MAT 425: Problem Set 7

Due on May 5, 2023

Professor Paul Minter

Evan Dogariu
Collaborators: Reuel Williams and David Shustin

Problem 1

Solution

Proof. Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded operator on a separable Hilbert space $\mathcal{H}$ which is diagonal with respect to an orthonormal basis $\{\varphi_k\}_k$, with $T\varphi_k = \lambda_k$.

$(\implies)$ Suppose first that T is compact. For all $\epsilon > 0$, define the index set of all eigenvalues with magnitude larger than ϵ as

$$A_\epsilon := \{k : |\lambda_k| \geq \epsilon\}$$

We want to show that for all ϵ, the set A_ϵ is finite. So, suppose by way of contradiction that there exists an ϵ such that A_ϵ has infinitely many distinct elements. Consider the (infinite) sequence

$$(T\varphi_k)_{k \in A_\epsilon}$$

Since T is bounded, we know that this sequence is also bounded. By the definition of compactness, this sequence must then have some convergent subsequence indexed by $(k_n)_n \subset A_\epsilon$. Since $\mathcal{H}$ is a Hilbert space and is therefore complete, this subsequence is Cauchy. The Cauchy criterion then reads that there exists an N such that for all $n, m > N$,

$$||T\varphi_{k_n} - T\varphi_{k_m}||_{\mathcal{H}} < \epsilon \qquad (k_m, k_n \in A_\epsilon)$$

However, we have that for all $k_m, k_n \in A_\epsilon$,

$$\begin{aligned}
||T\varphi_{k_n} - T\varphi_{k_m}||_{\mathcal{H}}^2 &= ||\lambda_{k_n}\varphi_{k_n} - \lambda_{k_m}\varphi_{k_m}||_{\mathcal{H}}^2 \\
&= ||\lambda_{k_n}\varphi_{k_n}||_{\mathcal{H}}^2 + ||\lambda_{k_m}\varphi_{k_m}||_{\mathcal{H}}^2 - 2Re\,\langle \lambda_{k_n}\varphi_{k_n}, \lambda_{k_m}\varphi_{k_m}\rangle_{\mathcal{H}} \\
&= ||\lambda_{k_n}\varphi_{k_n}||_{\mathcal{H}}^2 + ||\lambda_{k_m}\varphi_{k_m}||_{\mathcal{H}}^2 = |\lambda_{k_n}|^2 \cdot ||\varphi_{k_n}||_{\mathcal{H}}^2 + |\lambda_{k_m}|^2 \cdot ||\varphi_{k_m}||_{\mathcal{H}}^2 \\
&= |\lambda_{k_n}|^2 + |\lambda_{k_m}|^2 \\
&\geq 2\epsilon^2,
\end{aligned}$$

where the first line uses that the φ_k's are eigenvectors, the third line uses that the φ_k's are orthogonal, the fourth line uses that the φ_k's have unit norm, and the last line uses that k_m and k_n are in A_ϵ. Taking the square root, we have that for all $n, m > N$,

$$||T\varphi_{k_n} - T\varphi_{k_m}||_{\mathcal{H}} \geq \sqrt{2}\epsilon > \epsilon$$

This contradicts the Cauchy criterion, and so we find that A_ϵ must be finite for all ϵ; then, for all $\epsilon > 0$, it must be that $\max A_\epsilon < \infty$. We know that for all $n > \max A_\epsilon$, it is true that $n \notin A_\epsilon \implies |\lambda_n| < \epsilon$. Since such a property holds for all ϵ, we know that $|\lambda_k| \to 0$, and so $\lambda_k \to 0$.

$(\impliedby)$ Suppose now that $\lambda_k \to 0$. Let P_n be the orthogonal projection operator onto the subspace spanned by $\varphi_1, ..., \varphi_n$. We want to show that

$$||P_nT - T||_{op} \leq \sup_{m>n} |\lambda_m|$$

For any vector $v \in \mathcal{H}$, we can write $v = \sum_{k=1}^\infty a_k\varphi_k$ in terms of the basis, and so since P_n sends φ_k to 0 for $k > n$,

$$||(P_nT - T)v||_{\mathcal{H}}^2 = \left\| \left(\sum_{k=1}^n a_k\lambda_k\varphi_k\right) - \left(\sum_{k=1}^\infty a_k\lambda_k\varphi_k\right) \right\|_{\mathcal{H}}^2 = \left\| \sum_{k=n+1}^\infty a_k\lambda_k\varphi_k \right\|_{\mathcal{H}}^2$$

Since the φ_k's are an orthnormal basis, Parseval's identity gives that

$$||(P_nT - T)v||_{\mathcal{H}}^2 = \left\| \sum_{k=n+1}^\infty a_k\lambda_k\varphi_k \right\|_{\mathcal{H}}^2 = \sum_{k=n+1}^\infty |a_k|^2|\lambda_k|^2$$

Clearly, for each k we are summing over, $|\lambda_k| \leq \sup_{m>n} |\lambda_m|$, and so

$$||(P_nT - T)v||_{\mathcal{H}}^2 \leq \sum_{k=n+1}^{\infty} |a_k|^2 \left(\sup_{m>n} |\lambda_m|\right)^2 = \left(\sup_{m>n} |\lambda_m|\right)^2 \sum_{k=n+1}^{\infty} |a_k|^2$$

So, for all unit vectors $v \in \mathcal{H}$ with $||v||_{\mathcal{H}} = \sum_{k=1}^{\infty} |a_k|^2 = 1$, we have that

$$||(P_nT - T)v||_{\mathcal{H}}^2 \leq \left(\sup_{m>n} |\lambda_m|\right)^2 \sum_{k=n+1}^{\infty} |a_k|^2 \leq \left(\sup_{m>n} |\lambda_m|\right)^2 \sum_{k=1}^{\infty} |a_k|^2 = \left(\sup_{m>n} |\lambda_m|\right)^2$$

Taking the square root of both sides, we get that for all unit vectors v,

$$||(P_nT - T)v||_{\mathcal{H}} \leq \sup_{m>n} |\lambda_m|$$

Since this holds for all unit v, it must hold in supremum, and so

$$||P_nT - T||_{op} = \sup_{||v||_{\mathcal{H}}=1} ||(P_nT - T)v||_{\mathcal{H}} \leq \sup_{m>n} |\lambda_m|$$

Now, we know that $\lambda_k \to 0$. Let $\epsilon > 0$. Then, there exists some N such that for all $m > N$, we know $|\lambda_m| \leq \epsilon$. Therefore, for all $n > N$,

$$||P_nT - T||_{op} \leq \sup_{m>n} |\lambda_m| \leq \epsilon,$$

where the last inequality holds since $m > n \implies m > N \implies |\lambda_m| \leq \epsilon$. Since such an N exists for all $\epsilon > 0$, we know that $||P_nT - T||_{op} \to 0$. However, note that each P_nT is a compact operator since its range has finite dimension (in particular, it has dimension n). So, by Proposition 6.1(ii), since we have a sequence $(P_nT)_n$ of compact operators with $||P_nT - T||_{op} \to 0$, we get that the bounded operator T is also compact.
■

Problem 2

Solution

Proof of (a). Let $T : \mathcal{H} \to \mathcal{H}$ be compact, and assume $\lambda \neq 0$. We want to show that the range of $\lambda I - T$ is closed. To that end, suppose that $(g_n)_n \subset \text{range}(\lambda I - T)$ is an arbitrary sequence of elements in the range that converges to some vector $g \in \mathcal{H}$; we want to show that $g \in \text{range}(\lambda I - T)$. For convenience, denote

$$V_\lambda := \ker(\lambda I - T)$$

We know that V_λ is closed since null spaces are closed. So, we can perform the orthogonal decomposition $\mathcal{H} = V_\lambda \oplus V_\lambda^\perp$. Now, since each g_n is in the range, we can write

$$g_n = (\lambda I - T) f_n$$

for some $f_n \in \mathcal{H}$. In this case, use the orthogonal decomposition to get $f_n = f_n^{(1)} + f_n^{(2)}$ with $f_n^{(1)} \in V_\lambda$ and $f_n^{(2)} \in V_\lambda^\perp$, which means

$$g_n = (\lambda I - T) f_n = (\lambda I - T) f_n^{(1)} + (\lambda I - T) f_n^{(2)} = (\lambda_I - T) f_n^{(2)}$$

This means that g_j is also the image of some element $f_n^{(2)} \in V_\lambda^\perp$, and so we can suppose without loss of generality that $f_n \in V_\lambda^\perp$ for all n.

Lemma 1. *The sequence $(f_n)_n$ is bounded.*

Proof of Lemma 1. Suppose by way of contradiction that $(f_n)_n$ is not bounded. Since $(g_n)_n$ is convergent, it is bounded, say by $||g_n||_\mathcal{H} \leq M$ for all n. For each n, define

$$v_n := (\lambda I - T)\left(\frac{f_n}{||f_n||_\mathcal{H}}\right) = \frac{1}{||f_n||_\mathcal{H}}(\lambda I - T) f_n = \frac{g_n}{||f_n||_\mathcal{H}}$$

Since $||v_n||_\mathcal{H} \leq M/||f_n||_\mathcal{H}$ and the f_n's get unboundedly large, we can always find a new arbitrarily small element of $(v_n)_n$. So, there exists some subsequence $(v_{n_k})_k$ such that $v_{n_k} \to 0$ as $k \to \infty$. However, note that the sequence $\left(T\left(\frac{f_{n_k}}{||f_{n_k}||_\mathcal{H}}\right)\right)_k$ is the image of a bounded sequence under a compact operator T; we must therefore have a convergent subsequence $T\left(\frac{f_{n_{kj}}}{||f_{n_{kj}}||_\mathcal{H}}\right) \to w$ as $j \to \infty$ for some $w \in \mathcal{H}$. Since

$$\lambda \frac{f_{n_k}}{||f_{n_k}||_\mathcal{H}} = T\left(\frac{f_{n_k}}{||f_{n_k}||_\mathcal{H}}\right) + v_{n_k},$$

taking the limit along the subsequence $(n_{kj})_j \subset (n_k)_k$ and dividing by $\lambda \neq 0$ reveals that, since $v_{n_{kj}} \to 0$ (it is a subsequence of $(v_{n_k})_k$),

$$\frac{f_{n_{kj}}}{||f_{n_{kj}}||_\mathcal{H}} \to \frac{w}{\lambda}$$

Since $f_{n_{kj}} \in V_\lambda^\perp$ for all j and orthogonal complements are closed subspaces, we find that $\frac{w}{\lambda} \in V_\lambda^\perp$ as well. However, since T is bounded and therefore continuous, $T\left(\frac{f_{n_{kj}}}{||f_{n_{kj}}||_\mathcal{H}}\right) \to T\left(\frac{w}{\lambda}\right)$. By uniqueness of limits,

$$w = T\left(\frac{w}{\lambda}\right) \implies Tw = \lambda w \implies w \in V_\lambda$$

So, since $w \in V_\lambda$ and $w \in V_\lambda^\perp$, we know that $w = 0$. Therefore, $\frac{f_{n_{kj}}}{||f_{n_{kj}}||_\mathcal{H}} \to 0$. This is a contradiction, since a sequence of unit vectors can't approach 0. So, $(f_n)_n$ is bounded. ■

Problem 2 continued on next page...

Now, since $(f_n)_n$ is bounded and T is compact, there exists a subsequence $(f_{n_k})_k$ such that $(Tf_{n_k})_k$ converges, say $Tf_{n_k} \to h \in \mathcal{H}$ as $k \to \infty$. By definition of g_n and the fact that $\lambda \neq 0$,

$$g_{n_k} = \lambda f_{n_k} - Tf_{n_k} \implies f_{n_k} = \frac{g_{n_k} + Tf_{n_k}}{\lambda} \implies \lim_{k\to\infty} f_{n_k} = \frac{g+h}{\lambda} \in \mathcal{H},$$

where we used the fact that $Tf_{n_k} \to h$ and $g_{n_k} \to g$ as $k \to \infty$ (since $g_n \to g$, so does any subsequence). Define

$$f := \frac{g+h}{\lambda}$$

such that $f_{n_k} \to f$ as $k \to \infty$. Then, T is compact $\implies$ T is bounded $\implies$ T is continuous, and so $Tf_{n_k} \to Tf$ as $k \to \infty$ (continuous functions inherit limits). So, the facts $f_{n_k} \to f$ and $Tf_{n_k} \to Tf$ together imply that $(\lambda I - T)f_{n_k} \to (\lambda I - T)f$ as $k \to \infty$. However, $g_{n_k} = (\lambda I - T)f_{n_k} \to g$; by uniqueness of limits, this gives that

$$g = (\lambda I - T)f$$

In particular, we get that $g \in \text{range}(\lambda I - T)$. Then, $\text{range}(\lambda I - T)$ contains all of its limit points and is therefore closed. ■

Proof of (b). Suppose now that $\lambda = 0$. We wish to find a compact operator $T : \mathcal{H} \to \mathcal{H}$ such that range $(\lambda I - T) =$ range $(-T)$ is not closed; for simplicity, let us instead find a T for which $V :=$ range (T) is not closed, as the negation of this T will have the desired property. Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$, and define

$$T(\varphi_k) := \frac{1}{k}\varphi_{k+1}$$

We know from Problem 5 that T is compact, and so all we must do is find a sequence of vectors in V such that their limit is not in V. To this end, consider the sequence $(f_n)_n \subset \mathcal{H}$ given by

$$f_n := \sum_{k=1}^{n} \varphi_k \implies Tf_n = \sum_{k=1}^{n} \frac{\varphi_{k+1}}{k}$$

Clearly, the sequence $(Tf_n)_n$ is contained in V by definition. Furthermore, the sequence has a limit which is equal to

$$\lim_{n\to\infty} Tf_n = \sum_{k=1}^{\infty} \frac{\varphi_{k+1}}{k} =: g \implies ||g||_{\mathcal{H}}^2 = \sum_{k=1}^{\infty} \left|\frac{1}{k}\right|^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

where we used Parseval's identity and the p-series to show that $||g||_{\mathcal{H}}^2 < \infty$. So, the infinite sum defining g converges to a vector in $\mathcal{H}$, and we can say $Tf_n \to g$. However, it can be shown that $g \notin V$. Suppose by way of contradiction that $g \in V$; that is, suppose that $g = Tf$ for some $f \in \mathcal{H}$. Write $f = \sum_{k=1}^{\infty} a_k \varphi_k$. This means that

$$\sum_{k=1}^{\infty} \frac{a_k}{k}\varphi_{k+1} = Tf = g = \sum_{k=1}^{\infty} \frac{1}{k}\varphi_{k+1} \implies 0 = Tf - g = \sum_{k=1}^{\infty} \frac{a_k - 1}{k}\varphi_k$$

Since the φ_k's are linearly independent, the only way for this sum to be equal to 0 is if all the coefficients are 0, which means that $a_k = 1$ for all $k \geq 1$. However, Parseval's identity reveals that

$$||f||_{\mathcal{H}}^2 = \sum_{k=1}^{\infty} |a_k|^2 = \sum_{k=1}^{\infty} 1 = \infty$$

This is a contradiction since we assumed that $f \in \mathcal{H}$, and so it must be that $g \notin V$. We have thus found a compact operator T for which a sequence in range (T) converges to a vector not in range (T), and so range (T) is not closed. This means that range $(T - \lambda I)$ is also not closed when $\lambda = 0$, which in turn means that range $(\lambda I - T)$ is not closed. So, the result from (a) doesn't always hold when $\lambda = 0$. ■

Proof of (c). ($\implies$) Suppose first that the range of $\lambda I - T$ is all of $\mathcal{H}$. We want to show that $h \notin \ker(\overline{\lambda} I - T^*)$ for all nonzero h, as this will prove that the kernel is trivial. To this end, let $h \in \mathcal{H}$ be an arbitrary nonzero vector. We want to show that $(\overline{\lambda} I - T^*)h \neq 0$. So, note that since the range of $\lambda I - T$ is all of $\mathcal{H}$, then there exists some $f \in \mathcal{H}$ such that $(\lambda I - T)f = h$. We have that the adjoint of $\overline{\lambda} I - T^*$ is simply $(\overline{\lambda} I - T^*)^* = \lambda I - T$; to see this note that for all $u, v \in \mathcal{H}$,

$$\left\langle u, (\overline{\lambda} I - T^*)v \right\rangle_{\mathcal{H}} = \left\langle u, \overline{\lambda} v \right\rangle_{\mathcal{H}} - \langle u, T^* v \rangle_{\mathcal{H}} = \lambda \langle u, v \rangle_{\mathcal{H}} - \langle Tu, v \rangle_{\mathcal{H}} = \langle (\lambda I - T)u, v \rangle_{\mathcal{H}}$$

Using this, we get

$$\begin{aligned} \left\langle (\overline{\lambda} I - T^*)h, f \right\rangle_{\mathcal{H}} &= \left\langle (\overline{\lambda} I - T^*)(\lambda I - T)f, f \right\rangle_{\mathcal{H}} \\ &= \langle \lambda f - Tf, \lambda f - Tf \rangle_{\mathcal{H}} \\ &= ||\lambda f - Tf||_{\mathcal{H}}^2 = ||h||_{\mathcal{H}}^2 \neq 0, \end{aligned}$$

So, we see that $\left\langle (\overline{\lambda} I - T^*)h, f \right\rangle_{\mathcal{H}} \neq 0$, which in particular means that $(\overline{\lambda} I - T^*)h \neq 0$ (if it were 0, we would not have been able to find a vector f that it isn't orthogonal to). Therefore, $h \notin \ker(\overline{\lambda} I - T^*)$; since this holds for all arbitrary nonzero h, we find that $\overline{\lambda} I - T^*$ has trivial kernel.

($\impliedby$) We will prove the contrapositive of the reverse direction. For notation, define $V := \text{range}(\lambda I - T)$. Suppose that $V \neq \mathcal{H}$ (i.e. the range of $\lambda I - T$ is not all of $\mathcal{H}$). By part (a), V is closed. So, we can decompose

$$\mathcal{H} = V \oplus V^\perp$$

Since $V \neq \mathcal{H}$, there must be some nonzero element of $V^\perp$ (if $V^\perp$ were to equal $\{0\}$, then $\mathcal{H} = V \oplus V^\perp = V \neq \mathcal{H}$, a contradiction). So, let $f \in V^\perp$ be nonzero. Define $g := (\overline{\lambda} I - T^*)f$. Since f is orthogonal to all elements of the range of $\lambda I - T$, in particular we must have

$$\langle (\lambda I - T)g, f \rangle_{\mathcal{H}} = 0 \implies \langle g, (\lambda I - T)^* f \rangle_{\mathcal{H}} = 0$$

We know from the proof of the other direction $\lambda I - T$ and $\overline{\lambda} I - T^*$ are adjoints, and so we get that

$$\left\langle g, (\overline{\lambda} I - T^*)f \right\rangle_{\mathcal{H}} = 0 \implies \langle g, g \rangle_{\mathcal{H}} = 0 \implies g = 0$$

This means that $(\overline{\lambda} I - T^*)f = 0$, and so $f \in \ker(\overline{\lambda} I - T^*)$. Since f is nonzero, this means that $\overline{\lambda} I - T^*$ has nontrivial kernel, and we are done. ■

Problem 3

Solution

Proof of (a). Define the function

$$K(z) := i(\text{sign}(z)\pi - z)$$

on the interval $[-\pi, \pi)$ and extend it periodically over all of $\mathbb{R}$. For an $f \in L^1([-\pi, \pi])$ that is extended over $\mathbb{R}$ periodically, we have

$$(Tf)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(x-y)f(y)dy = \frac{i}{2\pi}\int_{-\pi}^{\pi} \pi\text{sign}(x-y)f(y)dy - \frac{i}{2\pi}\int_{-\pi}^{\pi}(x-y)f(y)dy$$

Since $x - y > 0$ for y over the interval $[-\pi, x)$ and $x - y < 0$ for y over the inteval $(x, \pi]$, we split the first integral and get that

$$\begin{aligned}(Tf)(x) &= \frac{i}{2\pi}\int_{-\pi}^{x} \pi f(y)dy - \frac{i}{2\pi}\int_{x}^{\pi} \pi f(y)dy - \frac{i}{2\pi}\int_{-\pi}^{\pi}(x-y)f(y)dy \\ &= \frac{i}{2}\int_{-\pi}^{x} f(y)dy + \frac{i}{2}\int_{\pi}^{x} f(y)dy + \frac{i}{2\pi}\int_{-\pi}^{\pi} yf(y)dy - x\frac{i}{2\pi}\int_{-\pi}^{\pi} f(y)dy\end{aligned}$$

We know that the functions $\int_{-\pi}^{x} f(y)dy$ and $\int_{\pi}^{x} f(y)dy$ are absolutely continuous w.r.t x by Proposition 1.12(ii) from Chapter 2; furthermore, both $\int_{-\pi}^{\pi} yf(y)dy$ and $\int_{-\pi}^{\pi} f(y)dy$ are constant w.r.t. x . So, the function $x\int_{-\pi}^{\pi} f(y)dy$ is a linear function w.r.t. x, and is therefore absolutely continuous. This means that Tf is a linear combination of 3 absolutely continuous functions and a constant, and must itself be absolutely continuous. To see that a linear combination of absolutely continuous functions is absolutely continuous, note that the variation over a partition $\sum_{k=1}^{N} |F(b_k) - F(a_k)|$ can be split by the triangle inequality, and each term can be bounded separately since the constituents are absolutely continuous. Take the minimum δ required to uniformly bound the constituent variations such that the total is $< \epsilon$, and the sum is therefore also absolutely continous. Thus, $F = Tf$ is absolutely continuous.

Suppose now that $\int_{-\pi}^{\pi} f(y)dy = 0$. By the converse part of Theorem 3.11 of Chapter 3, we know that F' exists a.e., and that we can simply differentiate with the rule that

$$\frac{d}{dx}\int_{a}^{x} f(y)dy = f(x) \quad a.e.$$

So, since $\int_{-\pi}^{\pi} yf(y)dy$ is a constant and contributes 0 to the derivative and $\int_{-\pi}^{\pi} f(y)dy = 0 \implies x\int_{-\pi}^{\pi} f(y)dy = 0$, the linearity of the derivative grants that

$$F'(x) = \frac{i}{2}f(x) + \frac{i}{2}f(x) + 0 - 0 = if(x) \quad a.e.$$

■

Proof of (b). Note that if we can show that

$$\widetilde{K}(x, y) := K(x-y) = i(\text{sign}(x-y)\pi - (x-y))$$

is in $L^2([-\pi, \pi] \times [-\pi, \pi])$, then we get for free that T is a Hilbert-Schmidt operator on $L^2([-\pi, \pi])$, which automatically implies compactness of T as an operator on $L^2([-\pi, \pi])$. Now, by Proposition 3.9 of Chapter 2, $\widetilde{K}$ is measurable on $[-\pi, \pi] \times [-\pi, \pi]$. Since $|\widetilde{K}|^2$ is nonnegative, we can apply Tonelli's Theorem (Theorem 3.2 of Chapter 2) to see that

$$||\widetilde{K}||^2_{L^2([-\pi,\pi]\times[-\pi,\pi])} = \int_{[-\pi,\pi]\times[-\pi,\pi]} |\widetilde{K}|^2 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\text{sign}(x-y)\pi - (x-y)|^2 dxdy$$

For $x, y \in [-\pi, \pi]$, we know by the triangle inequality that

$$|\text{sign}(x-y)\pi - (x-y)| \leq |\text{sign}(x-y)\pi| + |x-y| = \pi + |x-y| \leq \pi + |x| + |y| \leq 3\pi$$

So,

$$||\widetilde{K}||^2_{L^2([-\pi,\pi]\times[-\pi,\pi])} \leq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(3\pi)^2 dxdy = 9\pi^2 \cdot 2\pi \cdot 2\pi = 36\pi^4 < \infty$$

In particular, $\widetilde{K} \in L^2([-\pi,\pi]\times[-\pi,\pi])$, and so T is a valid Hilbert-Schmidt operator on $L^2([-\pi,\pi])$, which immediately means that T is compact. To see that T is self-adjoint on $L^2([-\pi,\pi])$, note that for all $f, g \in L^2([-\pi,\pi])$,

$$\begin{aligned}\langle Tf, g\rangle_{L^2([-\pi,\pi])} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}K(x-y)f(y)dy\right)\cdot\overline{g(x)}dx \\ &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}K(x-y)f(y)\overline{g(x)}dydx\end{aligned}$$

Now, note that $\overline{K(x-y)} = -K(x-y) = K(y-x)$ because $\bar{i} = -i$ and the function $K(\cdot)$ is odd. Therefore,

$$\langle Tf, g\rangle_{L^2([-\pi,\pi])} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(y)\overline{K(y-x)g(x)}dydx$$

If we switch the order of integration (we know that $|K(y-x)| \leq 3\pi$ and that $f, g \in L^2$, and so we can apply Fubini's Theorem), we get that

$$\begin{aligned}\langle Tf, g\rangle_{L^2([-\pi,\pi])} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}f(y)\int_{-\pi}^{\pi}\overline{K(y-x)g(x)}dxdy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)\overline{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}K(y-x)g(x)dx\right)}dy \\ &= \langle f, Tg\rangle_{L^2([-\pi,\pi])},\end{aligned}$$

where we used that the complex conjugate of an integral is the integral of the conjugate of the integrand (which can easily be seen by splitting into real and imaginary parts). Since this holds for all $f, g \in L^2([-\pi,\pi])$, we see that T is self-adjoint. ∎

Proof of (c). ($\Longrightarrow$) Suppose first that $\varphi \in L^2([-\pi,\pi])$ is an eigenfunction of T. From part (b), we know that $T\varphi$ is absolutely continuous (and so differentiable a.e.). Also, if we define

$$\phi(x) := \varphi(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(y)dy \implies \int_{-\pi}^{\pi}\phi(x)dx = 0,$$

then the second result from part (b) yields

$$(T\phi)'(x) = i\phi(x) \quad a.e.$$

Now, if we let $C := \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(y)dy \implies \phi = \varphi - C$, then $\forall x$,

$$(T\phi)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}K(y)\phi(x-y)dy = \frac{1}{2\pi}\int_{-\pi}^{\pi}K(y)\varphi(x-y)dy - \frac{C}{2\pi}\int_{-\pi}^{\pi}K(y)dy$$

The first term on the right hand side is none other than $(T\varphi)(x)$, and the second term evaluates to 0 by the oddness of K (the fact that T sends constant functions to 0 is something we will use again). So,

$$(T\phi)(x) = (T\varphi)(x) \implies (T\phi)'(x) = (T\varphi)'(x)$$

In particular, this means that $T\varphi$ is differentiable a.e., and that since $\phi(x) = \varphi(x) - C$,

$$(T\phi)'(x) = i\phi(x) \quad a.e. \implies (T\varphi)'(x) = i\phi(x) = i\varphi(x) - iC \quad a.e.$$

However, since φ is an eigenfunction of T, then

$$(T\varphi)(x) = \lambda\varphi(x) \quad a.e.$$

If $\lambda = 0$, then $T\varphi = 0 \implies (T\varphi)' = 0 \implies i\varphi - iC = 0 \implies \varphi = C$, and so φ equals the function 1 up to a constant multiple. If $\lambda \neq 0$, then the previous facts imply that

$$\lambda\varphi'(x) = i\varphi(x) - iC \quad a.e. \implies \varphi'(x) = \frac{i}{\lambda}\varphi(x) - \frac{iC}{\lambda} \quad a.e.$$

So, φ is differentiable a.e., and satisfies the above differential equation. However, note that

$$C = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(y)dy = \frac{1}{2\pi\lambda}\int_{-\pi}^{\pi} (T\varphi)(y)dy = \frac{1}{2\pi\lambda}\langle T\varphi, 1\rangle_{L^2([-\pi,\pi])} = \frac{1}{2\pi\lambda}\langle \varphi, T1\rangle_{L^2([-\pi,\pi])} = 0,$$

where the above result is because T is symmetric and we know that T sends constant functions to 0. Therefore, $C = 0$, and

$$\varphi'(x) = \frac{i}{\lambda}\varphi(x)$$

We know this differential equation to have the unique (up to a constant multiple) solution of

$$\varphi(x) = e^{ix/\lambda}$$

The last step is to note that since $C = 0$ and $\lambda \neq 0$, we can compute

$$0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix/\lambda}dx = \frac{1}{2\pi}\left[\frac{\lambda}{i}e^{ix/\lambda}\right]_{-\pi}^{\pi} \implies e^{i\pi/\lambda} - e^{-i\pi/\lambda} = 0$$

$$\implies 2i\sin(\pi/\lambda) = 0 \implies \sin(\pi/\lambda) = 0$$

The above reveals that $\pi/\lambda = n\pi$ for some $n \in \mathbb{Z}$, which means that $\lambda = \frac{1}{n}$ for some $n \in \mathbb{Z}$. To sum up, we have shown that for an eigenfunction φ of T with eigenvalue λ, either φ is constant with $\lambda = 0$, or $\varphi(x) \propto e^{inx}$ and $\lambda = \frac{1}{n}$ for some $n \in \mathbb{Z} \setminus \{0\}$.

($\impliedby$) To show the other direction, suppose first that $\lambda = 0$ and $\varphi(x) = C$ for some $C \in \mathbb{C}$. Then, as we have seen in the previous part, T sends constant functions to 0, and so

$$T\varphi = 0 = \lambda\varphi,$$

and φ is an eigenfunction of T with eigenvalue λ. Suppose next that $n \in \mathbb{Z} \setminus \{0\}$ and $\varphi(x) = Ce^{inx}$ for some $C \in \mathbb{C}$ nonzero. Then, we can compute using the expression from part (a) that

$$(T\varphi)(x) = \frac{Ci}{2}\int_{-\pi}^{x} e^{iny}dy + \frac{Ci}{2}\int_{\pi}^{x} e^{iny}dy + \frac{Ci}{2\pi}\int_{-\pi}^{\pi} ye^{iny}dy - x\frac{Ci}{2\pi}\int_{-\pi}^{\pi} e^{iny}dy$$

Starting with the fourth integral first,

$$\int_{-\pi}^{\pi} e^{iny}dy = \left[\frac{1}{in}e^{iny}\right]_{-\pi}^{\pi} = \frac{1}{in}\left(e^{in\pi} - e^{-in\pi}\right) = \frac{2}{n}\sin(n\pi) = 0$$

For the first and second integrals,

$$\int_{a}^{x} e^{iny}dy = \left[\frac{1}{in}e^{iny}\right]_{a}^{x} = \frac{1}{in}\left(e^{inx} - e^{ina}\right)$$

So,

$$\frac{Ci}{2}\int_{-\pi}^{x} e^{iny}dy + \frac{Ci}{2}\int_{\pi}^{x} e^{iny}dy = \frac{Ci}{2}\left(\frac{1}{in}\left(e^{inx} - e^{-in\pi}\right) + \frac{1}{in}\left(e^{inx} - e^{ina\pi}\right)\right)$$
$$= \frac{Ci}{2}\cdot\frac{2}{in}e^{inx} = \frac{C}{n}e^{inx}$$

For the third integral, integration by parts yields

$$\int_{-\pi}^{\pi} ye^{iny}dy = \left[-\frac{i}{n}e^{iny}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{i}{n}e^{iny}dy$$

Both of these terms evaluate to 0, as we have seen before. In total, we find that

$$(T\varphi)(x) = \frac{C}{n}e^{inx} + 0 + 0 = \frac{1}{n}\varphi(x)$$

So, Ce^{inx} is an eigenfunction of T with eigenvalue $\frac{1}{n}$, as desired. ■

Proof of (d). From part (b), since T is symmetric and compact, there is an orthonormal basis of $L^2([-\pi,\pi])$ consisting of eigenfunctions of T. However, from part (c), those eigenfunctions are exactly $\{e^{inx}\}_{n\in\mathbb{Z}}$. Since e^{inx} and e^{imx} have different eigenvalues for $n \neq m$ (unless $n = 0$ and $m = 1$), they are orthogonal. When $n = 0$ and $m = 1$, they can be computed to be orthogonal via

$$\left\langle 1, e^{ix}\right\rangle_{L^2([-\pi,\pi])} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x)dx + \frac{i}{2\pi}\int_{-\pi}^{\pi}\sin(x)dx = \frac{1}{2\pi}[\sin(x)]_{-\pi}^{\pi} + 0 = 0$$

They are also all certainly normalized, as for all $n \in \mathbb{Z}$ we have

$$||e^{inx}||^2_{L^2([-\pi,\pi])} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|e^{inx}|^2dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}1dx = 1$$

So, the eigenfunctions of T form an orthonormal set, and from the spectral theorem there is *some* orthonormal basis of $L^2([-\pi,\pi])$ consisting of eigenfunctions of T; therefore, the orthonormal basis of $L^2([-\pi,\pi])$ must be precisely the eigenfunctions $\{e^{inx}\}_{n\in\mathbb{Z}}$. ■

Problem 4

Solution

Proof of (a). Let $\mathcal{H} := L^2([0,1])$. Let $T : \mathcal{H} \to \mathcal{H}$ be defined by

$$T(f)(t) = tf(t)$$

Note first that for all $f, g \in \mathcal{H}$ and all $\alpha, \beta \in \mathbb{C}$,

$$T(\alpha f + \beta g)(t) = t(\alpha f + \beta g)(t) = \alpha t f(t) + \beta t g(t) = \alpha T(f)(t) + \beta T(g)(t) \qquad \forall t \in [0,1],$$

and so T is linear. Also, for all $f \in \mathcal{H}$, we have

$$||Tf||_{\mathcal{H}}^2 = \int_{[0,1]} |tf(t)|^2 dt \leq \int_{[0,1]} |f(t)|^2 dt = ||f||_{\mathcal{H}}^2,$$

where for the inequality we used that $t^2 \leq 1$ over $[0,1]$. So, T is bounded. Next, for all $f, g \in \mathcal{H}$ we know

$$\langle Tf, g \rangle_{\mathcal{H}} = \int_{[0,1]} tf(t) \cdot \overline{g(t)} dt = \int_{[0,1]} f(t) \cdot \overline{tg(t)} dt = \langle f, Tg \rangle_{\mathcal{H}},$$

where for the second equality we used that $t = \bar{t}$ for $t \in [0,1]$. This reveals that $T = T^*$. The last thing to show is that T is not compact. To do so, we will first prove a silly little lemma (that we totally won't use for part (b) hehe).

Lemma 2. *T has no eigenvalues and no eigenvectors.*

Proof of Lemma 2. Suppose by way of contradiction that T has an eigenvalue λ. Then, there is some corresponding eigenvector $f \neq 0$ (0 cannot be an eigenvector) such that (being careful to note that equality in $\mathcal{H}$ means equality over the domain $[0,1]$ up to a set of zero measure)

$$Tf = \lambda f \implies tf(t) = \lambda f(t) \quad \text{for a.e. } t \in [0,1]$$

Since $f \neq 0$ and $tf(t) = \lambda f(t)$ a.e., there must be a set E of positive measure such that $f(t) \neq 0$ and $tf(t) = \lambda f(t)$ for all $t \in E$. Then, for all $t \in E$ we can divide by $f(t)$ to see that $t = \lambda$ over E, and so $E \subset \{t : t = \lambda\}$. However, the set $\{t : t = \lambda\}$ trivially has 0 measure (it can be contained in an arbitrarily small ball around λ, and so it cannot have a positive measure), contradicting that E has positive measure. So, we see that T can have no eigenvalue λ (note that this logic also disallows the possibility $\lambda = 0$). Therefore, it can't have any eigenvectors either. ■

Now, suppose by way of contradiction that T is compact. Certainly $T \neq 0$, and we have already shown that $T = T^*$. Then, by Lemma 6.5, we find that either $||T||_{op}$ or $-||T||_{op}$ is an eigenvalue of T. However, this contradicts the result of Lemma 2, and so T cannot be compact. ■

Proof of (b). I lied. By Lemma 2, we know that T has no eigenvectors. ■

Problem 5

Solution

Proof. Let $\{\varphi_k\}_k$ be an orthonormal basis for $\mathcal{H}$. Let $T : \mathcal{H} \to \mathcal{H}$ be defined by

$$T(\varphi_k) = \frac{1}{k}\varphi_{k+1}$$

Note that this operator is certainly linear, as it is defined on a basis and extended linearly. Furthermore, for every $f = \sum_{k=1}^{\infty} a_k \varphi_k$, Parseval's identity yields

$$||Tf||_{\mathcal{H}}^2 = \left\| \sum_{k=1}^{\infty} \frac{a_k}{k}\varphi_{k+1} \right\|_{\mathcal{H}}^2 = \sum_{k=1}^{\infty} \left| \frac{a_k}{k} \right|^2 \le \sum_{k=1}^{\infty} |a_k|^2 = ||f||_{\mathcal{H}}^2,$$

where the inequality is because $k \ge 1 \implies \frac{1}{k^2} \le 1$. So, $||T||_{op} \le 1$, and so T is bounded. Now, for all $n \in \mathbb{N}$, define the projection operator P_n onto the subspace of $\mathcal{H}$ spanned by $\{\varphi_1, ..., \varphi_n\}$. Then, $P_n T$ is a compact operator for all n since it bounded and has finite rank. For all unit vectors $f = \sum_{k=1}^{\infty} a_k \varphi_k$ with $\sum_k |a_k|^2 = 1$,

$$||(P_n T - T)f||_{\mathcal{H}}^2 = \left\| (P_n - I) \sum_{k=1}^{\infty} \frac{a_k}{k}\varphi_{k+1} \right\|_{\mathcal{H}}^2 = \left\| \sum_{k=n}^{\infty} \frac{a_k}{k}\varphi_{k+1} \right\|_{\mathcal{H}}^2 = \sum_{k=n}^{\infty} \frac{|a_k|^2}{k^2}$$

We know that $|a_k|^2 \le 1$ for all k because f is a unit vector. So, for all unit vectors f, we have

$$||(P_n T - T)f||_{\mathcal{H}}^2 \le \sum_{k=n}^{\infty} \frac{1}{k^2}$$

Then, the supremum over all unit vectors still has this property. In other words,

$$||P_n T - T||_{op}^2 \le \sum_{k=n}^{\infty} \frac{1}{k^2}$$

Since the right hand side is a tail of a convergent sum, it must go to 0 as $n \to \infty$. Therefore,

$$||P_n T - T||_{op} \to 0$$

This means that T is the limit of a sequence $(P_n T)_n$ of compact operators (in the topology induced by the operator norm). By Proposition 6.1(ii), since T is bounded we know that T is therefore compact.

We now wish to show that T has no eigenvectors. Suppose by way of contradiction that $g = \sum_{k=1}^{\infty} a_k \varphi_k \neq 0$ is an eigenvector of T, say with eigenvalue λ. Then,

$$\begin{aligned} Tg = \lambda g &\implies \sum_{k=1}^{\infty} \frac{a_k}{k}\varphi_{k+1} = \sum_{k=1}^{\infty} \lambda a_k \varphi_k \\ &\implies \sum_{k=2}^{\infty} \frac{a_{k-1}}{k-1}\varphi_k = \sum_{k=1}^{\infty} \lambda a_k \varphi_k \\ &\implies \lambda a_1 \varphi_1 + \sum_{k=2}^{\infty} \left(\lambda a_k - \frac{a_{k-1}}{k-1} \right) \varphi_k = 0 \end{aligned}$$

Since the φ_k's are linearly independent (they form a basis), we must have that all their coefficients are 0. In particular, we require

$$\lambda a_1 = 0 \qquad \text{and} \qquad \lambda a_k - \frac{a_{k-1}}{k-1} = 0 \quad \forall k \ge 2$$

Suppose first that $\lambda \neq 0$; the first statement then requires $a_1 = 0$. Since $a_1 = 0$, the second statement with $k = 2$ requires that $a_2 = 0$. Similarly, the second statement with $k = 3$ requires that $a_3 = 0$ since $a_2 = 9$. Proceeding inductively, we find that $a_k = 0$ for all $k \geq 1$, and so $g = 0$. This contradicts that g is an eigenvector. Suppose now that $\lambda = 0$. The second statement then yields that $a_{k-1} = 0$ for all $k \geq 2 \implies a_k = 0 \; \forall k \geq 1$, resulting in the same contradiction regardless of the value of λ. Therefore, T has no eigenvectors. ■

Problem 6

Solution

Proof of (a). Let T_1 and T_2 be two linear, self-adjoint, compact operators that commute. Since T_1 is self-adjoint and compact, the spectral theorem tells us that there is an orthonormal basis $\{\varphi_k\}_k$ of $\mathcal{H}$ composed of eigenvectors of T_1. Let $\{\lambda_k\}_k$ be the corresponding eigenvalues such that $T_1\varphi_k = \lambda_k\varphi_k$. Let $\{\widetilde{\lambda_n}\}_n$ be the set of distinct eigenvalues. Then, we know that the eigenspaces $\{\ker(\widetilde{\lambda_n}I - T_1)\}_n$ are finite dimensional for all n, and are orthogonal for $i \neq j$ by Lemmas 6.3 and 6.4. Furthermore, since each element of the orthonormal basis $\{\varphi_k\}_k$ is contained in exactly one such eigenspace, we know that we must have the decomposition

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \ker(\widetilde{\lambda_n}I - T_1)$$

Let $V_n := \ker(\widetilde{\lambda_n}I - T_1)$ denote the eigenspace corresponding to the n^{th} distinct eigenvalue of T_1; then, $V_n \subset \mathcal{H}$ is a closed subspace since null spaces are closed, and so V_n is itself a Hilbert space. Then, for all $f \in V_n$, we see that because T_1 and T_2 commute,

$$T_1(T_2f) = T_2(T_1f) = T_2(\widetilde{\lambda_n}f) = \widetilde{\lambda_n}T_2f \implies (\widetilde{\lambda_n}I - T_1)T_2f = 0 \implies T_2f \in V_n$$

So, $T_2|_{V_n} : V_n \to V_n$ is an operator on the Hilbert space V_n. Certainly, this operator is linear, bounded, and self-adjoint since T_2 is. Compactness also follows clearly, since any sequence $\{f_l\}_l$ of bounded vectors in V_n yields a convergent subsequence $\{T_2f_{l_m}\}_m$; since T_2 maps vectors from V_n to V_n, which is closed, we see that this subsequence converges in V_n as well. This means that $T_2|_{V_n}$ is self-adjoint and compact, and so by the spectral theorem there exists an orthonormal basis $\{\phi_s^n\}_{s=1}^{\dim V_n}$ of V_n corresponding of eigenvectors of S_n. However, each element of V_n is automatically an eigenvector of T_1 by construction, and so these $\phi_s^{(n)}$'s are eigenvectors of both T_1 and T_2. We clam that

$$B := \bigcup_{n=1}^{\infty} \{\phi_s^{(n)}\}_{s=1}^{\dim V_n}$$

is an orthonormal basis for $\mathcal{H}$. Certainly, each vector in B has unit norm by construction. Furthermore, for any two $\phi_s^{(n)}$ and $\phi_p^{(m)}$ in B, they are orthogonal if $n \neq m$ because eigenspaces are orthogonal by Lemma 6.3, and they are orthogonal if $n = m$ and $s \neq p$ because they are part of an orthonormal set in V_n. So, B forms an orthonormal set in $\mathcal{H}$. To see that it is a basis, note that it is the union of bases $\{\phi_s^{(n)}\}_{s=1}^{\dim V_n}$ of the V_n's, which partition the space $\mathcal{H}$ by a direct sum. So, any element of $f \in \mathcal{H}$ is uniquely representable as a sum $f = \sum_{n=1}^{\infty} v_n$, where each $v_n \in V_n$. Since each v_n is in turn uniquely representable as a finite linear combination of elements of $\{\phi_s^{(n)}\}_{s=1}^{\dim V_n}$ since $\{\phi_s^{(n)}\}_{s=1}^{\dim V_n}$ is a basis of V_n, we see that each f is uniquely representable as an infinite linear combination of elements of B. This means that B is an orthonormal basis of $\mathcal{H}$, formed by vectors that are eigenvectors of both T_1 and T_2. Therefore, T_1 and T_2 are simultaneously diagonalizable. ■

Proof of (b). Suppose that $T : \mathcal{H} \to \mathcal{H}$ is normal and compact. Let

$$T_1 := \frac{T + T^*}{2} \qquad \text{and} \qquad T_2 = \frac{iT^* - iT}{2}$$

Lemma 3. *The sum of compact operators is compact.*

Proof of Lemma 3. Let S, T be compact. Let $(f_n)_n$ be a bounded sequence in $\mathcal{H}$. Then, $(Sf_n)_n$ has a convergent subsequence, say $(Sf_{n_k})_k$ by compactness. Now, consider $(f_{n_k})_k$, which is a bounded sequence in $\mathcal{H}$; compactness of T grants that $(Tf_{n_k})_k$ has a convergent subsequence, say $(Tf_{n_{k_j}})_j$. Since $(n_{k_j})_j$ is a subsequence of $(n_k)_k$, we know that both $(Sf_{n_{k_j}})_j$ and $(Tf_{n_{k_j}})_j$ must converge. Therefore, $((S+T)f_{n_{k_j}})_j$ must also converge. So, for all bounded $(f_n)_n \subset \mathcal{H}$, there is a subsequence $(f_{n_{k_j}})_j$ such that $((S+T)f_{n_{k_j}})_j$

Problem 6 continued on next page...

converges. Therefore, $S+T$ is compact. ■

Lemma 3 and Proposition 6.1(iv) (T compact $\iff$ T^* compact) tell us that T_1 and T_2 are both compact. Furthermore, they are both self-adjoint. To see this, note that

$$T_1^* = \frac{(T+T^*)}{2} = \frac{T^*+T}{2} = T_1$$

and

$$T_2^* = \frac{(iT-iT^*)^*}{2} = \frac{(iT)^*+(-iT^*)^*}{2} = \frac{-iT^*+iT}{2} = T_2,$$

where we used the fact that $(cS)^* = \bar{c}S^*$ for all $c \in \mathbb{C}$. Lastly, we can note that because $TT^* = T^*T$,

$$T_1T_2 = \frac{i}{4}(T+T^*)(T^*-T) = \frac{i}{4}(TT^* - T^2 + (T^*)^2 - T^*T) = \frac{i}{4}((T^*)^2 - T^2)$$

and

$$T_2T_1 = \frac{i}{4}(T^*-T)(T+T^*) = \frac{i}{4}(T^*T + (T^*)^2 - T^2 - TT^*) = \frac{i}{4}((T^*)^2 - T^2)$$

So, $T_1T_2 = T_2T_1$, and therefore T_1 and T_2 are two compact, self-adjoint, commuting linear operators. By part (a), they are simultaneously diagonalizable, say w.r.t. an orthonormal basis $\{\varphi_k\}_k$. Lastly, since we can write

$$T_1 + iT_2 = \frac{T+T^*}{2} + \frac{-T^*+T}{2} = T,$$

we find that T is the linear combination of two operators that are diagonal w.r.t. $\{\varphi_k\}_k$. In particular, T itself must then be diagonal w.r.t. $\{\varphi_k\}_k$, which means T is diagonalizable. ■

Proof of (c). Let $T : \mathcal{H} \to \mathcal{H}$ be compact. Let $\lambda \in \mathbb{C}$, and suppose that $U := \lambda I - T$ is unitary. This means in particular that $\lambda I - T$ is injective, which means that there is no nonzero $f \in \mathcal{H}$ such that $Tf = \lambda f$; put differently, this means that λ is not an eigenvalue of T. Furthermore, $\lambda I - T$ unitary means that

$$(\lambda I - T)^{-1} = (\lambda I - T)^* = \overline{\lambda} I - T^*,$$

where we evaluated the adjoint just like we did in the proof of Problem 2(c) (this is a consequence of the fact that U unitary implies $\langle f, U^{-1}g\rangle_{\mathcal{H}} = \langle Uf, UU^{-1}g\rangle_{\mathcal{H}} = \langle Uf, g\rangle_{\mathcal{H}} \implies U^{-1} = U^*$). This relationship grants the following two equalities

$$(\lambda I - T)(\overline{\lambda} I - T^*) = I \implies |\lambda|^2 I - \lambda T^* - \overline{\lambda} T + TT^* = I$$
$$(\overline{\lambda} I - T^*)(\lambda I - T) = I \implies |\lambda|^2 I - \lambda T^* - \overline{\lambda} T + T^*T = I$$

Subtracting the two above equations reveals that $TT^* = T^*T$; in other words, T is a compact, normal operator. The result from part (b) reveals that there is an orthonormal basis $\{\varphi_k\}_k$ of $\mathcal{H}$ consisting of eigenvectors of T. Let $\{\lambda_k\}_k$ be the corresponding eigenvalues; since λ is not an eigenvalue of T, we know that $\lambda \notin \{\lambda_k\}_k$. Now, for each basis vector φ_k,

$$U\varphi_k = (\lambda I - T)\varphi_k = \lambda\varphi_k - T\varphi_k = \lambda\varphi_k - \lambda_k\varphi_k = (\lambda - \lambda_k)\varphi_k$$

Therefore, φ_k is an eigenvector of U with eigenvalue $\lambda - \lambda_k \neq 0$. Since this holds for each k and $\{\varphi_k\}_k$ is an orthonormal basis of $\mathcal{H}$, we have found an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of U. Thus, U can be diagonalized. ■

Problem 7

Solution

Proof of (a). Suppose that $T : \mathcal{H} \to \mathcal{H}$ is compact and $\lambda \neq 0$. We want to show that

$$\lambda I - T \text{ is injective} \iff \overline{\lambda} I - T^* \text{ is injective}$$

To do so, we will show the contrapositives of both directions.

($\implies$) Suppose first that $\overline{\lambda} I - T^*$ is **not** injective. Then, there is some nonzero vector $f \in \mathcal{H}$ such that

$$(\overline{\lambda} I - T^*) f = 0 \implies T^* f = \overline{\lambda} f$$

For all nonzero vectors $g \in \mathcal{H}$, we have that

$$\langle \lambda g - Tg, f \rangle_{\mathcal{H}} = \lambda \langle g, f \rangle_{\mathcal{H}} - \langle Tg, f \rangle_{\mathcal{H}} = \left\langle g, \overline{\lambda} f \right\rangle_{\mathcal{H}} - \langle g, T^* f \rangle_{\mathcal{H}} = \langle g, T^* f \rangle_{\mathcal{H}} - \langle g, T^* f \rangle_{\mathcal{H}} = 0$$

So, f is orthogonal to the range of $\lambda I - T$, which from Problem 2(a) we know is closed. **FINISH THIS, something something Riesz lemma??? i dont get it man** ■

Proof of (b). From Problem 2(c), we know that the range of $\lambda I - T$ is all of $\mathcal{H}$ if and only if the null space of $\overline{\lambda} I - T^*$ is trivial. In other words,

$$\lambda I - T \text{ is surjective} \iff \overline{\lambda} I - T^* \text{ is injective}$$

From part (a), though, we know that

$$\lambda I - T \text{ is injective} \iff \overline{\lambda} I - T^* \text{ is injective}$$

Combining these two, we arrive at the desired conclusion that

$$\lambda I - T \text{ is injective} \iff \lambda I - T \text{ is surjective}$$

■

Problem 8

Solution

Proof of (a). Suppose $\mathcal{H}$ is a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a bounded operator with $||T||_{op} < 1$. For notation, let $M := ||T||_{op} < 1$. Consider the operator $W : \mathcal{H} \to \mathcal{H}$ given by

$$W := I + T + T^2 + ... = I + \sum_{n=1}^{\infty} T^n \implies W(f) = f + \sum_{n=1}^{\infty} T^n f$$

Note first that this is indeed a valid map from $\mathcal{H} \to \mathcal{H}$, since for every $f \in \mathcal{H}$, the sum converges. To see this, note that by definition of the operator norm,

$$||T^n f||_{\mathcal{H}} = ||T(T^{n-1} f)||_{\mathcal{H}} \leq M \cdot ||T^{n-1} f||_{\mathcal{H}}$$

for all $f \in \mathcal{H}$ and all $n > 1$. So, by induction (with the base case $||Tf||_{\mathcal{H}} \leq M \cdot ||f||_{\mathcal{H}}$) we have that

$$||T^n f||_{\mathcal{H}} \leq M^n \cdot ||f||_{\mathcal{H}}$$

Therefore, by the triangle inequality,

$$||Wf||_{\mathcal{H}} = \left\| f + \sum_{n=1}^{\infty} T^n f \right\|_{\mathcal{H}} \leq ||f||_{\mathcal{H}} = \sum_{n=1}^{\infty} ||T^n f||_{\mathcal{H}} \leq ||f||_{\mathcal{H}} \sum_{n=0}^{\infty} M^n < \infty,$$

where the last sum is a geometric series with $M < 1$, and is therefore finite. So, W indeed maps vectors of $\mathcal{H}$ to other vectors in $\mathcal{H}$, and is therefore a valid map. We would like to show that W is an inverse of the operator $I - T$. To that end, note that since $TW = T + T^2 + ...$, we get

$$(I - T)W = IW - TW = W - TW = I + \sum_{n=1}^{\infty} T^n - \sum_{n=1}^{\infty} T^n = I,$$

Next, note that by the distributive property of operators,

$$\begin{aligned} W(I - T) &= \left(I + \sum_{n=1}^{\infty} T^n \right) (I - T) = I(I - T) + \sum_{n=1}^{\infty} T^n (I - T) \\ &= I - T + \sum_{n=1}^{\infty} T^n - \sum_{n=1}^{\infty} T^{n+1} \\ &= I - T + \sum_{n=1}^{\infty} T^n - \sum_{n=2}^{\infty} T^n \\ &= I - \sum_{n=1}^{\infty} T^n + \sum_{n=1}^{\infty} T^n \\ &= I, \end{aligned}$$

where we distributed in the first two lines and relabeled the index of the second summation in the third line. So, we find that $(I - T)W = W(I - T) = I$, and so W is an inverse of $I - T$. In particular, this means that $I - T$ is invertible. ■

Proof of (b). Suppose now that $\mathcal{H}$ is finite dimensional. Let T be any arbitrary bounded operator. Let $\epsilon > 0$. We wish to find an invertible operator $S : \mathcal{H} \to \mathcal{H}$ with $||T - S||_{op} < \epsilon$. Note first that T can have only finitely many distinct eigenvalues: if T had infinitely many distinct eigenvalues, by Lemma 6.3(ii) there would be an infinite orthogonal set in $\mathcal{H}$, contradicting that $\mathcal{H}$ is finite dimensional. So, there must exist

Problem 8 continued on next page...

some nonzero $\lambda \in \mathbb{C}$ with $|\lambda| < \epsilon$ such that λ is **not** an eigenvalue of T. Since this λ is not an eigenvalue of T, we know that there is no nonzero vector $f \in \mathcal{H}$ such that $Tf = \lambda f$, and so there is no nonzero vector f such that $(\lambda I - T)f = 0$. In other words the kernel of $\lambda I - T$ is trivial, and so $\lambda I - T$ is injective. Now, we can say that T is compact, since T is of finite rank (range $T \subset \mathcal{H} \implies \dim(\text{range } T) \leq \dim(\mathcal{H}) < \infty$). This allows us to apply the Fredholm alternative (Problem 7(b)) to say that since T is compact and $\lambda \neq 0$ is such that $\lambda I - T$ is injective, then $\lambda I - T$ is also surjective, which means that it is invertible. Set $S := T - \lambda I$; then, S is invertible as well. We compute

$$||T - S||_{op} = ||T - (T - \lambda I)||_{op} = ||\lambda I||_{op} = |\lambda| \cdot ||I||_{op} = |\lambda| < \epsilon$$

So, there is an invertible S such that $||T - S||_{op} < \epsilon$. Since this holds for any $\epsilon > 0$, we see that we can arbitrarily approximate T with invertible operators. Since T was an arbitrary bounded operator, we find that the set of invertible operators is dense in the set of bounded operators. ■

Proof of (c). The result from (b) fails when $\mathcal{H}$ is infinite dimensional. For a counterexample, let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$ and consider the left and right shift operators $T_L, T_R : \mathcal{H} \to \mathcal{H}$ that map

$$T_L(\varphi_k) := \begin{cases} \varphi_{k-1} & k > 1 \\ 0 & k = 1 \end{cases}$$

and

$$T_R(\varphi_k) := \varphi_{k+1} \qquad \forall k \in \mathbb{N}$$

Clearly both T_L and T_R are not invertible, since $T_L(\varphi_1) = 0$ (and so T_L is not injective since $\ker T_L \neq \emptyset$) and there is no vector in $\mathcal{H}$ that gets mapped to φ_1 under T_R (and so T_R is not surjective). T_L is, however, bounded because for all vectors $f = \sum_{k=1}^{\infty} a_k \varphi_k$ in $\mathcal{H}$,

$$||T_L f||_{\mathcal{H}}^2 = \left\| \sum_{k=2}^{\infty} a_k \varphi_{k-1} \right\|_{\mathcal{H}}^2 = \sum_{k=2}^{\infty} |a_k|^2 = ||f||_{\mathcal{H}}^2 - |a_1|^2 \leq ||f||_{\mathcal{H}}^2 \implies ||T_L||_{op} \leq 1,$$

where we used Parseval's identity for the second equality. Similarly, we can compute that $||T_R||_{op} \leq 1$. Furthermore, we know that for all vectors $f = \sum_{k=1}^{\infty} a_k \varphi_k$ in $\mathcal{H}$, we have

$$T_L(T_R f) = T_L\left(\sum_{k=1}^{\infty} a_k \varphi_{k+1}\right) = \sum_{k=1}^{\infty} a_k T_L(\varphi_{k+1}) = \sum_{k=1}^{\infty} a_k \varphi_k = f,$$

and so $T_L T_R = I$. Now, we want to show that there is no invertible operator that approximates T_L arbitrarily well. Suppose by way of contradiction that there was an invertible operator S such that

$$||T_L - S||_{op} < 1$$

For all $f \in \mathcal{H}$,

$$||T_R(T_L - S)f||_{\mathcal{H}} \leq ||T_R||_{op} \cdot ||(T_L - S)f||_{\mathcal{H}} \leq ||T_R||_{op} \cdot ||T_L - S||_{op} \cdot ||f||_{\mathcal{H}},$$

and so the operator $T_R(T_L - S)$ has operator norm

$$||T_R(T_L - S)||_{op} \leq ||T_R||_{op} \cdot ||T_L - S||_{op} < 1$$

Now, we would have by part (a) that $W := I - T_R(T_L - S) = I + T_R S - T_R T_L$ is invertible. Observe that since $T_L T_R = I$,

$$T_L W = T_L(I + T_R S - T_R T_L) = T_L + T_L T_R S - T_L T_R T_L = T_L + S - T_L = S$$

$$\implies T_L = SW^{-1}$$

Since T_L is not invertible but both S and W are, this is a contradiction. So, there cannot be an invertible operator S such that $||T_L - S||_{op} < 1$, which certainly means that we cannot approximate the bounded operator T_L arbitrarily well with invertible operators. So, the set of invertible operators cannot be dense in the set of bounded operators when $\mathcal{H}$ is infinite dimensional. ■