpha}$$

Using the a-priori bound and a Schur complement $\mathcal{H} = \ell^2(\Lambda_L) \oplus \ell^2(\mathbb{Z} \setminus \Lambda_L)$, we can see that finite-volume FMC $\Rightarrow$ ∞-volume FMC via

② $$\mathbb{E}\left[|G(x,y;z)|^s\right] \leq C\, \mathbb{E}\left[|\underset{\substack{\uparrow \\ \text{indep of } L}}{G_L}(x,y;z)|^s\right]$$ (see Ch. 11 of Aizenman - Warzel)

Furthermore, the $|x-y|$ behavior is controlled by the $|0-L|$ behavior:

③ $$\mathbb{E}\left[|G_L(x,y;z)|^s\right] \leq \underset{\substack{\uparrow \\ \text{indep of } L}}{C}\, \mathbb{E}\left[|G_L(0,\underset{\substack{\uparrow \\ \text{something at the edge of the box}}}{L};z)|^s\right]$$

Lastly, we use the following fact.

(4) Lemma:

If g is an integral kernel satisfying
$$g(x,y) \le \gamma \sum_{z \in \partial \Omega} g(x,z)\, g(z,y) \quad \text{for } \gamma \text{ suff. small},$$
then suff. fast poly decay of g $\Rightarrow$ exponential decay of g

Using (1)-(4), we do the following:

Define $S_E := \{ w : \operatorname{dist}(\sigma(H_L(w)), E) \ge C L^{-\beta} \}$. Within S_E, Combes-Thomas yields $|G(0,L;E)|^s \le \frac{2^s}{L^{-s\beta}} \exp(-C s L^{-\beta} L)$. Also, $\mathbb{P}\{\chi_{S_E^c}\} \le \tilde{C} L^{-\alpha}$

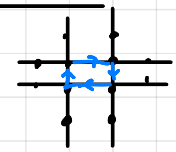
So,
$$\mathbb{E}\left[|G_L(0,L;E)|^s \right] = \mathbb{E}\left[|G_L(0,L;E)|^s \chi_{S_E} \right] + \mathbb{E}\left[|G_L(0,L;E)|^s \chi_{S_E^c} \right]$$
$$\overset{\text{C.S.}}{\le} \frac{2^s}{L^{-s\beta}} \exp(-C s L^{1-\beta}) + \tilde{C} L^{-s\alpha p}$$

and so we get a decay of fractional moments.

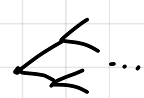
Thus, we get localization for the E for which we may prove (1): these are exactly the Lifschitz tails!

IV - Complete localization in 1D: transfer matrix approach

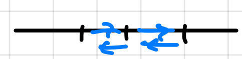
Intuitively, localization comes about from quantum interference: randomness from other places and the past affect state.



This is why we have been able to show delocalization on tree graphs: there are no cycles and many directions to distribute randomness.



In 1D, this effect is seen to the max, since there aren't as many dimensions in which to distribute the randomness.



In 1D,
$$H\psi = z\psi \iff 2d\,\psi_n - \psi_{n-1} - \psi_{n+1} + \lambda w_n \psi_n = z\psi_n \quad \forall n \in \mathbb{Z}$$
$$\iff \psi_{n+1} = -(z - 2d - \lambda w_n)\psi_n - \psi_{n-1}$$

Lifting $\Psi_n := \begin{bmatrix} \psi_{n+1} \\ \psi_n \end{bmatrix}$,
$$H\psi = z\psi \iff \Psi_n = \underbrace{\begin{bmatrix} -(z-2d-\lambda w_n) & -1 \\ 1 & 0 \end{bmatrix}}_{=: A_n(z)} \Psi_{n-1}$$

These $A_n(z)$ are the transfer matrices, and we have that $\varphi_n = \left(\prod_{j=1}^{n} A_j(z)\right) \varphi_0$

From conservation of prob. current, we see that the transfer matrices are symplectic: $A_n(z)^T \Omega A_n(z) = \Omega$ with $\Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

So, $A_n(z)^{-1} = \Omega^{-1} A_n(z)^T \Omega \Rightarrow \dots \Rightarrow$ eigenvalues are symmetric about S^1.

Thus, the system may be modeled via large products of iid random matrices.

Products of Random Matrices

Consider $\{B_n\}_{n\in\mathbb{Z}}$ iid random matrices of size $W \times W$.
For $j \in [w]$, we define the Lyapunov exponents

$$\gamma_j := \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\log \sigma_j(B_n \dots B_1)\right]$$

$\sigma_j(M) = j^{th}$ singular value
$\sigma_1(M) = \|M\|$
$\sigma_w(M) = \|M^{-1}\|^{-1}$

In the 1D Anderson model, $W=2$ and $\sigma_1(M) = \dfrac{1}{\sigma_2(M)}$ by symplectic condition.
So, $\gamma_1(z) = -\gamma_2(z)$.

- If $\gamma_1(z) > 0$, we expect $|\psi_n| \lesssim e^{-\gamma_1(z) n} \Rightarrow$ localized
- If $\gamma_1(z) = 0$, we expect $\nexists$ exp. decay $\xRightarrow{\text{contrapositive of lemma}} \nexists$ poly decay $\Rightarrow$ deloc.

The Furstenberg theory gives an answer to when there are simple Lyapunov exponents: its precisely when $\{B_n\}_n$ fills an open subset of the group they belong to (the symplectic group): this can't happen for the 1D Anderson model since only one matrix element depends on the randomness.

So, $\gamma_1(z) \neq \gamma_2(z) \Rightarrow \gamma_1(z) > 0$.

Fill in 3/19

Intro to Topo. Insulators

3/21 -

Quick recap on topological insulators:

Analytically, we have been using the condition for $H \in \mathcal{B}(\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N)$ of

$$\|H_{xy}\| \le c\, e^{-m\|x-y\|} \iff H \text{ localized}$$

We seek a topological classification.

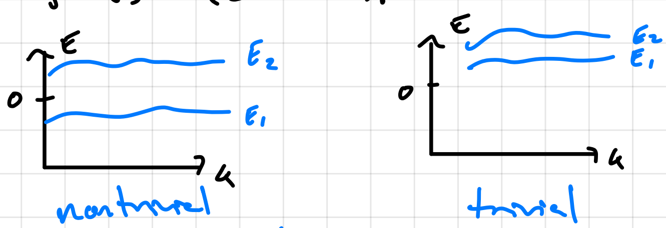
Periodic, 2 DOF (both can be relaxed)

For illustration, let H be periodic, i.e. $H_{x+z,y+z} = H_{x,y}$.
Since the F.T. diagonalizes H, we have a symbol

$$h : \underset{\text{cont.}}{\mathbb{T}^d \to} \{A \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \mid \underbrace{\sigma(A) \not\ni 0}_{\text{if } 0 \notin \sigma(A), \text{ then C.T. gives that } A \text{ is local.}}\}$$

The set of such h's is $\cong$ the space of local Hamiltonians.
Use the compact open topology on {symbols} (L^∞ norm).

In the case $N=2$, $d=1$, we want



So, the space in $d=1$ has

$$h : S^1 \to \{\underset{\text{S.A.}}{A} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : \underset{\text{two of the eigenvalues}}{E_1 < 0 < E_2}\} \cong S^2$$

The $\cong$ is an alg. top. fact. We know $C_*[S^1 \to S^2] \underset{\text{homeo}}{\cong} \{0\}$

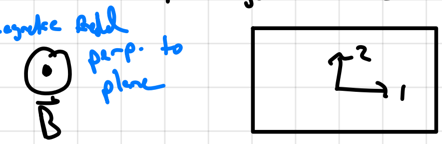
So, there is $\boxed{\text{nothing interesting in 1D}}$.

In 2D, $C[\mathbb{T}^2 \to S^2] \cong \underset{\text{Chern \#}}{\mathbb{Z}}$

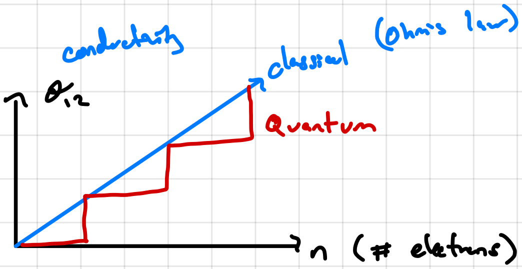
Quantum Hall Effect (1979)

2DEG (low temp) system has

large magnetic field perp. to plane



2D



Classical Computation

We can do the classical computation: for a path $\gamma : \mathbb{R} \to \mathbb{R}^2 \cong \mathbb{C}$, we have the ODE

time $\to$ space

resistivity, <0

$$\ddot{\gamma}(t) = E(\gamma) + \dot{\gamma} \wedge B(\gamma) + r\dot{\gamma}$$

in const electric and magnetic fields

$E(\gamma) = E_0 e_1$

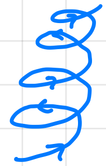
$B(\gamma) = B_0 e_3$, $B_0 \in \mathbb{R}$

This has the solution

$$\gamma(t) = \frac{-E_0}{r - iB_0} t + \left(e^{(r-iB_0)t} - 1\right)\left(\frac{E_0}{(r-iB_0)^2} + \frac{1}{r-iB_0}\dot{\gamma}(0)\right) + \gamma(0)$$

- if $r = E_0 = 0$, $\frac{1}{B_0}$ is the cyclotron radius and we get circular motion $\frac{\dot{\gamma}(0)}{B_0} = r$

- If $E_0, B_0 \neq 0$, $r = 0$, we get the Hall effect: there is net movement in the 2^{nd} direction despite constant electric field in e_1 direction.



- In equilibrium, $\ddot{\gamma} = 0 \Rightarrow -(r - iB_0)\dot{\gamma} = E_0$. For 2D current density $j = n\dot{\gamma}$, which by Ohm's law $j = \sigma E$ gives

$$E = \frac{1}{\sigma} n \dot{\gamma} \Rightarrow \sigma = \frac{-n}{r - iB_0} \in \mathbb{C}$$

(Note that in the above, we have used the perspective common to 2D:

$$E \in \mathbb{R}^2, \quad j \in \mathbb{R}^2, \quad \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \quad j = \sigma E \quad \rightsquigarrow \quad E \in \mathbb{C}, \quad j \in \mathbb{C}, \quad \sigma \in \mathbb{C}, \quad j = \sigma E$$
)

- If $B_0 = 0$, $\sigma = -\frac{n}{r} \in \mathbb{R}$ and everything behaves as usual (i.e. resistivity $\sim \frac{1}{\sigma}$)

- If $B_0 \neq 0$, then σ doesn't blow up as $r \to 0$. Instead,

$$\lim_{r \to 0} \sigma = -i \frac{n}{B_0} \in i\mathbb{R}.$$ We call $\sigma_{Hall} = -\frac{n}{B_0}$ is the longitudinal Hall conductivity.

Quantum Computation

In 2D, we have $H = (P-A)^2 + E_0 X = (-i\nabla - A)^2 \in \mathcal{B}(L^2(\mathbb{R}^2))$, with a gauge choice s.t. $\mathrm{curl}(A) = B_0$ is constant:

$$A(x) = \frac{1}{2} B_0 \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \qquad\qquad A(x) = B_0 \begin{bmatrix} -x_2 \\ 0 \end{bmatrix} \quad \text{or} \quad B_0 \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

symmetric gauge   Landau gauge

Recall that in the classical computation, to get σ we sought the velocity $\dot{\gamma}$. Here, we want $V = i[H, X]$ (since $\partial_t \langle A(t)\rangle_\psi = \langle i[H,A]\rangle_\psi$)

In the second Landau gauge,

$$H = (P-A)^2 + E_0 X_1 = P_1^2 + (P_2 - B_0 X_1)^2 + E_0 X_1$$

There is no dependence on X_2, and so it's 2-translation-invariant. By a partial F.T. in the second coord,

$$\hat{H}(k_2) = P_1^2 + (k_2 - B_0 X_1)^2 + E_0 X_1 \overset{\substack{\text{complete}\\\text{the square}}}{=} P_1^2 + B_0^2\left(X_1 - \frac{k_2}{B_0} + \frac{E_0}{2B_0^2}\right)^2 + \frac{E_0}{B_0} k_2 - \frac{E_0^2}{4B_0^2}$$

This is solvable with $E_j(k_2) = B_0(2j+1) + \frac{E_0}{B_0} k_2 - \frac{E_0}{4B_0^2}$ $\quad (j \in \{0, 1, 2, \dots\})$ (shifted SHO)

However, it's difficult to make sense of this. ✗

Instead, we will do perturbation theory in E_0 and eventually use Kubo. So, we first solve the unperturbed setting: the Landau Hamiltonian

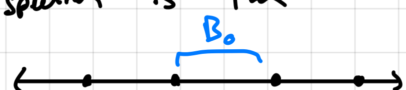
$$H_0 = (P-A)^2 \underset{\substack{\uparrow\\\text{symmetric}\\\text{gauge}}}{=} P^2 + \tfrac{1}{4} B_0 X^2 - B_0 L_3, \qquad L_3 = X_1 P_2 - X_2 P_1 \text{ (angular momentum)}$$

By a change of coords $Z := X_1 + i X_2$ (and $|Z|^2 = X^2$)
$D := \frac{1}{2}(P_1 - iP_2)$ (and $|D|^2 = \frac{1}{4}P^2$) $\Rightarrow L_3 = ZD + Z^* D^*$

Since $[D, Z] = \mathbb{1}$, we get $L_3 = 2\,\mathrm{Re}\{ZD\} - \mathbb{1}$, and so

$$H_0 = 4|D|^2 + \frac{B_0^2}{4}|Z|^2 - B_0(2\,\mathrm{Re}\{ZD\} - \mathbb{1}) = \Big|\underbrace{\frac{B_0}{2} Z - 2D^*}_{=:A}\Big|^2 + B_0 \mathbb{1}$$

We may show $[A^*, A] = B_0 \mathbb{1}$, and so it's a ladder operator and $H = A^* A + B_0 \mathbb{1}$. Via two 45-degree rotations, we modified it to a <u>single harmonic oscillator</u>. It turns out that the spectrum is then



each is a copy of an independent SHO, and so is ∞-degenerate

We call each of these a "Landau level"

Dropping the constants B_0 from above,

$$D \equiv \frac{i}{2}(P_1 - iP_2) = \frac{i}{2}(-i\partial_1 - \partial_2) = i\,\partial_{\bar z} \Rightarrow A = -\exp(-\tfrac{1}{2}|z|^2)\,\partial_{\bar z}\exp(\tfrac{1}{2}|z|^2)$$

To find ground state, we need $A\Psi = 0 \iff \partial_{\bar z}\exp(\tfrac{1}{2}|z|^2)\Psi(z) = 0$
Letting $\psi(z) = \exp(-\tfrac{1}{2}|z|)\,f(z)$, then $\underbrace{\partial_{\bar z} f(z) = 0}_{\text{Cauchy-Riemann equations!}}$

So, the first Landau level is

$$\operatorname{span}\left\{ f(z)\,e^{-\frac{1}{2}|z|^2} : f \text{ holo.} \right\}$$

A particular choice of $f(z)$ as monomials allows $\Psi_{0m}(z) = \frac{z^m}{\sqrt{\pi\, m!}}\exp(-\tfrac{1}{2}|z|^2)$ holo for $m \in \mathbb{N} \cup \{0\}$
for $m \geq 0$. These satisfy $L_3\Psi_{0,m} = m\Psi_{0,m}$, and so the first L.L. has angular momentum ≥ 0. More generally,

$$\boxed{n^{\text{th}} \text{ Landau level has ang. mom. } \geq -n}$$

For a Landau level at fixed n, the Hilbert space of states is $\cong \ell^2(\mathbb{N})$

Fill ~ 3/26

3/28 - Properties of θ_{Hall}

Recall that for the quantum Hall effect and the double commutator formula

$$\theta_{Hall} = i\,\mathrm{tr}\left(P\left[[\Lambda_1, P], [\Lambda_2, P]\right]\right) \qquad \text{(DCF)}$$

From this,

1. H has spectral gap @ $E_F \Longrightarrow P = \chi_{(-\infty, E_F)}(H)$ is local
2. $[\Lambda_j, P]$ is local and $\|[\Lambda_j, P]_{xy}\|$ has decay in $|x_j|, |y_j|$ separately
3. $[\Lambda_1, P][\Lambda_2, P] \in \mathcal{J}_1(\mathcal{H})$
4. Using position operators, we need to use the trace/unit volume

$$\theta_{Hall} = i\,\mathrm{trpuv}\left(P[[X_1,P],[X_2,P]]\right) \overset{P \text{ periodic}}{=} \frac{i}{(2\pi)^2}\int_{k\in\mathbb{T}^d} dk\ \hat{P}(k)\, \varepsilon_{ij}(\partial_i \hat{P})(k)(\partial_j \hat{P})(k)$$

note $\widehat{[X_j,P]}(k) = i(\partial_{k_j}\hat{P})(k)$

Since $\hat{P}(k) = \sum_{j=1}^{n} \psi_j(k) \otimes \psi_j(k)^*$, we get the Berry curve formula

$$\boxed{\theta_{Hall} = \int_{k\in\mathbb{T}^d} \sum_{j=1}^{n} \varepsilon_{\alpha\beta}\, \partial_\alpha \langle \psi_j(k), \partial_\beta \psi_j(k)\rangle}$$

$$\left(\varepsilon_{ij} \equiv \begin{cases} 1 & i<j \\ -1 & j<i \\ 0 & \text{else}\end{cases}\right)$$

Levi-Civita

A very famous paper by TKNN '82 proved that this evaluates to an integer (i.e. Chern #).

We will prove integrality of the DCF, which is also more general.

Integrality of Double-Commutator Formula (Fredholm)

Lemma: $P\left[[\Lambda_1, P], [\Lambda_2, P]\right] = [P\Lambda_1 P,\ P\Lambda_2 P]$

Proof: $\varepsilon_{ij}\, P[\Lambda_i,P][\Lambda_j,P] \overset{\text{symmetric terms get killed by } \varepsilon_{ij} \text{ sum}}{=} \varepsilon_{ij}\, P\Lambda_i P\Lambda_j P$ $\square$

Note that $AB \in \mathcal{J}_1$, then $\mathrm{tr}([A,B]) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$.
Since $P\Lambda_1 P\Lambda_2 P$ is not trace class, $\theta_{Hall} \neq 0$.

Now, some Fredholm stuff.

Theorem: (Fedosov formula) ← scaffolding for index Atiyah-Singer

If $F \in \mathcal{F}(\mathcal{H})$ (Fredholm) and G is a parametrix with $[F,G] \in \mathcal{J}_1(\mathcal{H})$, then

$$\mathrm{index}(F) = \mathrm{tr}([F,G])$$

Theorem: (Atkinson)

If F has a parametrix, then $F \in \mathcal{F}(\mathcal{H})$.

The magic is the following formula:

> **Theorem:** (Baby AS Index)
>
> If Q is S.A. proj and U unitary with $[U,Q] \in \mathcal{J}_1(\mathcal{H})$, then
>
> $$\underset{\text{(geometry)}}{\mathrm{tr}(U^*[Q,U])} = \mathrm{index}(QUQ + Q^\perp)$$
> $$= \dim \ker(QUQ + Q^\perp) \quad \text{(topology)}$$

compare with $\oint \frac{f'}{f} = \mathrm{index}(f)$

since $[Q,U] \approx f'$

and $U^* \approx \frac{1}{f}$

Proof of theorem: (from Avron, Seiler, Simon '94 "Charge deficiency")

Let $Q_U := QUQ + Q^\perp$. We show that $Q_{U^*} = QU^*Q + Q^\perp$ is a parametrix: we WTS $\mathbb{1} - (Q_{U^*})Q_U \in \mathcal{K}(\mathcal{H})$.

$$\mathbb{1} - (Q_{U^*})(Q_U) = Q + Q^\perp - (QU^*QUQ + Q^\perp)$$
$$= Q - QU^*QUQ = Q(\underbrace{\mathbb{1}}_{=U^*U} - U^*QU)Q$$
$$= QU^*\underbrace{(\mathbb{1}-Q)}_{Q^\perp}UQ$$
$$= QU^*Q^\perp[U,Q] \in \mathcal{K}(\mathcal{H}) \quad \text{since } [U,Q] \text{ compact.}$$

So, Q_U is Fredholm. Applying Fedosov with parametrix Q_{U^*}. Thus,

$$\mathrm{index}(Q_U) = \mathrm{tr}((Q_U)(Q_{U^*}) - (Q_{U^*})(Q_U))$$
$$= \mathrm{tr}(QUQU^*Q - Q + Q - QU^*QUQ)$$

We showed above that $QUQU^*Q - Q \in \mathcal{J}_1(\mathcal{H})$, and so, letting $R := U^*QU$ be another S.A. proj,

$$\mathrm{index}(Q_U) = \mathrm{tr}(RQR - R) - \mathrm{tr}(QRQ - Q)$$

Note that $[Q,(Q-R)^2] = [R,(Q-R)^2] = 0$ since $Q(Q-R)^2 = Q - QR - QRQ + QR = (Q-R)^2 Q$. Then,

$$(RQR-R) - (QRQ-Q) = (Q-R)^3 = (Q-U^*RQ)^3 = (U^*[U,Q])^3$$
$$\Rightarrow \operatorname{index}(QU) = \operatorname{tr}((U^*[U,Q])^3)$$

We are almost done, and all we must show is that we can use the 1st power instead of the 3rd:

$$(Q-R)^3 = Q - R - QRQ + RQR = Q - R - [QR, RQ]$$
$$= Q - R - [QR,[R,Q-R]]$$

Since $Q-R \in \mathcal{J}_1(\mathcal{H})$, then $[QR,[R,Q-R]] = 0$ since it's $[A,B]$ with $B \in \mathcal{J}_1$.
So,

$$\operatorname{tr}((Q-R)^3) = \operatorname{tr}(Q-R) = \operatorname{tr}(U^*[U,Q])$$

$\square$

Now, the main result.

<u>Theorem:</u> $(\sigma_{Hall} \in \mathbb{Z})$

We have
$$\sigma_{Hall} = i\operatorname{tr}(P[[\Lambda_1,P],[\Lambda_2,P]])$$
$$= \frac{1}{2\pi}\operatorname{index}(\Lambda_1 \exp(-2\pi i P\Lambda_2 P)\Lambda_1 + \Lambda_1^{\perp})$$
$$\in \frac{1}{2\pi}\mathbb{Z}.$$

K-theory index

<u>Proof:</u> Let us note that from the DCF and the first lemma,

$$\sigma_{Hall} = i\operatorname{tr}([P\Lambda_1 P, P\Lambda_2 P]) \overset{\text{no-op}}{=} \frac{1}{2\pi}\int_{\alpha=0}^{2\pi} d\alpha \; i\operatorname{tr}([P\Lambda_1 P, P\Lambda_2 P])$$

$$\underset{\substack{\text{cyclicity of} \\ \text{trace \& everything} \\ \text{is } \in \mathcal{J}_1(\mathcal{H})}}{=} \frac{i}{2\pi}\int_{\alpha=0}^{2\pi} \operatorname{tr}(e^{-i\alpha P\Lambda_2 P}[P\Lambda_1 P, P\Lambda_2 P]e^{i\alpha P\Lambda_2 P})\, d\alpha$$

Since $e^{-i\alpha A}[B,A]e^{i\alpha A} = i\partial_\alpha e^{-i\alpha A}Be^{i\alpha A}$, the fundamental thm. of calc gives

$$= \frac{1}{2\pi}\operatorname{tr}(e^{-2\pi i P\Lambda_2 P}P\Lambda_1 P e^{2\pi i P\Lambda_2 P} - P\Lambda_1 P)$$
$$= \frac{1}{2\pi}\operatorname{tr}(e^{-2\pi i P\Lambda_2 P}[P\Lambda_1 P, e^{2\pi i P\Lambda_2 P}])$$

Since $[P\Lambda_2 P, P] = 0$, we know $[e^{2\pi i P\Lambda_2 P}, P] = 0$, and so

$$= \frac{1}{2\pi}\operatorname{tr}(Pe^{-2\pi i P\Lambda_2 P}P[\Lambda_1, e^{-2\pi i P\Lambda_2 P}])$$

We may write $e^{-2\pi i P\Lambda_2 P} = P e^{-2\pi i P\Lambda_2 P} + P^\perp$ by unitarity, and so

$$= \frac{1}{2\pi} \operatorname{tr}\left(e^{-2\pi i P\Lambda_2 P} [\Lambda_1, e^{2\pi i P\Lambda_2 P}] \right)$$

By Baby AS,

$$= \frac{1}{2\pi} \operatorname{index}\left(\Lambda_1 e^{-2\pi i P\Lambda_2 P} \Lambda_1 + \Lambda_1^\perp \right) \in \frac{1}{2\pi} \mathbb{Z}. \qquad \square$$

Calculating σ_{Hall} - Loughlin Flux Formula

The above formulae are good to prove things but not to compute the Chern #. We go a different route.

$\rightsquigarrow$ radial electric field, measure "current" as # (density) of e^- going $\to \infty$

Let $U := \exp(i \arg(X_1 + iX_2))$ be the unitary associated with flux insertion at the origin over one period. Note that $P - U^*PU$ is not in $\mathcal{J}_1$, but it is $\in \mathcal{J}_3$. So, we expect

$$\sigma_{Hall} = \frac{1}{2\pi} \operatorname{tr}\left((P - U^*PU)^3 \right) = \dots = \frac{1}{2\pi} \operatorname{index}(PUP + P^\perp).$$

We can now calculate the Landau Hamiltonian's Chern # !

Prop:

Let P be a proj. onto one Landau level.

finish

LL $n \geq 0$ has ang. mom. $\ell \geq -n \implies \operatorname{im}(P) \cong \ell^2(\mathbb{Z}_{\geq -n})$

Let $\Theta := \arg(X_1 + iX_2)$ be the polar angle position op. Then,

- Ang. mom. is conjugate var. to Θ
- Θ generates the angular momentum shifts (like how e^{iX} is momentum shift by 1)
- $(\Theta f)(r, \varphi) = \varphi f(r, \varphi)$ (polar coords)

4/2 -

Recall from last time that for the IQHE with

- $H \in \mathcal{B}(\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^N)$ local, gapped @ E_F
- $P \equiv \chi_{(-\infty, E_F)}(H)$ local

we were able to show

projections to upper and right half-planes

$$\theta_{Hall} = i\, \mathrm{tr}\Big(\underbrace{P\big[[\Lambda_1, P],[\Lambda_2, P]\big]}_{\text{trace-class}}\Big) = \frac{1}{2\pi} \underbrace{\mathrm{index}\left(\Lambda_1 e^{-2\pi i P \Lambda_2 P} \Lambda_1 + \Lambda_1^\perp\right)}_{\text{Kitaev index}}$$

$$\in \frac{1}{2\pi}\mathbb{Z}$$

We will now look at the Laughlin index, which is more commonly used in mathematical physics.

Def: (Laughlin Flux Insertion)

Define $U := \exp(i \arg(x_1 + i x_2))$ to be the Laughlin flux insertion.

Theorem: (Laughlin Index)

We have $\theta_{Hall} = \frac{1}{2\pi} \mathrm{index}(PUP + P^\perp)$

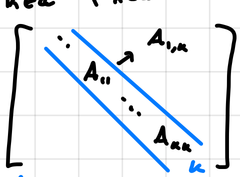
Proof: Recall from last lecture that if $[P, U] \in \mathcal{K}(\mathcal{H})$, then $PUP + P^\perp =: \mathbb{P}U \in \mathcal{F}(\mathcal{H})$ (earlier, we knew $[\Lambda_1, e^{-2\pi i P \Lambda_2 P}] \in \mathcal{J}_1(\mathcal{H})$). It turns out that $[P, U] \in \mathcal{J}_3(\mathcal{H})$ but not trace-class. We need the following lemmas:

Lemma: If
$$\begin{cases} [Q, W] \in \mathcal{J}_1(\mathcal{H}) \ \text{then} \ \mathrm{index}(QW) = \mathrm{tr}(W^*[W, Q]) \\ [Q, W] \in \mathcal{J}_3(\mathcal{H}) \ \text{then} \ \mathrm{index}(QW) = \mathrm{tr}(W^*[W, Q]^3) \end{cases}$$

Avron, Seiler, Simon '94 noncommutative geometry!

Schatten p-norm

Lemma: $\|A\| \equiv \mathrm{tr}(|A|^p)^{1/p} \leq \sum_{k \in \mathbb{Z}^d} \left(\sum_{x \in \mathbb{Z}^d} \|A_{x, x+k}\|^p \right)^{1/p}$ $\forall A \in \mathcal{B}(\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N)$, $A_{xy} = \langle \delta_x, A \delta_y \rangle$

Proof of lemma: Let $A =$ 

locality ⇒ concentrates on bands

and so

$A = \sum_{k \in \mathbb{Z}^d} A^{(k)}$ & $(A^{(k)})_{xy} = A_{xy} \delta_{x-y,k}$

$\Rightarrow \|A\|_p \leq \sum_{k \in \mathbb{Z}^d} \|A^{(k)}\|_p$

We compute

$$\|A^{(k)}\|_p^p = tr(|A^{(k)}|^p) = \| \, |A^{(k)}|^2 \|_{p/2}^{p/2}$$

Then,

$$(|A^{(k)}|^2)_{xy} = (A^{(k)*}A^{(k)})_{xy} = \sum_{\tilde{x}} ((A^{(k)})^*)_{x,\tilde{x}} (A^{(k)})_{\tilde{x}y}$$

$$= \sum_{\tilde{x}} (A_{\tilde{x}x} \delta_{\tilde{x}-x,k})^* A_{\tilde{x}y} \delta_{\tilde{x}-y,k} = \delta_{xy} |A_{x+k,x}|^2$$

So, $|A^{(k)}|^2$ is diagonal, yielding $(|A^{(k)}|^2)_{xy}^{p/2} = \delta_{xy} |A_{x+k,x}|^2$

$\square$

The next claim is to show the following:

<u>Lemma:</u> If P is a local projection and U is Laughlin, <u>then</u> $[P,U] \in \mathcal{J}_3(\mathcal{H})$.

<u>Proof:</u> The previous lemma gives $\|[P,U]\|_3 \le \sum_{k\in\mathbb{Z}^2} \left(\sum_{x\in\mathbb{Z}^2} \|[P,U]_{x,x+k}\|^3\right)^{\frac{1}{3}}$. Locality of P gives summability in k via

$$[P,U]_{xy} = (PU-UP)_{xy} = (\delta_x, (PU-UP)\delta_y) = (U_{yy}-U_{xx}) P_{xy}$$

$$\Rightarrow \|[P,U]_{x,x+k}\| \le \|P_{x,x+k}\| \, \|U_{x+k,x+k} - U_{x,x}\| \underset{\text{locality}}{\le} C e^{-\mu\|k\|} \|U_{x+k,x+k} - U_{x,x}\|$$

We will use the fact that for $f: \mathbb{Z}^2 \to \mathbb{C}$ given by $(x_1,x_2) \to e^{i\arg(x_1+ix_2)}$, $\exists D \in (0,\infty)$ s.t.

$$|f(x)-f(y)| \le D \frac{\|x-y\|}{1+\|x\|}$$

Then, $\|U_{x+k,x+k} - U_{x,x}\| = |f(x+k)-f(x)| \le D\frac{\|k\|}{1+\|x\|}$. We need the third exponent since $\frac{1}{1+|x|}$ isn't integrable in 2D, but $\frac{1}{(1+|x|)^3}$ is. So, $[P,U] \in \mathcal{J}_3(\mathcal{H})$.

$\square$

The main theorem then follows.

$\square$

We can also <u>directly connect</u> the Kitaev and Laughlin indices, without reference to the DCF which may not always hold. This proof uses direct homotopy.

<u>Prop:</u>

$$\text{index}(PU) = \text{index}(\Lambda_1 e^{-2\pi i P\Lambda_2 P}\Lambda_1 + \Lambda_1^{\perp}).$$
$$=: \text{index}(\Lambda_1 e^{-2\pi i P\Lambda_2 P})$$

Fredholm + A.S.-index book: Bleeker & Booss

Proof: For $F\in\mathcal{F}(\mathcal{H})$, we know $\operatorname{index}(F+G)=\operatorname{index}(F)$ if

① $\|G\|$ is sufficiently small (Dieudonné)
② G is compact (Atkinson?)

Let $f:\mathbb{Z}^2\to\mathbb{C}$ be sending $(x_1,x_2)\mapsto e^{i\arg(x_1+ix_2)}$ as before.

Step 1: Change $f(X)$ to $f(X-a)$ for some $a\in\mathbb{C}\setminus\mathbb{Z}^2$, positive and large. Norm-cont. deformation to change

$$[0,1]\ni t\mapsto \mathbb{P}f(X+(1-t)a)$$

Step 2: Let $\varphi: S^1\to S^1$ be a continuous fn with winding number $\operatorname{wind}(\varphi)=+1$ that does all its winding in a small window

This closes the circle to an arc in a norm-continuous homotopy. So,

$$\mathbb{P}f(X-a)-\mathbb{P}e^{i\varphi(\arg(X-a))}\in\mathcal{K}(\mathcal{H})$$

Step 3: $\operatorname{index}(\mathbb{P}e^{i\varphi(\arg(x+a))})=\operatorname{index}(\Lambda_1\mathbb{P}e^{i\varphi(\arg(x+a))})$ since their difference is

c.w. $\mathbb{P}$ vs P

$$\Lambda_1\mathbb{P}e^{\cdots}-\mathbb{P}e^{\cdots}=\Lambda_1\mathbb{P}e^{\cdots}\Lambda_1+\Lambda_1^\perp-\mathbb{P}e^{\cdots}$$

Since

$$\begin{aligned}\mathbb{P}e^{\cdots}&=(\Lambda_1+\Lambda_1^\perp)\mathbb{P}e^{\cdots}(\Lambda_1+\Lambda_1^\perp)\\&=\Lambda_1\mathbb{P}e^{\cdots}\Lambda_1+\Lambda_1^\perp\mathbb{P}e^{\cdots}\Lambda_1^\perp+\Lambda_1\mathbb{P}e^{\cdots}\Lambda_1^\perp+\Lambda_1^\perp\mathbb{P}e^{\cdots}\end{aligned}$$

$$\Rightarrow \text{diff}=\underline{\Lambda_1^\perp(1-\mathbb{P}e^{\cdots})\Lambda_1^\perp}+\Lambda_1\mathbb{P}e^{\cdots}\Lambda_1^\perp+\Lambda_1^\perp\mathbb{P}e^{\cdots}\Lambda_1$$

$$=\Lambda_1^\perp(P+P^\perp-Pe^{\cdots}P-P^\perp)\Lambda_1^\perp=\Lambda_1^\perp P(1-e^{\cdots})P\Lambda_1^\perp$$

$$=\Lambda_1^\perp[P,1-e^{\cdots}]P\Lambda_1^\perp+\underbrace{\Lambda_1^\perp(1-e^{\cdots})P\Lambda_1^\perp}$$

$=0$ since $e^{\cdots}$ is 1 outside a cone, so projection left $=0$.

The other extra parts are dealt with similarly. So,

$$\text{diff}=\Lambda_1^\perp[P,1-e^{\cdots}]P\Lambda_1^\perp\in\mathcal{K}(\mathcal{H})\quad\text{by step 2.}$$

Step 4: Add another flux on the left. Specifically, consider the new unitary $\tilde{U}:=e^{i\varphi(\arg(X-a))}e^{-i\varphi(\arg(X+a))}\equiv e^{i\xi(X)}$

The difference is now

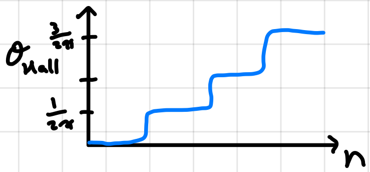
$$\Lambda_1\mathbb{P}e^{\cdots}-\Lambda_1\mathbb{P}\tilde{U}=\Lambda_1Pe^{i\varphi(\arg(x-a))}(1-e^{-i\varphi(\arg(x+a))})P\Lambda_1\ \ldots\in\mathcal{K}(\mathcal{H})$$

Step 5: $\Lambda_1\mathbb{P}e^{i\xi(X)}-\Lambda_1\mathbb{P}e^{iP\xi(X)P}\in\mathcal{K}(\mathcal{H})$

Step 6: Deform $\xi(X)$ to $-2\pi i\Lambda_2(X)$ norm-continuously. □

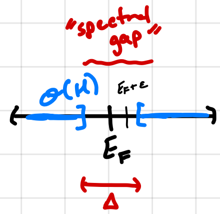
4/4- IQHE cont.

We saw so far:



$E_F \notin \sigma(H)$

$P_{E_F} := \chi_{(-\infty, E_F)}(H)$
$\equiv \chi_{(-\infty, E_F+\varepsilon)}(H)$ if $E_F+\varepsilon \in \Delta$



$\sigma_{Hall} = i\,\mathrm{tr}\left(P[[\Lambda_1, P],[\Lambda_2, P]]\right)$
$= \frac{1}{2\pi}\,\mathrm{index}\,(PUP + P^\perp)$
$=: \frac{1}{2\pi}\,\mathrm{index}\,(PU)$

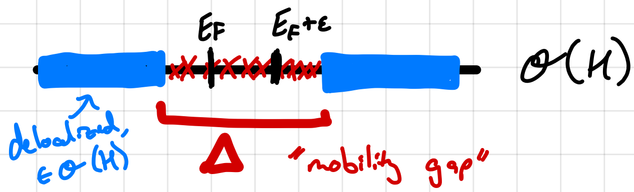
From the above, we immediately see

① $P = 0, 1 \Rightarrow \sigma_{Hall} = 0$

② If $[P, X] = 0$, then P commutes with functions of X and so $\sigma_{Hall} = 0$.

③ If P has finite-rank image or kernel, then $\sigma_{Hall} = 0$.

However, changing E_F to $E_{F+\varepsilon}$ doesn't change P under the spectral gap assumption. To see something interesting, we need to allow pure point eigenvalues in the spectral gap Δ, introducing the following picture that allows us to continuously vary E_F:



The above picture is about the IQHE under the disordered model.

① almost surely, the eigenvalues in Δ are simple

② Since σ_{Hall} is discrete, the only way for it to change is at points where σ_{Hall} doesn't exist
So, the blue bands can be taken to be delocalized.

$\Big($ $\Leftrightarrow PU$ not Fredholm
$\Leftrightarrow [P, U]$ not compact
$\Leftrightarrow DC$ is not $\in \mathcal{J}_1$
$\overset{?}{\Leftrightarrow} P$ is not localized $\Big)$

This is weird; it seems that $\sigma_{Hall} \neq 0 \Rightarrow \exists$ deloc., but in 2D we had seen complete localization. A more complete picture is this:

- complete deloc in 2D for time-reversal invariant (TRI) bosonic systems
- in $d \geq 3$, may have deloc.

Def. (TRI)

Let $\Theta : \ell^2(\mathbb{Z}^2 \to \mathbb{Z}^2)$ be the time-reversal operator, which is simply an antiunitary operator, i.e. $\langle \Theta\varphi, \Theta\psi\rangle = \overline{\langle \varphi, \psi\rangle}$ s.t. $\Theta\Theta^* = \Theta^*\Theta = 1$ and Θ is anti-$\mathbb{C}$-linear.

There are in general two versions : $\Theta^2 = \begin{cases} +1 & \text{bosonic} \\ -1 & \text{fermionic} \end{cases}$

We may choose Θ as complex conjugation $(\dots, \varphi_x, \varphi_{x+1}, \dots) \mapsto (\dots, \overline{\varphi_x}, \overline{\varphi_{x+1}}, \dots)$ since the evolution e^{itH} gets conjugated by Θ to e^{-itH}.

We say the system is TRI if $[H, \Theta] = 0$.

Note that if Θ is complex conjugation, then

① $[H,\Theta] = 0$ for the Anderson model since $[H,\Theta] = 0 \iff H_{xy} \in \mathbb{R} \quad \forall x,y$

② By measurable functional calculus, $[P, \Theta] = 0$

③ In fact, $[\Theta, X] = 0$ with $\Theta U \Theta = \Theta e^{i\arg(X)}\Theta = e^{-i\arg(X)} = U^*$

From the above, $\Theta(\mathbb{P}U)\Theta = \Theta(PUP + P^\perp)\Theta = PU^*P + P^\perp = \mathbb{P}U^*$.
Since the Fredholm index has $\operatorname{index}(F) = -\operatorname{index}(F^*)$ and $\operatorname{index}(AB) = \operatorname{index}(A) + \operatorname{index}(B)$, and so

$$\operatorname{index}(\mathbb{P}U) = -\operatorname{index}(\mathbb{P}U^*) = -2\operatorname{index}(\Theta) - \operatorname{index}(\mathbb{P}U) = -\operatorname{index}(\mathbb{P}U)$$
$$\Rightarrow \sigma_{Hall} = 0$$

There are several useful models to apply this:

① Disordered Landau on $L^2(\mathbb{R}^2)$: $H = (P-A)^2 + \lambda V_\omega(X)$, $A(x) = \frac{B_0}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$
$\Rightarrow$ one of the eigenstates in each mobility gap Δ is "delocalized"

② Harper model on $\ell^2(\mathbb{Z}^2)$: $H_{xy} = \delta_{\|x-y\|} e^{i\varphi_{xy}}$

Beyer's model

③ On $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2 \cong \ell^2(\mathbb{Z}^2) \oplus \ell^2(\mathbb{Z}^2)$ with $H = \begin{bmatrix} A & B^* \\ B & -A \end{bmatrix}$
and $[X_2, B] = 0$, $[X_1, A] = 0$.
Then,
$$\hat{A}(k_2) = \aleph \tanh(k_2) \xRightarrow{B = R_1 \text{ right shift}} \mathrm{Chern}_0(H) = \operatorname{tr}(B^*[\Lambda_1, B]) = -1$$
With disorder, this gives the picture above.

Properties of Chern # w.r.t. disorder

Prop:

If P, Q are local S.A. projections s.t. $P \perp Q$, then
$$\operatorname{index}\left((P+Q)_{bb}\, U\right) = \operatorname{index}(PU) + \operatorname{index}(QU)$$

Proof: • $\operatorname{ind} PU + \operatorname{ind} QU = \operatorname{ind}\left((PU)(QU)\right) = \operatorname{ind}\left(PUPQ^{\perp} + P^{\perp}QUQ + P^{\perp}Q^{\perp}\right)$
$$= \operatorname{ind}\Big(PUP - \cancel{PUPQ} + QUQ - \cancel{PQUQ} + \underbrace{(1-P)(1-Q)}_{1-(P+Q)}\Big)$$

• $(P+Q)_{bb}\, U = (P+Q)U(P+Q) + 1 - (P+Q) = PUP + QUQ + PUQ + QUP + 1 - (P+Q)$

So, we must show that $PUQ + QUP \in \mathcal{K}(\mathcal{H})$, which holds since $PUQ = [P,U]Q \in \mathcal{K}(\mathcal{H})$ by assumption.

$\square$

Def: (SULE basis)

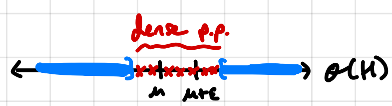
We say that $\{\varphi_n\}_n$ is a "SULE" ONB for V iff $\exists\, \{x_n\}_n \subseteq \mathbb{Z}^d$ "localization centers" s.t. $\forall \varepsilon > 0$, $\exists\, C_\varepsilon < \infty$ s.t. $\|\varphi_n(x)\| \le C_\varepsilon e^{-\mu\|x-y\| + \varepsilon\|x_n\|}$ $(x \in \mathbb{Z}^d)$

For such a setup, we have the summability
$$\sum_{n\in\mathbb{N}} (1+\|x_n\|)^{-d-\delta} < \infty \qquad (\forall \delta > 0)$$

see del Rio, Jito, ..., Last, Simon for proof of $\left|\{n\in\mathbb{N} \mid \|x_n\| \le L\}\right| \lesssim L^d$

If a S.A. proj. P has that $\operatorname{im}(P)$ has a SULE basis, then we say P is fully localized.

Prop:

Let $P_\mu := \chi_{(-\infty,\mu)}(H)$. Then, if 

, then
$$\operatorname{Chern}(P_\mu) = \operatorname{Chern}(P_{\mu+\varepsilon}).$$

Proof: $P_{\mu+\varepsilon} = P_\mu + Q$ with $Q := \chi_{[\mu,\mu+\varepsilon)}(H)$ by the functional calculus. So, we must show that $\operatorname{Chern}(Q) = 0$.

Lemma: Q fully localized $\Rightarrow \operatorname{Chern}(Q) = 0$

Proof: When H exhibits Anderson localization in Δ,
$$\xRightarrow{\text{FMC}} \sup_{\varepsilon>0} \mathbb{E}\left[\|G(x,y;E+i\varepsilon)\|^s\right] \le C e^{-\mu\|x-y\|} \qquad (E\in\Delta)$$
$$\xRightarrow{\text{exp. decay of mer. func. calc}} \sup_{f\in B_1(\Delta)} \mathbb{E}\left[\|f(H)_{xy}\|\right] \le C e^{-\mu\|x-y\|}$$
$$\Rightarrow \text{almost-surely}, \quad \forall\varepsilon>0\ \exists C_\varepsilon<\infty \text{ s.t. } \|f(H)_{xy}\| \le C_\varepsilon e^{-\mu\|x-y\|+\varepsilon\|x\|}$$

We WTS $\mathrm{ind}(QU)=0$, or equivalently show that QU is compactly away from invertible. Define V unitary as follows:

$$V\varphi_n := e^{i\arg(x_n)}\varphi_n \quad , \qquad V\varphi=\varphi \quad \text{if} \quad \varphi\in \mathrm{im}(Q^\perp)$$

V is clearly unitary and $QU-V\in K(\mathcal{H}) \iff (U-V)Q\in K(\mathcal{H})$
We can show that $(U-V)Q$ is p-Schatten for p suff. large:

$$\|(U-V)Q\|_p \le \sum_{k\in\mathbb{Z}^2}\left(\sum_{x\in\mathbb{Z}^2}\|((U-V)Q)_{x,x+k}\|^p\right)^{1/p} \qquad \text{(earlier lemma)}$$

Also,

$$((U-V)Q)_{xy} \overset{f(z)=e^{i\arg(z)}}{=} \sum_{n=1}^{\infty}(f(x)-f(x_n))\varphi_n(x)\overline{\varphi_n(y)}$$

We know $|f(x)-f(y)|\le D\frac{\|x-y\|}{1+\|x\|}$ from last time, which along with the SULE estimate gives summability in k,x,n. So, $(U-V)Q$ is p-Schatten and so compact.

$\square$

$\square$

fill in w/a

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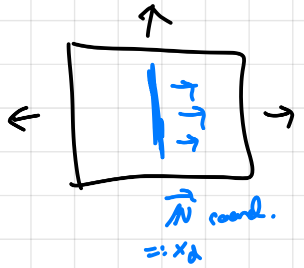
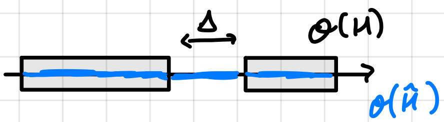
Recall that we want to model edge physics in bounded systems.



Adjust $\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^N \rightsquigarrow \ell^2(\mathbb{Z}^{d-1}\times\mathbb{N})\otimes\mathbb{C}^N$, the simplest boundary we could introduce.

We saw last time that

$\ell^2(\mathbb{Z}^d)$	$\rightsquigarrow$	$\ell^2(\mathbb{Z}^{d-1}\times\mathbb{N})$
local	$\to$	local
insulator	$\to$	<u>not</u> an insulator
$\sigma(H)\cap\Delta=\emptyset$	$\to$	$\sigma(H)\cap\Delta=\Delta$



We will introduce a functional calculus for (as regular as possible) fns supported on Δ, as this will let us understand edge systems coming from truncating spectrally-gapped bulk systems.

Def: (Bulk Gap)

We say a local edge Hamiltonian $\hat{H}=\hat{H}^*\in\mathcal{B}(\ell^2(\mathbb{Z}^{d-1}\otimes\mathbb{N})\otimes\mathbb{C}^N)$ has a bulk gap within $\Delta\subseteq\mathbb{R}$ iff $\forall$ smooth $g:\mathbb{R}\to\mathbb{C}$ with $\operatorname{supp}(g)\subseteq\Delta$,

$$\|g(H)_{xy}\|\leq Ce^{-\mu\|x-y\|-\nu(x_d+y_d)}\qquad\forall x,y\in\mathbb{Z}^{d-1}\times\mathbb{N}$$

Smooth Functional Calculus (Dynkin, Helffer-Sjöstrand, Hunziker-Sigal)

Let $f:\mathbb{R}\to\mathbb{C}$ be smooth & compactly-supported and $A=A^*\in\mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ separable. The goal is, as always, to define $f(A)$.

Consider the Wirtinger derivative $\partial_{\bar{z}}\equiv\partial_x+i\partial_y$ and $\underset{\text{Cauchy-Riemann eq.}}{\text{CRE}}\iff\partial_{\bar{z}}g=0$

Let $\chi:\mathbb{R}\to\mathbb{R}$ be even, smooth, compactly supported, with $\chi|_{B_\delta(0)}\equiv1$ for some $\delta>0$ (χ is basically a bump). Fix $N\in\mathbb{N}$. ← large

Def: (Quasi-analytic extension of f)

We define $\tilde{f}:\mathbb{C}\to\mathbb{C}$ via $\tilde{f}(x+iy):=\chi(y)\sum_{k=0}^{N}f^{(k)}(x)\dfrac{(iy)^k}{k!}$

to be the quasi-analytic extension of f depending on χ and N.

Observe the following:

① $\tilde{f}(x) = f(x) \quad \forall x \in \mathbb{R}$ since $k \neq 0$ terms vanish $\Rightarrow$ extension!

② $\tilde{f}$ obeys the CRE on $\mathbb{R}$, i.e. $(\partial_{\bar z} f)|_{\mathbb{R}} = 0$. To see this,

$$(\partial_{\bar z} \tilde{f})(x+iy) = (\partial_x + i\partial_y)\, \chi(y) \sum_{k=0}^{N} f^{(k)}(x) \frac{(iy)^k}{k!}$$

$$= \sum_{k=0}^{N} f^{(k+1)}(x) \frac{(iy)^k}{k!} + i f^{(k)}(x)\, k\, \frac{i^k y^{k-1}}{k!}$$

$$= \sum_{k=1}^{N+1} f^{(k)}(x) \frac{(iy)^{k-1}}{(k-1)!} - \sum_{k=0}^{N} f^{(k)}(x) \frac{(iy)^{k-1}}{(k-1)!}$$

$$= \chi(y) f^{(N+1)}(x) \frac{(iy)^N}{N!} + i\, \chi'(y) \sum_{k=0}^{N} f^{(k)}(x) \frac{(iy)^k}{k!}$$

When $|y| < \delta$, $= f^{(N+1)}(x) \frac{(iy)^N}{N!} \overset{y=0}{=} 0$

Note that $\partial_{\bar z} \tilde{f}$ is compactly-supported in $\mathbb{C}$.

③ f compact spt. $\Rightarrow f^{(k)}$ compact spt.

④ Analogously to the Cauchy integral formula, we have

Prop:

integral over $\mathbb{R}\times\mathbb{R}$, not contour

$$f(a) = \frac{1}{2\pi} \int_{z \in \mathbb{C}} (\partial_{\bar z} \tilde{f})(z)\, (a-z)^{-1}\, dz \qquad (\forall a \in \mathbb{R})$$

$$\equiv \frac{1}{2\pi} \lim_{\epsilon \downarrow 0} \int_{\{|\mathrm{Im}\{z\}| > \epsilon\}} dz\, (\partial_{\bar z} \tilde{f})(z)\, (a-z)^{-1}$$

Proof: We only need to consider as $y \to 0$. We have

$$|(\partial_{\bar z} \tilde{f})(x+iy)| \leq \frac{1}{N!} \overbrace{|\chi(y)|}^{\leq 1}\, |y|^N\, \overbrace{|f^{(N+1)}(x)|}^{\text{cpt. spt.}} + \sum_{k=0}^{N} \frac{|y|^k}{k!} |f^{(k)}(x)|\, \underbrace{|\chi'(y)|}_{=0 \text{ for } |y|<\delta}$$

$$\leq \frac{C_N(f)}{N!} |y|^N \quad \leftarrow \text{some constant}$$

Also, $|a-z|^{-1} \leq |y|^{-1}$. Thus,

(★) $$\left| \frac{1}{2\pi} \int_{z \in \mathbb{C}} (\partial_{\bar z} \tilde{f})(z)\, (a-z)^{-1}\, dz \right| \leq \frac{1}{2\pi} \int_{x+iy \in \underset{\text{compact}}{K}} \frac{C_N(f)}{N!} |y|^{N-1}\, dx\, dy < \infty$$

for $N \geq 1$, and so the integral converges absolutely!

Define

$$f_\epsilon(a) := \int_{|y|>\epsilon} (\partial_{\bar z} \tilde{f})(x+iy)\, (a-x-iy)^{-1}\, dx\, dy$$

We claim $f_\epsilon \to f$ pointwise.

Integration by parts w.r.t. $\partial_{\bar z}$ (which is just Stokes' in 2D) gives that $f_\varepsilon(a) = \frac{1}{2\pi i}\int_{x\in\mathbb{R}} \left[\tilde f(x+iy)(a-x-iy)\right]_{y=-\varepsilon}^{\varepsilon} dx$ since $(a-z)^{-1}$ is holomorphic. Since $\tilde f(x\pm i\varepsilon) = f(x) \pm i\varepsilon f'(x) + O(\varepsilon^2)$,

$$\Rightarrow f_\varepsilon(a) = \int_{x\in\mathbb{R}} f(x)\,\frac{1}{\pi}\,\mathrm{Im}\left\{\frac{1}{a-x-i\varepsilon}\right\} dx$$

$$+ \int_{x\in\mathbb{R}} f'(x)\,\frac{1}{2\pi}\,\varepsilon\,\underbrace{\left((x-a-i\varepsilon)^{-1} + (x-a+i\varepsilon)^{-1}\right)}_{\frac{1}{w}+\frac{1}{\bar w} = \frac{2\mathrm{Re}\{w\}}{|w|^2}}$$

$$= \frac{\varepsilon}{\pi}\frac{a-x}{(a-x)^2+\varepsilon^2} \xrightarrow{\varepsilon\to 0} \begin{cases} 0 & x = a \\ 0 & x \neq a \end{cases}$$

So, $f_\varepsilon(a) \to f(a)$. $\square$

Def (Smooth F'al Calc): ★

We may always define $f_\varepsilon(A) := \frac{1}{2\pi}\int_{|\mathrm{Im}\{z\}|>\varepsilon} (\partial_{\bar z}\tilde f)(z)\,(A - z\mathbb{1})^{-1}\, dz$ since we have the resolvent away from the real line. Pointwise convergence of $f_\varepsilon \to f$ tells us that $f_\varepsilon(A) \to f(A)$ strongly, where the RHS is understood here via measurable f'al calc. In fact, it can be boosted to operator-norm convergence. So, we get:

$f : \mathbb{R}\to\mathbb{C}$ smooth, compactly-supported $\Longrightarrow f(A) \equiv \frac{1}{2\pi}\int_{z\in\mathbb{C}} (\partial_{\bar z}\tilde f)(z)\,(A-z\mathbb{1})^{-1}\,dz$

converges in op. norm

Theorem (Smooth Preserves Locality):

Let $A = A^* \in \mathcal{B}(\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^N)$ be local and $f:\mathbb{R}\to\mathbb{C}$ smooth and compactly supported. Then, $\exists\,\mu>0$ s.t. $\forall N\in\mathbb{N}$, $\exists\, C_N<\infty$ s.t.

$$\|f(A)_{xy}\| \le C_N\left(1+\mu\|x-y\|\right)^{-N}$$

Proof: $f(A)_{xy} = \frac{1}{2\pi}\int_{z\in\mathbb{C}} (\partial_{\bar z}\tilde f)(z)\,(A-z\mathbb{1})^{-1}_{xy}\,dz$

integral converges weakly

$$\Rightarrow \|f(A)_{xy}\| \overset{(*)}{\le} \frac{1}{2\pi}\int_{z\in K} dz\,\frac{C_N(f)}{N!}\,|\mathrm{Im}\{z\}|^N\,\frac{2}{|\mathrm{Im}\{z\}|}\,e^{-\mu|\mathrm{Im}\{z\}|\,\|x-y\|}$$

$$\le 4\frac{C_N(f)}{\pi N!}\|A\|\int_{\alpha=0}^{\infty} d\alpha\,\alpha^{N-1}e^{-\alpha\mu\|x-y\|}$$

$$= \frac{4\,C_N(f)}{\pi N!}\|A\|\,(N-1)!\,(\mu\|x-y\|)^{-N}$$ if $x\neq y$ and something else that's regular if $x=y$.

$\square$

Theorem: (Smooth Preserves Bulk Decay):

Let $H = H^* \in \mathcal{B}(\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N)$ be local with a spectral gap on $\Delta \subseteq \mathbb{R}$. Let $J : \ell^2(\mathbb{Z}^{d-1} \times \mathbb{N}) \to \ell^2(\mathbb{Z}^d)$ be the $\hookrightarrow$ partial isometry and let $g : \mathbb{R} \to \mathbb{C}$ smooth with $\operatorname{supp}(g) \subseteq \Delta$.
If $\hat{H} \in \mathcal{B}(\ell^2(\mathbb{Z}^{d-1} \times \mathbb{N}) \otimes \mathbb{C}^N)$, then

$$\|(\hat{H} - J^* H J)_{xy}\| \leq C e^{-\mu \|x-y\| - \nu (x_d + y_d)} \Rightarrow \|g(\hat{H})_{xy}\| \leq \ldots$$

Proof: By def, $(\underbrace{J^* H J}_{=:\hat{H}})_{xy} = H_{xy}$ if $x_d, y_d > 0$. Comparing $g(\hat{H})$ with $g(H)$,

$$g(\hat{H})_{xy} - g(H)_{xy} \overset{x_d, y_d > 0}{=} \frac{1}{2\pi} \int_{z \in \mathbb{C}} (\partial_{\bar{z}} \tilde{g})(z) \left[(\hat{H} - z\mathbb{1})^{-1}_{xy} - (H - z\mathbb{1})^{-1}_{xy} \right] dz$$

$$\overset{\text{resolvent id.}}{=} \ldots \quad \text{finish}$$

$\square$

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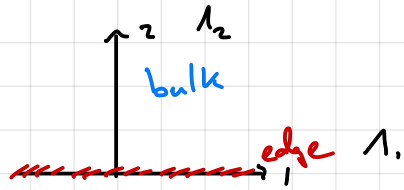


$\mathcal{H} \xleftarrow{J \text{ partial iso}} \hat{\mathcal{H}}$

Recall the picture on $\ell^2(\mathbb{Z}^2)$ or $\ell^2(\mathbb{Z}\times\mathbb{N})$:

We say A is local and decays in direction j iff

$$\|A_{xy}\| \le C e^{-\mu\|x-y\| - \nu(|x_j|+|y_j|)}$$

Note that if A is local, then $[A, \Lambda_j]$ decays in direction j.

Some facts:

- if $H \in \mathcal{B}(\mathcal{H})$ is local and gapped, and g is smooth and supported on the gap, then $g(J^* H J)$ decays into bulk
- if A local, B decays in dir. j, then AB, BA, B^* decays in dir. j
- A decays in all directions $\Rightarrow$ A is trace-class

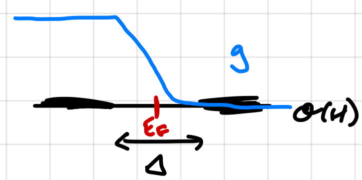
Def: (edge Hamiltonian)

$\hat{H} \in \mathcal{B}(\hat{\mathcal{H}})$ is an edge Hamiltonian if $\hat{H} = J^* H J$ for some $H \in \mathcal{B}(\mathcal{H})$ that is local, gapped, and decays into the bulk.

Def: (edge Hall conductivity)

We define the edge Hall conductivity to be

$$\hat{\sigma}_{Hall} = i \operatorname{tr}\left(g'(\hat{H}) \, [\hat{H}, \Lambda_1] \right)$$



for a smooth approximation g to $\chi_{(-\infty, E_F)}$.

Theorem: (Kellendonk, Richter, Schulz-Baldes '99):

We have $$\hat{\sigma}_{Hall} = \frac{1}{2\pi} \operatorname{index}\left(\Lambda, e^{-2\pi i g(\hat{H})}\right)$$

Proof: check notes :)

$\square$

Theorem: (Bulk-edge correspondence)

So, $\hat{H}$ local and $\hat{H} - J^* H J$ decays into bulk

$$\hat{\sigma}_{Hall} = \sigma_{Hall} \quad \text{when} \quad H \in \mathcal{B}(\ell^2(\mathbb{Z}^2)) \quad \text{local \& gapped}$$

Proof: We get there using:

Theorem: (Fonseca-Shapiro-Sheta-Wang-Yamakawa '20)

$$\hat{\sigma}_{Hall} = \frac{1}{2\pi} \operatorname{index}_{\hat{\mathcal{H}}}\left(\Lambda, e^{-2\pi i g(\hat{H})}\right)$$

$$\sigma_{Hall} = \frac{1}{2\pi} \operatorname{index}_{\mathcal{H}}\left(\Lambda, e^{-2\pi i P \Lambda_2 P}\right)$$

agrees on $\sigma(H)$

$$P = \chi_{(-\infty, E)}(H) = g(H)$$

With this, we WTS we can replace $P \Lambda_2 P$ with $\Lambda_2 P \Lambda_2$ in σ_{Hall}, which we get since

$$P\Lambda_2 P - \Lambda_2 P \Lambda_2 = [P, \Lambda_2] P + \Lambda_2 P \Lambda_2^{\perp}$$

decays into the bulk.

$$= [P, \Lambda_2] P + [\Lambda_2, P] \Lambda_2^{\perp}$$

Next, we WTS we can replace $\Lambda_2 g(H) \Lambda_2$ with $g(J^* H J)$. To do so:

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Def:

$\dim\ker\Lambda, \dim\operatorname{im}\Lambda = \infty$

Let $\mathcal{H}$ be a Hilbert space and Λ a nontrivial proj.
We say $A \in \mathcal{B}(\mathcal{H})$ is Λ-local if $[A,\Lambda]$ is compact.

Let $\mathcal{L}(\Lambda)$ denote the space of Λ-local ops.

Prop:

If $A,B \in \mathcal{L}(\Lambda)$, then

- $AB,\ A+B,\ A^* \in \mathcal{L}(\Lambda)$
- if A normal, $f: \sigma(A) \to \mathbb{C}$ continuous, then $f(A) \in \mathcal{L}(\Lambda)$

Let $\mathcal{U} := \{u \in \mathcal{B}(\mathcal{H}) : u \text{ unitary}\}$. Then,

(unitaries) **Theorem:** (Kuiper Theorem)

$$\underbrace{\pi_0}_{\#\text{ connected components}}(\mathcal{U}) = 0$$

Proof: Let $u \in \mathcal{U}$, and so $\sigma(u) \subseteq S^1$. Find $f: \sigma(u) \to \mathbb{R}$ bdd. s.t. $e^{if(\lambda)} = \lambda$, and so $u = e^{if(u)}$. Letting $\gamma: [0,1] \to \mathcal{U}$ be given by $t \mapsto e^{itf(u)}$, this is a continuous path from $\mathbb{1}$ to u. Since this holds $\forall u \in \mathcal{U}$, $\pi_0(\mathcal{U}) = 0$.

$\square$

Remark: This proof will fail for $\mathcal{U} \cap \mathcal{L}(\Lambda)$ since we cannot guarantee $\gamma(t) \in \mathcal{L}(\Lambda)$ (which happens since we cannot guarantee that f is continuous in the case $\sigma(u) = S^1$.
So, perhaps $\pi_0(\mathcal{U} \cap \mathcal{L}(\Lambda)) \neq 0$, and indeed this is true.

(local unitaries) **Theorem: (Shapiro and the grad student):**

related to $\pi_0(\mathcal{F}(\mathcal{H})) = \mathbb{Z}$

$$\pi_0(\mathcal{U} \cap \mathcal{L}(\Lambda)) = \mathbb{Z}$$

Proof: We want to find a correspondence between path-connected components of $\mathcal{U} \cap \mathcal{L}(\Lambda)$ and the value of index Λu.

By continuity of the index, we already know that if $u \xrightarrow{\mathcal{U} \cap \mathcal{L}(\Lambda)} v$, then index Λu = index Λv. We will show the converse.

By the log. property of the index, index(Λu) = index(Λv) then

$\mathrm{index}(\Lambda U V^*) = 0$ and $UV^* \rightsquigarrow \mathbb{1} \Rightarrow U \rightsquigarrow V$.
So, it suffices to show that $\mathrm{index}(\Lambda U) = 0 \Rightarrow U \overset{\mathcal{U}\cap\mathcal{L}(\Lambda)}{\rightsquigarrow} \mathbb{1}$.

Suppose $U \in \mathcal{U} \cap \mathcal{L}(\Lambda)$ is s.t. $\mathrm{index}(\Lambda U) = 0$.
Decompose $\mathcal{H} = \mathrm{im}\,\Lambda^\perp \oplus \mathrm{im}\,\Lambda$, and write

$$U = \begin{bmatrix} U_{LL} & U_{LR} \\ U_{RL} & U_{RR} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{bmatrix}$$

$$\Rightarrow [U, \Lambda] = \begin{bmatrix} 0 & U_{LR} \\ -U_{RL} & 0 \end{bmatrix} \overset{U \in \mathcal{L}(\Lambda)}{\Longrightarrow} U_{LR}, U_{RL} \in \mathcal{K}(\mathcal{H})$$

local $\Leftrightarrow$ off-diags are cpt.

Since U is unitary, $UU^* = \mathbb{1}$, from which we see $U_{LL}U_{LL}^* - \mathbb{1}$, $U_{LL}^* U_{LL} - \mathbb{1} \in \mathcal{K}(\mathcal{H})$.
... for RR

U_{LL} is "essentially unitary"

So, U_{RR} is Fredholm with $\mathrm{index}(U_{RR}) = 0$.

Lemma:

If $Z \in \mathcal{B}(\mathcal{H})$ has $Z^*Z - \mathbb{1}$, $ZZ^* - \mathbb{1} \in \mathcal{K}(\mathcal{H})$ and $\mathrm{index}(Z) = 0$,
then $\exists Y \in \mathcal{U}$ s.t. $Z - Y \in \mathcal{K}(\mathcal{H})$.

Apply this lemma to U_{RR} and U_{LL} to get $B_{RR}, B_{LL} \in \mathcal{U}$ with

$$\begin{bmatrix} U_{LL} & 0 \\ 0 & U_{RR} \end{bmatrix} - \underbrace{\begin{bmatrix} B_{LL} & 0 \\ 0 & B_{RR} \end{bmatrix}}_{=: B \in \mathcal{L}(\Lambda)} \in \mathcal{K}(\mathcal{H}) \Rightarrow U = B + K = \underbrace{(\mathbb{1} + KB^*)}_{=: A} B \quad \text{for } K \in \mathcal{K}(\mathcal{H}).$$

(no off-diags)

So, $U = AB$ where $A = \mathbb{1} + C$ with $C \in \mathcal{K}(\mathcal{H})$ and $B \in \mathcal{U} \cap \mathcal{L}(\Lambda)$.

this means A has p.p. spectrum w/ accumulation only at 1

Applying the prev result to B_{LL} and B_{RR}, $B \overset{\mathcal{U}\cap\mathcal{L}}{\rightsquigarrow} \mathbb{1}$ since the off-diagonals stay 0. Since $\sigma(A) \subseteq S^1$, then we may find $f : \sigma(A) \to \mathbb{C}$ continuous s.t. $A = e^{i f(A)}$. Continuity preserves locality, and so $A \overset{\mathcal{U}\cap\mathcal{L}}{\rightsquigarrow} \mathbb{1}$.
Thus, $AB \overset{\mathcal{U}\cap\mathcal{L}}{\rightsquigarrow} \mathbb{1}$.

$\square$

We stop and note that π_0(self-adj. unitaries) is infinite. However, if we restrict to nontrivial unitaries we get more.

Theorem:

(nontrivial SA unitaries)

$\dim \ker(U \pm \mathbb{1}) = \infty$, i.e. $\sigma_{ess}(U) = \{\pm 1\}$

$$\pi_0(\text{self-adjoint, nontrivial unitaries}) = 0$$

Proof: Write $\mathcal{H} = (\ker U + \mathbb{1}) \oplus (\ker U - \mathbb{1})$. Let $U, V \in$ S.A. nontrivial unitaries.

Since $\dim(\ker U+\mathbb{1}) = \dim(\ker V+\mathbb{1}) = \infty$, then $\exists\, W: \ker(U+\mathbb{1}) \to \ker(V+\mathbb{1})$ unitary. We have $U = W^* V W$ (check this). Then, since unitaries are path-connected, $W \overset{\mathcal{U}}{\rightsquigarrow} \mathbb{1}$ along W_t. Defining $U_t := W_t^* V W_t$, we see that $U \rightsquigarrow V$.

$\square$

Def: (Λ- nontrivial)

Let $U \in \mathcal{U} \cap \mathcal{L}(\Lambda)$ be S.A. We say U is Λ-nontrivial if

$$\sigma_{ess}(\Lambda U \Lambda) = \sigma_{ess}(\Lambda^\perp U \Lambda^\perp) = \{\pm 1\}$$

I.e. U acts nontrivially on both $\operatorname{im}\Lambda$ and $\operatorname{im}\Lambda^\perp$.

Theorem:

$$\pi_0\left(\{U \in \mathcal{U}\cap\mathcal{L}(\Lambda) : U^* = U,\ U \text{ nontrivial},\ U\ \Lambda\text{-nontrivial}\}\right) = 0$$

Proof: As before, write $U = \begin{bmatrix} X & A \\ A^* & Y \end{bmatrix}$, X, Y S.A.. We have the properties

(i) $\|U\| = 1 \Rightarrow \|X\|, \|Y\| \leq 1$

(ii) $U \in \mathcal{U} \Rightarrow \begin{array}{l} AA^* = \mathbb{1} - Y^2 \\ A^*A = \mathbb{1} - X^2 \\ XA = -AY \end{array}$

(iii) $U \in \mathcal{L}(\Lambda) \Rightarrow A$ compact

So, since X and Y are essentially-unitary, they have spectra that can accumulate at ± 1 only, and are isolated in $(-1,1)$. Thus, if $f: [-1,1] \to \mathbb{R}$ is continuous at ± 1, then $f|_{\sigma(X)}$ is continuous since $\sigma(X) \cap (-1,1)$ is isolated. If we let $\operatorname{sgn}(x) := \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$,

then $XA = -AY \Rightarrow \operatorname{sgn}(X)A = -A\operatorname{sgn}(Y)$ and so $\chi_{\{0\}}(X)A = A\operatorname{sgn}(Y)$

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Classification of 1-D Insulators

Recall that for periodic systems, we had a correspondence with maps from $\mathbb{T}^d \to Gr_n(\mathbb{C}^N)$. Since locality $\Leftrightarrow$ this map is continuous, we can study the topological structure: we find that the connected components of $\{f: \mathbb{T}^d \to \mathcal{M}\}$ correspond to homotopy classes of $\mathcal{M}$, and so

$$\pi_0(S^d \to \mathcal{M}) = \pi_d(\mathcal{M}) \implies \exists \text{ classification scheme}$$

Toward the non-periodic setting, there has been a program to apply methods from noncommutative geometry and K-theory (the C^* alg. type): "Oh, so none of you know K-theory?" - Shapiro

* Jean Bellissard (since 90's): K-theory in condensed matter physics
* G. Thiang (2015): Ph.D. thesis explores Kitaev table at level of K-theory

However, it is generally tough to apply these ideas, and so there is a goal to do it in a functional-analytic way. In 1D, this is already done.

Functional-Analytic Approach

Let $\mathcal{H}$ be a separable Hilbert space, and Λ a fixed S.A. projection.

* Assume Λ is nontrivial (i.e. $\dim(\ker \Lambda) = \dim(\operatorname{im} \Lambda) = \infty$)
* Define the subspace $\mathcal{L}(\Lambda) \equiv \mathcal{L} := \{A \in \mathcal{B}(\mathcal{H}) : [A, \Lambda] \in \mathcal{K}(\mathcal{H})\}$ "essentially commute"
* **Claim:** $\mathcal{L}(\Lambda)$ is a C^* algebra.

  Proof: Algebraic structure is inherited from $B(\mathcal{H})$, and so we must show $\mathcal{L}(\Lambda)$ is op. norm-closed. If $A_n \to A$ for $(A_n)_n \subseteq \mathcal{L}(\Lambda)$, then $[A_n, \Lambda] \to [A, \Lambda] \Rightarrow [A, \Lambda] \in \mathcal{K}(\mathcal{H})$ since compact ops. are norm-closed. $\square$

* Since continuous functional calc. is closed in a C^*-alg, then $[A, A^*] = 0$ and $f: \mathbb{C} \to \mathbb{C}$ continuous means

$$[A, \Lambda] \in \mathcal{K}(\mathcal{H}) \implies [f(A), \Lambda] \in \mathcal{K}(\mathcal{H})$$

So, any continuous functional calculus preserves decaying into bulk. (subsumes the smooth f'al calc.)

Now, define the linear operator $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ sending $A \mapsto \Lambda A \Lambda + \Lambda^\perp$. Λ defines a $\mathbb{Z}_2$-grading of $\mathcal{H}$ via $\mathcal{H} = \underbrace{\ker(\Lambda)}_{=: \mathcal{H}_L} \oplus \underbrace{\operatorname{im}(\Lambda)}_{=: \mathcal{H}_R}$ and so

$$A = \begin{bmatrix} A_{LL} & A_{LR} \\ A_{RL} & A_{RR} \end{bmatrix} \overset{\Lambda}{\longmapsto} \begin{bmatrix} \mathbb{1}_L & 0 \\ 0 & A_{RR} \end{bmatrix}$$

We note $\Lambda(\mathcal{U}(\mathcal{H}) \cap \mathcal{L}(\Lambda)) \subseteq \mathcal{F}(\mathcal{H})$ since ΛU^* is a parametrix for ΛU:

$$\mathbb{1} - (\Lambda U^*)(\Lambda U) = \Lambda - \Lambda U^* \Lambda U \Lambda = \Lambda U^* (\mathbb{1} - \Lambda) U \Lambda = \underbrace{[\Lambda, U^*]}_{\in \mathcal{K}(\mathcal{H})} \Lambda^\perp U \Lambda$$

⊛ So, we may define the $\mathbb{Z}$-index $\operatorname{ind}_\Lambda : \mathcal{U}(\mathcal{H}) \cap \mathcal{L}(\Lambda) \to \mathbb{Z}$.

⊛ This has $\operatorname{ind}_\Lambda(UV) = \operatorname{ind}_\Lambda(U) + \operatorname{ind}_\Lambda(V)$

Symmetries:

Let $C, J \in \mathcal{B}(\mathcal{H})$ be anti-unitary ops. s.t. $C^2 = -J^2 = \mathbb{1}$

$\Rightarrow$ invertible

$\underline{\mathbb{R}\text{-structure}}$: $\mathcal{H}_{\mathbb{R}} := \{ \varphi \in \mathcal{H} : C\varphi = \varphi \}$

quaternions

$\underline{\mathbb{H}\text{-structure}}$: generatures of $\mathbb{H}$-alg.: $\mathbb{1}, i\mathbb{1}, J, iJ$

$\underline{\mathbb{C}\text{-structure}}$: $\mathcal{H}$ is born with this!

Define for $F \in \{C, J\}$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{H}\}$ the following:

- $\mathcal{B}_{\mathbb{F}}(\mathcal{H}) := \{ A \in \mathcal{B}(\mathcal{H}) : AF = FA \}$
- $\mathcal{B}_{*\mathbb{F}}(\mathcal{H}) := \{ A \in \mathcal{B}(\mathcal{H}) : AF = FA^* \}$ ← i.e. $FAF = \pm A^*$
- $\mathcal{B}_{i\mathbb{F}}(\mathcal{H}) := \{ A \in \mathcal{B}(\mathcal{H}) : AF = -FA \}$

Standing assumption: $[C, \Lambda] = [J, \Lambda] = 0$ (C, J are hyperlocal, should be unnecessary).

$\mathbb{Z}_2$ - index (Atiyah-Singer 1969)

$\underline{\text{Claim:}}$ $\operatorname{ind}\big|_{\mathcal{F}_{*\mathbb{F}}(\mathcal{H})} = 0$

$\underline{\text{Pf:}}$ index $A = \dim\ker F - \dim\ker F^* = -\operatorname{index} F^*$. Since $A \in \mathcal{F}_{*\mathbb{F}}(\mathcal{H})$, then these must be equal, and so they are 0. □

The same holds for S.A. operators:

$\underline{\text{Def } (\mathbb{Z}_2\text{-index})}$: $\operatorname{index}_2(A) := (\dim\ker F) \bmod 2 \in \mathbb{Z}_2$

$\operatorname{index}_{2,\Lambda} : \mathcal{U}(\mathcal{H}) \cap \mathcal{L}(\Lambda) \to \mathbb{Z}_2$ sends $U \mapsto \operatorname{ind}_2 \Lambda U$.

Claim (AS '96): $\mathrm{ind}_2|_{\mathcal{F}_{*\mathbb{H}}(\mathcal{H})}$ and $\mathrm{ind}_2|_{\mathcal{F}_{i\mathbb{R}}^{S.A.}(\mathcal{H})}$ are norm-cont. and compactly stable.

⚠ $\nexists$ a logarithmic law for ind_2.

$\pi_0(\mathcal{U}(\mathcal{H})\cap\mathcal{L}(\Lambda))$ - the 10 Bijections

Theorem: (5 bijections)

W.r.t. the operator norm topology,

① , ② $\mathrm{ind}_\Lambda : \pi_0(\mathcal{U}_{\mathbb{F}}\cap\mathcal{L}) \xrightarrow{\sim} \mathbb{Z}$ $\quad (\mathbb{F}\in\{\mathbb{R},\mathbb{C}\})$

③ $\mathrm{ind}_\Lambda : \pi_0(\mathcal{U}_{\mathbb{H}}\cap\mathcal{L}) \xrightarrow{\sim} 2\mathbb{Z}$ $\quad$ (i.e. they're even)

④ $\pi_0(\mathcal{U}_{*\mathbb{R}}\cap\mathcal{L}) \cong \{0\}$

⑤ $\mathrm{ind}_\Lambda : \pi_0(\mathcal{U}_{*\mathbb{H}}\cap\mathcal{L}) \xrightarrow{\sim} \mathbb{Z}_2$

Remarks:

- ② was CHO '82 JFA, which we covered last time.
- $\pi_0(\mathcal{U})\cong\{0\}$ (Kuiper '65) compared with ② shows that locality is crucial.
- Atiyah - Singer 1969 / Atiyah - Jänich 1965 showed $[\underbrace{M}_{\text{some manifold}} \to \mathcal{F}(\mathbb{C}^\infty)] \cong K_0(M)$ and so $\pi_0(\mathcal{F}(\mathcal{H}))=\mathbb{Z}$.

Def: (Self-adjoint unitaries)

Define $\mathcal{S}(\mathcal{H}) := \{A=A^*\in\mathcal{U}(\mathcal{H})\}$ to be the class of S.A. unitaries. If P is an orthogonal projection, then $\mathbb{1}-2P$ is a S.A. unitary.

Physics: $P\equiv\chi_{(-\infty,0)}(H)$ the Fermi projection at $E_F=0$, $\underbrace{\mathrm{sgn}(H)}_{1 \text{ if } x>0 \text{ else } -1}$ the flat Hamiltonian, then $\mathrm{sgn}(H)=\mathbb{1}-2P$ is S.A. unitary

Recalling Λ-nontriviality from last time, we have

Claim: $\mathcal{S}_{\Lambda\text{-nontrivial}} \subsetneq \mathcal{S}\cap\mathcal{L}(\Lambda)\subsetneq\mathcal{S}$

Pf: $U=\begin{bmatrix} U_{LL} & U_{LR}\\ U_{RL} & U_{RR}\end{bmatrix}$, and so $U-V\in\mathcal{K}(\mathcal{H}) \Rightarrow U_{LR}, U_{RL}\in\mathcal{K}(\mathcal{H})$. $\square$

Theorem: (5 more bijections)

W.r.t. op. norm topology,

started proving last time

⑥, ⑦, ⑧ $\quad \pi_0\left(S_{\mathbb{F}}^{1\text{-nontrivial}}\right) \cong \{0\} \qquad (\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\})$

⑨ $\quad \pi_0\left(S_{i\mathbb{H}}^{1\text{-nontrivial}}\right) \cong \{0\}$

⑩ $\quad \mathrm{ind}_{2,1}\left(S_{i\mathbb{R}}^{1\text{-nontrivial}}\right) \xrightarrow{\simeq} \mathbb{Z}_2$

Remarks:
- ⑥ comes from Andruchow et al. 2015 JFA
- Dropping locality in ⑥ is easy: for U, V,
  $W: \ker(U-\mathbb{1}) \oplus \ker(U+\mathbb{1}) \to \ker(V-\mathbb{1}) \oplus \ker(V+\mathbb{1})$
  has $W^* U W = V$.

Classification of 1-D insulators

Previously, we had

copy the rest from blackboard pics

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spectrally-gapped

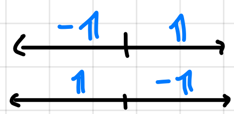
Write $\mathcal{I}_{0,N} = \{ H = H^* \in \mathcal{B}(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N) : H \text{ is exp-local and } 0 \notin \sigma(H) \}$ and equip it w/ the operator norm topology.

Idea: Relax exp-locality to $[\Lambda, H] \in \mathcal{K}$ for $\Lambda := \chi_{\{x>0\}}(X)$

Example:

$H = \Lambda - \Lambda^\perp$

$\tilde{H} = -\Lambda + \Lambda^\perp$



Claim: H and $\tilde{H}$ above are <u>not</u> path-connected in $\mathcal{I}_{0,N}$.

Proof: Suppose otherwise, i.e. suppose BWOC $\exists$ cont. $[0,1] \ni t \mapsto H_t \in \mathcal{I}_{0,N}$ s.t. $H_0 = H$, $H_1 = \tilde{H}$

Write $P_t := \frac{1}{2}(\mathbb{1} - \mathrm{sgn}(H_t)) \Rightarrow P_0 = \Lambda, \; P_1 = \Lambda^\perp$.

Fact: if $\exists$ cont. path connecting S.A. projections in a C^* alg., then they are equivalent up to conjugation by a unitary.

So, $\exists U \in \mathcal{U}(\mathcal{H}) \cap \mathcal{L}(\Lambda)$ s.t. $\Lambda = U \Lambda^\perp U^*$.

Writing $U = \begin{bmatrix} U_{LL} & U_{RL} \\ U_{LR} & U_{RR} \end{bmatrix}$, $[U, \Lambda] \in \mathcal{K} \iff U_{RL}, U_{LR} \in \mathcal{K}$.

So,

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{bmatrix} = \begin{bmatrix} U_{LL} & U_{RL} \\ U_{LR} & U_{RR} \end{bmatrix}^* \begin{bmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{LL} & U_{RL} \\ U_{LR} & U_{RR} \end{bmatrix}$$

$$\Rightarrow \mathbb{1} = U_{RL}^* U_{RL} \Rightarrow \mathbb{1} \in \mathcal{K} \quad \times.$$

$\square$

So, clearly we need to relax the definition slightly.

Def:

$H = H^* \in \mathcal{B}(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N)$ is a bulk-insulator iff:

$[H, \Lambda] \in \mathcal{K}$, $0 \notin \sigma(H)$, <u>and</u> $\Lambda \, \mathrm{sgn}(H) \Lambda$, $\Lambda^\perp \mathrm{sgn}(H) \Lambda^\perp$ are ess. nontrivial SAUs

Let $\mathcal{I}_{0,N}^B$ denote the set of bulk insulators.

Claim: $\mathcal{I}^B_{0,N}\overset{\text{"flat"}}{b}$ is a deformation retraction of $\mathcal{I}^B_{0,N}$.

Proof: $F(t,H) = (1-t)H + t\,\mathrm{sgn}(H)$ satisfies $\mathrm{sgn}(F(t,H)) = \mathrm{sgn}(H)$. note that this lets us check $F(t,H)\in\mathcal{I}^B_{0,N}$ via the $\mathcal{I}^B_{0N}$

□

The goal is to show $\underbrace{\pi_0(\mathcal{I}^B_{0,N}b)}_{\cong S_{1\text{-nontrivial}}} \cong \{0\}$

Andruchow et al. (2015) investigate

$$U = \begin{bmatrix} X & A \\ A^* & Y \end{bmatrix},\quad \tilde U = \begin{bmatrix} \tilde X & \tilde A \\ \tilde A^* & \tilde Y \end{bmatrix}$$

with $X, Y, \tilde X, \tilde Y$ S.A., $A, \tilde A$ compact

One wants to find path connecting $U \overset{S_{1\text{-nontrivial}}}{\rightsquigarrow} \tilde U$

If $V, \tilde V$ are two non-trivial SAUs, then $\exists\, W \in \mathcal{U}(\mathcal{H})$ s.t. $V = W^* \tilde V W$.
We want to decompose $\mathcal{H} = \ker(V+1)\oplus\ker(V-1) = \ker(\tilde V+1)\oplus\ker(\tilde V-1)$
and use Kuiper to connect $W \rightsquigarrow 1$ (Kuiper) on the diagonals.
So, it reduces to connecting U to a diagonal SAU within $S_{1\text{-nontrivial}}$.

... insert stuff here about intertwining eigenspaces of X, Y to each other via A ...

there is isomorphism between $\sigma(X)\setminus\{\pm1\}$ and $-\sigma(Y)\setminus\{\pm1\}$

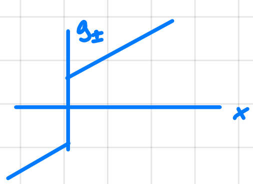
U is a SAU $\quad U^2 = 1 \iff \begin{cases} |A|^2 = 1 - Y^2 \\ |A^*|^2 = 1 - X^2 \\ XA = -AY \end{cases}$ for $A: \ker(Y-\lambda 1) \to \ker(X+\lambda 1)$, $|\lambda|<1$

Note that $\{\pm 1\} \overset{\text{ess. SAU}}{=} \sigma_{ess}(U) = \sigma_{ess}(X) \cup \sigma_{ess}(Y)$

and $\chi_{\{\lambda\}}(X)A = A\chi_{\{-\lambda\}}(Y)$

So, to construct a homotopy sending $\sigma(X) \to \{\pm 1\}$, we do the following:

- Let $f_\pm : \mathbb{R} \to \mathbb{R}$ send $x \mapsto \mathrm{sgn}(x) \pm \chi_{\{0\}}(x)$ (undefined at 0)
- Write $V := f_+(X) + f_-(Y) \in S_{1\text{-nontrivial}}$.



The problem reduces to showing $U \xrightarrow{S_{\text{nontrivial}}} V$.
Write $G := \frac{1}{2}(U+V) \Rightarrow GU = \dots = VG$

Note that $G \in \mathcal{F}(\mathcal{H})$ and $\mathrm{ind}_2 G = 0$. Defining $g_\pm(x) := x + f_\pm(x)$

$$\Rightarrow G - \underbrace{\tfrac{1}{2}(g_+(X)\oplus g_-(Y))}_{\text{invertible}} \in \mathcal{K} \to \mathrm{ind}_2(G) = 0$$

We claim even more: that G is itself invertible.

To see this, we WTS $\ker G = \{0\}$. Suppose $G\begin{bmatrix}\psi \\ \varphi\end{bmatrix} = 0$.
Since $G = \begin{bmatrix} \frac{1}{2} g_+(x) & A \\ A^* & \frac{1}{2} g_-(y) \end{bmatrix}$, $\Rightarrow \begin{cases} g_+(x)\psi + A\varphi = 0 & ① \\ g_-(y)\varphi + A^*\psi = 0 & ② \end{cases}$

So, $A^* g_+(x) = A^*\left(X + \mathrm{sgn}(x) + \chi_{\{0\}}(x)\right) = \left(-Y - \mathrm{sgn}(Y) + \chi_{\{0\}}(Y)\right) A^* = -g_-(Y) A^*$

Thus,

$$① \Rightarrow A^* g_+(x)\psi + |A|^2\varphi = 0 \Rightarrow 0 = -g_-(Y) A^*\psi + |A|^2\varphi$$

$$\overset{②}{\Rightarrow} 0 = g_-(Y)^2\varphi + |A|^2\varphi = \underbrace{(g_-(Y) + 1 - Y^2)}_{0 \notin \mathrm{im}(g_-(t) + 1 - t^2)}\varphi \Rightarrow \varphi = 0 \Rightarrow \psi = 0$$

So, G is invertible. We already knew $GU = VG$, and so

$$G^2 U = GVG = UG^2 \Rightarrow [G^2, U] = [G^2, V] = 0$$

$$\Rightarrow [\overset{G \text{ is S.A.}}{|G|}, U] = 0 \Rightarrow \dots \Rightarrow \underset{\text{"polar part of } G\text{"}}{\mathrm{pol}(G)}\, U = V\, \mathrm{pol}(G)$$

So, $\mathrm{pol}(G)$ is a SAU which conjugates U and V. Together,

$$U \xrightarrow{\mathrm{pol}(G) \rightsquigarrow 1} V \xrightarrow{\text{Kuiper}} \tilde{V} \xrightarrow{1 \rightsquigarrow \mathrm{pol}(\tilde{G})} \tilde{U}$$

So, the first entry in the Kitaev table is empty for 1D.
The full 10 bijections give the full 1D Kitaev table.

∎