MAT 425: Problem Set 7

Due on April 21, 2023

Professor Paul Minter

Evan Dogariu
Collaborators: Reuel Williams and David Shustin

Problem 1

Solution

Proof. Let us start by proving that $l^2(\mathbb{Z})$ with the given inner product is separable. To this end, let us define the following set of vectors:

$$S := \{v \in l^2(\mathbb{Z}) : v_n \in \mathbb{Q}[i] \quad \forall n \in \mathbb{Z}\}$$

Here, $\mathbb{Q}[i]$ is the set of all complex numbers $a + bi$ with rational coefficients $a, b \in \mathbb{Q}$ (algebraically, we get $\mathbb{Q}[i]$ by taking the quotient group $\mathbb{Q} \setminus [\{i\}]$); then, $\mathbb{Q}[i]$ is countable. Firstly, note that S is countable, since we can express

$$S = \bigcup_{k \in \mathbb{N}} \{v \in l^2(\mathbb{Z}) : v_n \in \mathbb{Q}[i] \quad \forall |n| < k \qquad \text{and} \qquad v_n = 0 \quad \forall |n| \geq k\}$$

Each constituent set $\{v \in l^2(\mathbb{Z}) : v_n \in \mathbb{Q}[i] \quad \forall |n| \leq k \qquad \text{and} \qquad v_n = 0 \quad \forall |n| > k\}$ is certainly countable, since it is the set of vectors of finite length, each coefficient having countably many possibilities (so, it is $\cong \mathbb{Q}[i]^{2k+1}$). So, since S is the countable union of countable sets, it is itself countable. We then want to prove that S is dense in $l^2(\mathbb{Z})$. So, let $a = (a_n)_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$ be arbitrary. Let $\epsilon > 0$. For each $a_n \in \mathbb{C}$, we can select $b_n \in \mathbb{Q}[i]$ such that

$$|a_n - b_n|^2 < \frac{\epsilon}{2^{|n|+2}}$$

by the fact that $\mathbb{Q}[i]$ is dense in $\mathbb{C}$. This means that, if we form the vector $b = (b_n)_{n \in \mathbb{Z}} \in S$, we get

$$||a - b||^2 = \langle a - b, a - b \rangle = \sum_{n \in \mathbb{Z}} (a_n - b_n)\overline{(a_n - b_n)} = \sum_{n \in \mathbb{Z}} |a_n - b_n|^2$$

$$< \sum_{n \in \mathbb{Z}} \frac{\epsilon}{2^{|n|+2}} \leq 2 \sum_{n \geq 0} \frac{\epsilon}{2^{|n|+2}} = \sum_{n \geq 0} \frac{\epsilon}{2^{n+1}} = \sum_{n \geq 1} \frac{\epsilon}{2^n} = \epsilon$$

So, since this holds for all $\epsilon > 0$, we can arbitrarily approximate a with S. Since this holds for all $a \in l^2(\mathbb{Z})$, we get that S is dense in $l^2(\mathbb{Z})$. Since S is countable, therefore $l^2(\mathbb{Z})$ is separable.

To see completeness of $l^2(\mathbb{Z})$, we must prove that every Cauchy sequence converges. Let $(a^{(n)})_n \subset l^2(\mathbb{Z})$ be a Cauchy sequence, where each $a^{(n)} = (a_k^{(n)})_k \in l^2(\mathbb{Z})$ (we use upper indices to label the elements of the Cauchy sequence, and lower indices to label the coordinates of each element). Then, the Cauchy criterion grants that there is some N such that for all $m, n > N$,

$$||a^{(n)} - a^{(m)}||^2 < \epsilon^2 \implies \sum_{k \in \mathbb{Z}} |a_k^{(n)} - a_k^{(m)}|^2 < \epsilon^2$$

In particular, this means that since each term in the sum is nonnegative, each individual term is also nonnegative; so, $|a_k^{(n)} - a_k^{(m)}|^2 < \epsilon^2 \implies |a_k^{(n)} - a_k^{(m)}| < \epsilon$ for all k. Therefore, since this holds for all ϵ, we see that for each k, the sequence $\{a_k^{(n)}\}_n$ is Cauchy in $\mathbb{C}$; this means that each one must converge to some element, say $a_k \in \mathbb{C}$. Form the vector $a := (a_1, a_2, ...)$; we want to show (1) that $a \in l^2(\mathbb{Z})$, and (2) that $a^{(n)} \to a$ in the norm. We will do (2) first. Let $\epsilon > 0$. Note that for all N, since we can pass limits through finite sums and $a_k^{(m)} \to a_k$ for all k, we get

$$\sum_{|k| \leq N} \left|a_k^{(n)} - a_k\right|^2 = \lim_{m \to \infty} \sum_{|k| \leq N} \left|a_k^{(n)} - a_k^{(m)}\right|^2 \leq \lim_{m \to \infty} \sum_{k \in \mathbb{Z}} \left|a_k^{(n)} - a_k^{(m)}\right|^2 = \lim_{m \to \infty} ||a^{(n)} - a^{(m)}||^2,$$

Since $\{a^{(n)}\}_n$ is Cauchy, we can select an M big enough such that for all $n > M$, the last term is $< \epsilon^2$; note that this value of M doesn't depend on N. This means that for all $n > M$,

$$\sum_{|k| \leq N} \left|a_k^{(n)} - a_k\right|^2 < \epsilon^2 \quad \forall N,$$

Problem 1 continued on next page...

which in particular means that it must hold in the limit. In other words, for all $n > M$,

$$||a^{(n)} - a||^2 = \sum_{k \in \mathbb{Z}} \left| a_k^{(n)} - a_k \right|^2 < \epsilon^2 \implies ||a^{(n)} - a|| < \epsilon$$

Since such an M exists for all ϵ, we get that $a^{(n)} \to a$ in the norm.

With (2) done, (1) comes clearly from the triangle inequality with

$$||a|| \leq ||a - a^{(n)}|| + ||a^{(n)}|| < \infty,$$

where the first term is bounded for large enough n because $a^{(n)} \to a$ in the norm, and the second term is bounded because $a^{(n)} \in l^2(\mathbb{Z})$ for all n. So, this Cauchy sequence converges in $l^2(\mathbb{Z})$, which means that $l^2(\mathbb{Z})$ is complete. ■

Problem 2

Solution

Proof of (a). We start with a convenient lemma.

Lemma 1. *Let $E = [a, b) \subset [0, \infty)$. Then, we have that, for all $\alpha > 0$, there is some constant $c_\alpha = \alpha \cdot m(B_1)$ such that*

$$\int_{|x| \in E} \frac{1}{|x|^\alpha} dx = c_\alpha \int_a^\infty \frac{1}{t^{\alpha+1}} \cdot (\min\{b, t\}^n - a^n) dt$$

Proof of Lemma 1. Note by the regular rules of Riemann integration that

$$\frac{1}{|x|^\alpha} = \int_{|x|}^\infty \frac{\alpha}{t^{\alpha+1}} dt$$

So, we get that by Tonelli's theorem, since $\frac{1}{|x|^\alpha}$ is nonnegative,

$$\int_{|x| \in E} \frac{1}{|x|^\alpha} dx = \int_{|x| \in E} \int_{|x|}^\infty \frac{\alpha}{t^{\alpha+1}} dt dx = \int_{\mathbb{R}^n} \mathbb{1}_{|x| \in E} \int_{\mathbb{R}} \mathbb{1}_{\{|x| \leq t\}} \frac{\alpha}{t^{\alpha+1}} dt dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \mathbb{1}_{|x| \in E} \mathbb{1}_{\{|x| \leq t\}} \frac{\alpha}{t^{\alpha+1}} dx dt = \int_0^\infty \int_{\mathbb{R}^n} \mathbb{1}_{|x| \in E \cap [0,t]} \frac{\alpha}{t^{\alpha+1}} dx dt$$

$$= \int_0^\infty \frac{\alpha}{t^{\alpha+1}} \cdot m(|x| \in E \cap [0, t]) dt$$

We can note that $E \cap [0, t] = \begin{cases} [a, \min\{b, t\}] & a \leq t \\ \emptyset & else \end{cases}$. So, denoting B_1 as the unit ball and noting the realtive scale invariance of the measure, we get

$$m(|x| \in E \cap [0, t]) = m(B_1) \cdot \begin{cases} (\min\{b, t\})^n - a^n & a \leq t \\ 0 & else \end{cases}$$

This gives that

$$\int_{|x| \in E} \frac{1}{|x|^\alpha} dx = \alpha \cdot m(B_1) \cdot \int_a^\infty \frac{1}{t^{\alpha+1}} \cdot (\min\{b, t\}^n - a^n) dt$$

■

From here, proving that both inclusions fail simply reduces to applications of the lemma. Consider the functions $f, g : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{|x|^n} & |x| \geq 1 \\ 0 & else \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{|x|^{n/2}} & |x| < 1 \\ 0 & else \end{cases}$$

We can compute the L^1 and L^2 norms of both of these functions easily, with $E_f := [a_f, b_f) = [1, \infty)$ and $E_g := [a_g, b_g) = [0, 1)$. We get from application of Lemma 1 and routine use of Riemann integration that

$$||f||_{L^1} = \int_{E_f} \frac{1}{|x|^n} dx = c_n \int_1^\infty \frac{1}{t^{n+1}} \cdot (t^n - 1) dt = c_n \int_1^\infty \frac{1}{t} dt - c_n \int_1^\infty \frac{1}{t^{n+1}} dt$$

$$= c_n \cdot \infty - c_n \cdot \left[\frac{-1}{n} t^{-n} \right]_1^\infty = c_n \cdot \infty - \frac{c_n}{n} = \infty \implies f \notin L^1(\mathbb{R}^n)$$

Similarly,

$$||f||_{L^2}^2 = \int_{E_f} \frac{1}{|x|^{2n}} dx = c_{2n} \int_1^\infty \frac{1}{t^{2n+1}} \cdot (t^n - 1) dt = c_{2n} \int_1^\infty \frac{1}{t^{n+1}} dt - c_{2n} \int_1^\infty \frac{1}{t^{2n+1}} dt$$
$$= c_{2n} \left[\frac{-1}{n} t^{-n} \right]_1^\infty - c_{2n} \cdot \left[\frac{-1}{2n} t^{-2n} \right]_1^\infty = \frac{c_{2n}}{2n} < \infty \implies f \in L^2(\mathbb{R}^n)$$

So, $f \in L^2(\mathbb{R}^n) \setminus L^1(\mathbb{R}^n)$ proves that $L^2(\mathbb{R}^n) \not\subset L^1(\mathbb{R}^n)$. Similar logic applies for g. We compute via the lemma and Riemann integration that

$$||g||_{L^1} = \int_{E_g} \frac{1}{|x|^{n/2}} dx = c_{n/2} \int_0^\infty \frac{1}{t^{n/2+1}} \cdot \min\{t, 1\}^n dt = c_{n/2} \int_0^1 t^{n/2-1} dt + c_{n/2} \int_1^\infty \frac{1}{t^{n/2+1}} dt$$
$$= c_{n/2} + c_{n/2} \cdot \left[\frac{-n}{2} t^{-n/2} \right]_1^\infty = c_{n/2} + \frac{c_{n/2} \cdot n}{2} < \infty \implies g \in L^1(\mathbb{R}^n)$$

and

$$||g||_{L^2}^2 = \int_{E_g} \frac{1}{|x|^n} dx = c_n \int_0^\infty \frac{1}{t^{n+1}} \cdot \min\{t, 1\}^n dt = c_n \int_0^1 \frac{1}{t} dt + c_n \int_1^\infty \frac{1}{t^{n+1}} dt$$
$$= c_n \cdot \infty + c_n \cdot \left[\frac{-1}{n} t^{-n} \right]_1^\infty = c_n \cdot \infty + \frac{c_n}{n} = \infty \implies g \notin L^2(\mathbb{R}^n)$$

So, $g \in L^1(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$ proves that $L^1(\mathbb{R}^n) \not\subset L^2(\mathbb{R}^n)$. Therefore, no such inclusion can hold. ■

Proof of (b). Suppose that $f \in L^2(\mathbb{R}^n) \implies |f| \in L^2(\mathbb{R}^n)$ is supported on a set $E \subset \mathbb{R}^n$ of finite measure (note that this means $\mathbb{1}_E \in L^2(\mathbb{R}^n)$). Then, we can observe that

$$||f||_{L^1} = \int_{\mathbb{R}^n} |f| = \int_{\mathbb{R}^n} |f| \cdot \mathbb{1}_E = \langle |f|, \mathbb{1}_E \rangle_{L^2}$$

By Cauchy-Schwarz and the fact that the L^2 norms of f and $|f|$ agree, we get

$$||f||_{L^1} = \langle |f|, \mathbb{1}_E \rangle_{L^2} \leq ||f||_{L^2} \cdot ||\mathbb{1}_E||_{L^2}$$

Since $||\mathbb{1}_E||_{L^2}^2 = \int_{\mathbb{R}^n} |\mathbb{1}_E|^2 = \int_{\mathbb{R}^n} \mathbb{1}_E = m(E)$, we conclude that

$$||f||_{L^1} \leq m(E)^{1/2} \cdot ||f||_{L^2},$$

and so $f \in L^2(\mathbb{R}^n) \implies f \in L^1(\mathbb{R}^n)$. ■

Proof of (c). Suppose now that $f \in L^1(\mathbb{R}^n)$ is bounded (i.e. $|f(x)| < M \quad \forall x \in \mathbb{R}^n$). Then, we can observe that

$$||f||_{L^2}^2 = \int_{\mathbb{R}^n} |f|^2 \leq \int_{\mathbb{R}^n} |f| \cdot M = M \int_{\mathbb{R}^n} |f| = M ||f||_{L^1}$$

Taking the square root, we conclude that

$$||f||_{L^2} \leq M^{1/2} \cdot ||f||_{L^1}^{1/2},$$

and so $f \in L^1(\mathbb{R}^n) \implies f \in L^2(\mathbb{R}^n)$. ■

Problem 3

Solution

Proof of (a). Write the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as given, and define a map $U : \mathcal{H}_1 \to \mathcal{H}_2$ that sends $F \mapsto f$ as determined by

$$f(x) := \frac{1}{\sqrt{\pi}(i+x)} F\left(\frac{i-x}{i+x}\right)$$

We wish to show that the map U is unitary. Firstly, observe that it certainly is linear, as we get that

$$U(\alpha F + \beta G) = \frac{1}{\sqrt{\pi}(i+x)}(\alpha F + \beta G)\left(\frac{i-x}{i+x}\right)$$

$$= \alpha\left(\frac{1}{\sqrt{\pi}(i+x)} F\left(\frac{i-x}{i+x}\right)\right) + \beta\left(\frac{1}{\sqrt{\pi}(i+x)} G\left(\frac{i-x}{i+x}\right)\right) = \alpha UF + \beta UG$$

Secondly, it is definitely injective, as it has a trivial kernel; to see this, note that the only way that $UF \equiv 0$ is if $F(i - x/i + x) = 0$ for all x, which only happens if $F \equiv 0$. So, in order to prove that U is unitary, it suffices to show that it is both surjective and norm-preserving.

Let $F \in L^2([-\pi, \pi])$ be arbitrary. To see norm-preserving, we can make use of the change of variables formula found in Exercise 21 of Chapter 3. To begin with, note that the function $x : \mathbb{R} \to \mathbb{R}$ given by $x(\theta) := \tan(\theta/2)$ is bounded and increasing on $[-a, a]$ for any $0 \le a < \pi$, since tan is monotonic and doesn't diverge over such intervals (interestingly, $x(\theta)$ is also surjective, which will come in handy later). Then, since we can express $x(\theta)$ as a difference of two bounded, monotonic functions (namely, $\tan(\theta/2)$ and 0), we get that x is absolutely continuous on $[-a, a]$ for all $a < \pi$. Furthermore, $x(\theta)$ is differentiable with

$$x'(\theta) = \frac{\sec^2(\theta/2)}{2} = \frac{1 + x(\theta)^2}{2}$$

This grants that, for all $a < \pi$, we can apply the change of variables formula on the second line to get that

$$\begin{aligned}\int_{\tan(-a/2)}^{\tan(a/2)} \frac{1}{\pi} \cdot \frac{1}{|i+x|^2}\left|F\left(\frac{i-x}{i+x}\right)\right|^2 dx &= \frac{1}{\pi}\int_{\tan(-a/2)}^{\tan(a/2)} \frac{1}{1+x^2}\left|F\left(\frac{i-x}{i+x}\right)\right|^2 dx \\ &= \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+x(\theta)^2}\left|F\left(\frac{i-x(\theta)}{i+x(\theta)}\right)\right|^2 \cdot x'(\theta) d\theta \\ &= \frac{1}{2\pi}\int_{-a}^{a}\left|F\left(\frac{i-\tan(\theta/2)}{i+\tan(\theta/2)}\right)\right|^2 d\theta\end{aligned}$$

Multiplying top and bottom by $-i\cos(\theta/2)$, we get

$$= \frac{1}{2\pi}\int_{-a}^{a}\left|F\left(\frac{\cos(\theta/2) + i\sin(\theta/2)}{\cos(\theta/2) - i\sin(\theta/2)}\right)\right|^2 d\theta = \frac{1}{2\pi}\int_{-a}^{a}\left|F\left(\frac{e^{i\theta/2}}{e^{-i\theta/2}}\right)\right|^2 d\theta = \frac{1}{2\pi}\int_{-a}^{a}\left|F\left(e^{i\theta}\right)\right|^2 d\theta$$

Taking the limit as $a \to \pi$ (which means $\tan(a/2) \to \infty$ and $\tan(-a/2) \to -\infty$), we get the relation

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \cdot \frac{1}{|i+x|^2}\left|F\left(\frac{i-x}{i+x}\right)\right|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|F\left(e^{i\theta}\right)\right|^2 d\theta$$

Note that the LHS is precisely equivalent to $||UF||^2_{\mathcal{H}_2}$, while the RHS is precisely equivalent to $\langle F, F\rangle_{\mathcal{H}_1} = ||F||^2_{\mathcal{H}_1}$. This proves immediately that U is norm-preserving, as desired. Note that this implies that $UF \in \mathcal{H}_2$ for every $F \in \mathcal{H}_1$, and so it is a valid mapping.

Problem 3 continued on next page. . .

To show that U is surjective, we will define a map V and prove that $V = U^{-1}$. Let $V : \mathcal{H}_2 \to h_1$ be the map that sends $f \to F$, where

$$F(e^{i\theta}) := \sqrt{\pi} \cdot (i + \tan(\theta/2)) \cdot f(\tan(\theta/2))$$

A very similar change of variables shows that $||Vf||_{\mathcal{H}_1} = ||f||_{\mathcal{H}_2}$ for every $f \in \mathcal{H}_2$, which reveals that $Vf \in \mathcal{H}_1$, and so V is a valid mapping. Now, for every $F \in \mathcal{H}_1$ we have that

$$\begin{aligned}
(VUF)(e^{i\theta}) &= \sqrt{\pi} \cdot (i + \tan(\theta/2)) \cdot (UF)(\tan(\theta/2)) \\
&= \sqrt{\pi} \cdot (i + \tan(\theta/2)) \cdot \frac{1}{\sqrt{\pi}(i + \tan(\theta/2))} F\left(\frac{i - \tan(\theta/2)}{i + \tan(\theta/2)}\right) \\
&= F\left(\frac{\cos(\theta/2) + i\sin(\theta/2)}{\cos(\theta/2) - i\sin(\theta/2)}\right) = F\left(\frac{e^{i\theta/2}}{e^{-i\theta/2}}\right) \\
&= F(e^{i\theta}) \quad \forall \theta
\end{aligned}$$

Similarly, we get that for every $f \in \mathcal{H}_2$,

$$\begin{aligned}
(UVf)(\tan(\theta/2)) &= \frac{1}{\sqrt{\pi}(i + \tan(\theta/2))}(Vf)(\tan(\theta/2)) \\
&= \frac{1}{\sqrt{\pi}(i + \tan(\theta/2))} \cdot \sqrt{\pi} \cdot (i + \tan(\theta/2)) \cdot f(\tan(\theta/2)) \\
&= f(\tan(\theta/2))
\end{aligned}$$

Since $x(\theta)$ is surjective, this means that $\tan(\theta/2)$ will hit every possible value for x as we vary θ, and so

$$(UVf)(x) = f(x) \quad \forall x$$

These two results prove that U and V are inverses, which in particular proves that they're both bijective. So, U is a bijective, norm-preserving linear map, and thus is a unitary operator. ■

Proof of (b). Now, we can note that the functions $\{F_n(e^{i\theta}) := e^{in\theta}\}_{n\in\mathbb{Z}} \subset \mathcal{H}_1$ actually form an orthonormal basis for $\mathcal{H}_1$. To see this, note that

$$\langle F_n, F_n \rangle_{\mathcal{H}_1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 d\theta = 1,$$

while for $m \neq n$ we get

$$\langle F_n, F_m \rangle_{\mathcal{H}_1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\theta}d\theta = \frac{1}{2\pi i(n-m)}\left[e^{i(n-m)\theta}\right]_{-\pi}^{\pi} = 0$$

So, the functions $\{F_n\}_{n\in\mathbb{Z}}$ form an set basis for $\mathcal{H}_1$. The fact that they form a basis comes from the well-known Fourier decomposition of $L^2([-\pi, \pi])$. By the properties of unitary maps, this means that the functions $\{UF_n\}_{n\in\mathbb{Z}}$ form an orthormal basis for $\mathcal{H}_2$. We can simply compute the image of these functions under U to see that

$$\left\{\frac{1}{\sqrt{\pi}}\frac{1}{i+x}\left(\frac{i-x}{i+x}\right)^n\right\}_{n\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(\mathbb{R})$, as desired. ■

Problem 4

Solution

Proof. Note first that $(S^\perp)^\perp \supset S$, since for every $f \in S$ we have $\langle f, g\rangle = 0$ for all $g \in S^\perp$, and so $f \in (S^\perp)^\perp$. Also, $(S^\perp)^\perp$ is closed since orthogonal complements are closed. Now, suppose by way of contradiction that $(S^\perp)^\perp$ is **not** the smallest closed subspace of $\mathcal{H}$ that contains S; that is, suppose that there is some smaller closed subspace V of $\mathcal{H}$ such that $S \subset V \subsetneq (S^\perp)^\perp$. Then, there must be some nonzero element $f \in (S^\perp)^\perp \backslash V$. Since V is a closed subspace, by Proposition 4.2 we get that $\mathcal{H} = V \oplus V^\perp$, which means that f decomposes into $f = f_V + f_{V^\perp}$ with $f_V \in V$ and $f_{V^\perp} \in V^\perp$; since $f \notin V$, we know that $f_{V^\perp} \neq 0$. Also, by the properties of subspaces, $f_{V^\perp} \in (S^\perp)^\perp$. So, we find a nonzero element in $(S^\perp)^\perp \cap V^\perp$; for ease of notation, we will call this element simply as f from now on. First, we will prove the following lemma:

Lemma 2. *If $A, B \subset \mathcal{H}$ are subspaces with $A \subset B$, then $B^\perp \subset A^\perp$.*

Proof of Lemma 2. Suppose that $x \in B^\perp$, which means that $\langle x, b\rangle = 0$ for each $b \in B$. In particular, this means that for every $a \in A \implies a \in B$, we have $\langle x, a\rangle = 0$. So, $x \in A^\perp$. Since this holds for every $x \in B^\perp$, we get $B^\perp \subset A^\perp$. ∎

Applying the lemma to $S \subset V$, we get that $V^\perp \subset S^\perp$. Applying it again, we get that $(S^\perp)^\perp \subset (V^\perp)^\perp$. This means that

$$f \in (S^\perp)^\perp \cap V^\perp \implies f \in (V^\perp)^\perp \cap V^\perp \implies f \in (V^\perp)^\perp \text{ and } f \in V^\perp$$

So, we must have that $f \equiv 0$, since the intersection of a closed subspace and its orthogonal complement is $\{0\}$. However, this contradicts our selection of f as nonzero, and the result follows. ∎

Problem 5

Solution

Throughout this problem, since S is closed, we can use the orthogonal decomposition $\mathcal{H} = S \oplus S^\perp$ to see that for every $f \in \mathcal{H}$, we have $f = Pf + (f - Pf)$, where $Pf \in S$ and $f - Pf \in S^\perp$.

Proof of (a). Note that $Pf \in S$ for all $f \in \mathcal{H}$ by the remark in the beginning of the problem. So, $P(Pf) = Pf$ for $f \in S$ by definition of P; it also holds that for $f \notin S$ we have $Pf = 0 \implies P(Pf) = 0 = Pf$, and so $P^2 = P$. Also, for every $f, g \in \mathcal{H}$, we have

$$\langle Pf, g\rangle = \langle Pf, Pg + (g - Pg)\rangle = \langle Pf, Pg\rangle + \langle Pf, g - Pg\rangle$$

Since $Pf \in S$ and $g - Pg \in S^\perp$, we know $\langle Pf, g - Pg\rangle = 0$. Similarly, $\langle f - Pf, Pg\rangle = 0$. So, by additivity of $\langle \cdot, \cdot\rangle$,

$$\langle Pf, g\rangle = \langle Pf, Pg\rangle + 0 = \langle Pf, Pg\rangle + \langle f - Pf, Pg\rangle = \langle f, Pg\rangle$$

Since this holds for every $f, g \in \mathcal{H}$ and the adjoint is unique, we have $P = P^*$. (Note that $P = P^*$ certainly has the other two properties of an adjoint trivially, as $||P||_{op} = ||P^*||_{op}$ and $(P^*)^* = P^* = P$). ■

Proof of (b). Suppose $P = P^* = P^2$ is a bounded (and therefore continuous by Proposition 5.2) operator on $\mathcal{H}$. Define $V := \ker P$ to be the kernel of P. Since P is continuous, V is closed (to see this, take any convergent sequence of points in the kernel of P and use that continuity means the limit will have value 0 and also be in the kernel). Therefore, V is the smallest closed subspace containing V; combined with the result from Problem 4, this gives us that $(V^\perp)^\perp = V$. Then, we can apply Theorem 2.4 and write the orthogonal decomposition

$$\mathcal{H} = V^\perp \oplus (V^\perp)^\perp$$

We want to show that P takes the form

$$Pf = \begin{cases} f & \text{if } f \in V^\perp \\ 0 & \text{if } f \in (V^\perp)^\perp = V \end{cases}$$

as this will prove that P is the projection operator for the closed subspace $V^\perp$ (it is closed since orthogonal subspaces are always closed). To show this, note that if $f \in (V^\perp)^\perp = V$, then trivially $Pf = 0$ (this is the definition of the space $V = \ker P$). So, it is left to show that for any $f \in V^\perp$ we have $Pf = f$. Suppose that $f \in V^\perp$. For all $g \in \mathcal{H}$, since $P = P^*$ we have

$$\langle Pf, g\rangle = \langle f, Pg\rangle$$

Since $g = Pg + (g - Pg)$ with $Pg \in V^\perp$ and $g - Pg \in (V^\perp)^\perp = V$, we also get

$$\langle f, g\rangle = \langle f, Pg + (g - Pg)\rangle = \langle f, Pg\rangle + \langle f, g - Pg\rangle = \langle f, Pg\rangle,$$

where $\langle f, g - Pg\rangle = 0$ because $f \in V^\perp$ and $g - Pg \in V$. This means that for every $g \in \mathcal{H}$ we have

$$\langle Pf, g\rangle = \langle f, g\rangle \implies \langle Pf - f, g\rangle = 0,$$

and so $Pf - f = 0 \implies Pf = f$. Since this holds for all $f \in V^\perp$, then P indeed takes the form as a projection onto the closed subspace $V^\perp$, and we are done. ■

Proof of (c). Let $S \subset \mathcal{H}$ be any closed subspace, and let P be the projection operator to S. We know from class that S is also a Hilbert space with the induced inner product, which means that $||\cdot||_S$ agrees with $||\cdot||_{\mathcal{H}}$ agrees for all elements of S. Because $\mathcal{H}$ is separable, let $\{f_k\}_{k=1}^\infty$ be a countable dense

Problem 5 continued on next page...

subset of $\mathcal{H}$. Then, note that $Pf_k \in S$ for each k by definition of projection operators; we want to show that $\{Pf_k\}_{k=1}^{\infty}$ is dense in S, as this will reveal that S has a countable dense subset and is therefore separable. To that end, let $f \in S$ be arbitrary. Let $\epsilon > 0$. We want to show that $||f - Pf_k||_S < \epsilon$ for some k. Select an $f_k \in \{f_k\}_{k=1}^{\infty}$ such that

$$||f - f_k||_{\mathcal{H}} < \epsilon$$

Such an f_k exists because $\{f_k\}_{k=1}^{\infty}$ is dense in $\mathcal{H}$. Since S is a closed subspace, we can use orthogonal decomposition (Theorem 2.4) to write any arbitrary $g \in \mathcal{H}$ as $g = Pg + (g - Pg)$, where $Pg \in S$ and $g - Pg \in S^{\perp}$. The Pythagorean Theorem grants that, since $Pg \perp g - Pg$, then

$$||g||_{\mathcal{H}}^2 = ||Pg||_{\mathcal{H}}^2 + ||g - Pg||_{\mathcal{H}}^2 + 2Re\langle Pg, g - Pg\rangle = ||Pg||_{\mathcal{H}}^2 + ||g - Pg||_{\mathcal{H}}^2,$$

and so

$$||Pg||_{\mathcal{H}}^2 \leq ||g||_{\mathcal{H}}^2 \implies ||Pg||_{\mathcal{H}} \leq ||g||_{\mathcal{H}} \quad \forall g \in \mathcal{H}$$

Note also that $f \in S \implies Pf = f$ by the logic used in earlier parts of the proof. Then, we have that since $||\cdot||_S$ agrees with $||\cdot||_{\mathcal{H}}$ on all elements of S,

$$||f - Pf_k||_S = ||Pf - Pf_k||_S = ||P(f - f_k)||_S = ||P(f - f_k)||_{\mathcal{H}} \leq ||f - f_k||_{\mathcal{H}} < \epsilon,$$

where the bound by ϵ comes from our selection of f_k. This shows that there exists a $Pf_k \in \{Pf_k\}_k \subset S$ such that $||f - Pf_k||_s < \epsilon$. Since this holds for all $\epsilon > 0$, we find that $\{Pf_k\}_k$ approximates f. Since this holds for all $f \in S$, we see that $\{Pf_k\}_k$ is dense in S. So, since S has a countable dense subset, it is therefore separable. ■

Problem 6

Solution

Proof. Let $\mathcal{H}, \mathcal{H}'$ be two completions of a pre-Hilbert space $\mathcal{H}_0$; they are therefore complete metric spaces. By Proposition 2.7, this gives us that (i) $\mathcal{H}_0 \subset \mathcal{H}$ and $\mathcal{H}_0 \subset \mathcal{H}'$, (ii) $\langle f, g\rangle_{\mathcal{H}_0} = \langle f, g\rangle_{\mathcal{H}} = \langle f, g\rangle_{\mathcal{H}'}$ whenever $f, g \in \mathcal{H}_0$, and (iii) $\mathcal{H}_0$ is dense in both $\mathcal{H}$ and $\mathcal{H}'$. We would like to construct a unitary mapping $U : \mathcal{H} \to \mathcal{H}'$ such that $Uf = f$ for all $f \in \mathcal{H}_0$. We construct it as follows:

1. Let $g \in \mathcal{H}$ be arbitrary.
2. Since $\mathcal{H}_0$ is dense in $\mathcal{H}$, there must exist some Cauchy sequence $\{f_n\}_n \subset \mathcal{H}_0$ that converges to g in $||\cdot||_{\mathcal{H}}$ (we can get arbitarily close by the density property and $\mathcal{H}$ is complete).
3. This Cauchy sequence $\{f_n\}_n \subset \mathcal{H}_0 \subset \mathcal{H}'$ must converge to some element $g' \in \mathcal{H}'$ because $\mathcal{H}'$ is also complete.
4. Define $Ug = g'$; that is, U maps limit points of Cauchy sequences in $\mathcal{H}_0$ from their limit point in $\mathcal{H}$ to their limit point in $\mathcal{H}'$.

We want to show that this is a unitary mapping that is the identity when restricted to $\mathcal{H}_0$. Firstly, note that it is indeed a mapping, since $Ug \in \mathcal{H}'$ for every $g \in \mathcal{H}$ by step 4 of the construction, and it is well-defined by uniqueness of limit points in $\mathcal{H}'$. Now, observe that for any $f \in \mathcal{H}_0$, we know that the Cauchy sequence $\{f_n\}_n$ converges to f in $||\cdot||_{\mathcal{H}}$ by step 2; however, since $f_n, f \in \mathcal{H}_0$ and $||\cdot||_{\mathcal{H}_0} = ||\cdot||_{\mathcal{H}} = ||\cdot||_{\mathcal{H}'}$ over $\mathcal{H}_0$ by property (ii), we must therefore have that $f_n \to f$ in $||\cdot||_{\mathcal{H}'}$. In particular, this means that $Uf = f$, since we defined Uf to be the element that $\{f_n\}_n$ converges to in $||\cdot||_{\mathcal{H}'}$. So, U restricted to $\mathcal{H}_0$ is indeed the identity.

Next, we would like to show that U is linear. To this end, let $\alpha \in \mathbb{C}$, and let $f \in \mathcal{H}$. There is some Cauchy sequence $\{f_n\}_n \subset \mathcal{H}_0$ such that $f_n \to f$ in $||\cdot||_{\mathcal{H}}$ by step 2. Note that this means that $\{\alpha f_n\}_n$ is still a Cauchy sequence in $\mathcal{H}_0$ (scaling by a fixed constant still allows elements to get arbitrarily close). To see that $\alpha f_n \to \alpha f$ in $||\cdot||_{\mathcal{H}}$, note that

$$||f_n - f||_{\mathcal{H}} = |1/\alpha| ||\alpha f_n - \alpha f||_{\mathcal{H}} \implies ||\alpha f_n - \alpha f||_{\mathcal{H}} \to 0 \text{ if } ||f_n - f||_{\mathcal{H}} \to 0$$

Now, we also knew that $f_n \to Uf$ in $||\cdot||_{\mathcal{H}'}$ by definition of Uf. Symmetric logic then shows that $\alpha f_n \to \alpha Uf$ in $||\cdot||_{\mathcal{H}'}$, and so $U(\alpha f) = \alpha Uf$ since αf and αUf are the $\mathcal{H}, \mathcal{H}'$ limit points of the same Cauchy sequence, respectively. Next, note that for two $f, g \in \mathcal{H}$ with corresponding Cauchy sequences $\{f_n\}_n, \{g_n\}_n \subset \mathcal{H}_0$ that converge to f, g in $||\cdot||_{\mathcal{H}}$ respectively, we get

$$||(f_n + g_n) - (f + g)||_{\mathcal{H}} = ||(f_n - f) + (g_n - g)||_{\mathcal{H}} \le ||f_n - f||_{\mathcal{H}} + ||g_n - g||_{\mathcal{H}} \to 0$$

So, $\{f_n + g_n\}_n \subset \mathcal{H}_0$ is a Cauchy sequence in $\mathcal{H}_0$ that converges in $||\cdot||_{\mathcal{H}}$ to $f + g$ (it is Cauchy by another application of the triangle inequality to the Cauchy criterion). Symmetric logic shows that

$$||(f_n + g_n) - (Uf + Ug)||_{\mathcal{H}'} = ||(f_n - Uf) + (g_n - Ug)||_{\mathcal{H}'} \le ||f_n - Uf||_{\mathcal{H}'} + ||g_n - Ug||_{\mathcal{H}'} \to 0$$

by the definitions of Uf, Ug as the elements of $\mathcal{H}'$ that $\{f_n\}_n, \{g_n\}_n$ converge to, respectively. So, this means that $U(f + g) = Uf + Ug$, and therefore that U is linear.

U is clearly bijective, as we can easily define its inverse by switching the roles of $\mathcal{H}$ and $\mathcal{H}'$ in the construction. This inverse will be both a left and right inverse by the uniqueness of limits, which grants us that

Problem 6 continued on next page...

U is bijective. So, all that remains is to prove that U preserves norms. To this end, let $f \in \mathcal{H}$. Note that the triangle inequality grants that, if $\{f_n\}_n$ is the Cauchy sequence in $\mathcal{H}_0$ that converges to f in $||\cdot||_{\mathcal{H}}$, then

$$||f||_{\mathcal{H}} \leq ||f - f_n||_{\mathcal{H}} + ||f_n||_{\mathcal{H}} \xrightarrow[n\to\infty]{} \lim_{n\to\infty} ||f_n||_{\mathcal{H}}$$

and

$$\lim_{n\to\infty} ||f_n||_{\mathcal{H}} \leq \lim_{n\to\infty} ||f_n - f||_{\mathcal{H}} + ||f|| + \mathcal{H} = ||f||_{\mathcal{H}}$$

So,

$$||f||_{\mathcal{H}} = \lim_{n\to\infty} ||f_n||_{\mathcal{H}}$$

Similar logic shows that

$$||Uf||_{\mathcal{H}'} = \lim_{n\to\infty} ||f_n||_{\mathcal{H}'}$$

However, by property (ii), we know that $\lim_{n\to\infty} ||f_n||_{\mathcal{H}} = \lim_{n\to\infty} ||f_n||_{\mathcal{H}'}$, and therefore that $||Uf||_{\mathcal{H}'} = ||f||_{\mathcal{H}}$. So, U preserves norms, and therefore it is unitary. ■

Problem 7

Solution

Proof. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$, where $\mathcal{H}_1$ is finite-dimensional, say with $\dim \mathcal{H}_1 = N$. Let $\{\varphi_k\}_{k=1}^N$ be an orthonormal basis for $\mathcal{H}_1$ (we know one must exist by Theorem 2.4). Let $f \in \mathcal{H}_1$ be arbitrary. Note that we can express $f = \sum_{k=1}^N a_k \varphi_k$ for some a_k's by the properties of a basis. Also, Parseval's identity gives that since the basis is orthonormal

$$||f||_{\mathcal{H}_1}^2 = \sum_{k=1}^N |a_k|^2$$

Therefore, by linearity of T and the triangle inequality,

$$||Tf||_{\mathcal{H}_2}^2 = \left\| T\left(\sum_{k=1}^N a_k \varphi_k\right) \right\|_{\mathcal{H}_2}^2 = \left\| \sum_{k=1}^N a_k T(\varphi_k) \right\|_{\mathcal{H}_2}^2 \leq \sum_{k=1}^N |a_k|^2 ||T(\varphi_k)||_{\mathcal{H}_2}^2$$

If we let

$$M^2 := \max_{k \in \{1,\ldots,N\}} ||T(\varphi_k)||_{\mathcal{H}_2}^2 < \infty,$$

then we get that

$$||Tf||_{\mathcal{H}_2}^2 \leq M \sum_{k=1}^N |a_k|^2 = M^2 ||f||_{\mathcal{H}_1}^2 \implies ||Tf||_{\mathcal{H}_2} \leq M ||f||_{\mathcal{H}_1}$$

Since this holds for every $f \in \mathcal{H}_1$ and the bound M doesn't depend on f, this reveals that T is bounded. ■

Problem 8

Solution

Proof of (a). Suppose that B is such that $||Tv|| \leq B||v||$ for all nonzero v, and so $\frac{||Tv||}{||v||} \leq B \implies \left\|T\left(\frac{v}{||v||}\right)\right\| \leq B$ for all nonzero v. Equivalently, $||Tw|| \leq B$ for all w with unit norm, since $\frac{v}{||v||}$ will always be unit norm. Since the conditions are equivalent for every such B, we certainly have

$$||T||_{op} := \inf\{B : ||Tv|| \leq B||v|| \text{ for all } v \in \mathcal{H}\} = \inf\{B : ||Tv|| \leq B \text{ for all } ||v|| = 1\}$$

Clearly, $||T||_{op} \geq ||Tv||$ for every unit vector $||v|| = 1$, since this is the case for every B, and so it must hold for the infimum over such B's. Therefore, it must hold over the supremum over unit vectors; that is,

$$||T||_{op} \geq \sup\{||Tv|| : ||v|| = 1\}$$

To see the other direction, note that $||Tv|| \leq \sup\{||Tv|| : ||v|| = 1\}$ for every unit vector $||v|| = 1$ by definition of supremum. So, the real number $\sup\{||Tv|| : ||v|| = 1\}$ upper bounds $||Tv||$ over the unit sphere; since $||T||_{op}$ is the infimum over such upper bounds, we must have

$$||T||_{op} \leq \sup\{||Tv|| : ||v|| = 1\}$$

by definition of infimum. The result follows. ■

Proof of (b). From part (a), we have that

$$||T_1 + T_2||_{op} = \sup\{||T_1 v + T_2 v|| : ||v|| = 1\}$$

Let $\epsilon > 0$ be arbitrary and let v be a unit vector $||v|| = 1$ such that

$$||T_1 + T_2||_{op} - \epsilon \leq ||T_1 v + T_2 v|| \leq ||T_1 + T_2||_{op}$$

Note that this can always be done, since we can realize values arbitrarily close to the supremum. Then, the triangle inequality and the fact that $||T_i v|| \leq ||T_i||_{op}$ for $i = 1, 2$ give

$$||T_1 + T_2||_{op} - \epsilon \leq ||T_1 v + T_2 v|| \leq ||T_1 v|| + ||T_2 v|| \leq ||T_1||_{op} + ||T_2||_{op}$$

So,

$$||T_1 + T_2||_{op} \leq ||T_1||_{op} + ||T_2||_{op} + \epsilon$$

for every $\epsilon > 0$. Taking $\epsilon \to 0$, we get the desired result. ■

Proof of (c). In order for $d(T_1, T_2) := ||T_1 - T_2||_{op}$ to be a metric, the following properties must hold for every $T_1, T_2, T_3 \in \mathcal{L}(\mathcal{H})$:

1. $d(T_1, T_2) \geq 0$
2. $d(T_1, T_2) = d(T_2, T_1)$
3. $d(T_1, T_3) \leq d(T_1, T_2) + d(T_2, T_3)$
4. $d(T_1, T_2) = 0 \iff T_1 = T_2$

The first condition holds trivially, as $|| \cdot ||_{\mathcal{H}}$ is always nonnegative, and so the supremum over such norms must also be nonnegative, which means that $|| \cdot ||_{op} \geq 0$. The second condition also holds easily, since $||T_1 v - T_2 v|| = ||T_2 v - T_1 v||$ for all $v \in \mathcal{H}$ means

$$||T_1 - T_2||_{op} = \sup\{||T_1 v - T_2 v|| : ||v|| = 1\} = \sup\{||T_2 v - T_1 v|| : ||v|| = 1\} = ||T_2 - T_1||_{op}$$

Problem 8 continued on next page...

For the third condition, we can apply the result from part b to see

$$||T_1 - T_3||_{op} = ||(T_1 - T_2) + (T_2 - T_3)||_{op} \leq ||T_1 - T_2||_{op} + ||T_2 - T_3||_{op}$$

For the last condition, suppose first that $d(T_1, T_2) = 0$. Then, nonnegativity of the norm $||\cdot||_{\mathcal{H}}$ yields

$$\sup\{||T_1 v - T_2 v|| : ||v|| = 1\} = 0 \implies ||T_1 v - T_2 v|| = 0 \text{ for all unit vectors } v$$

So, this means that $T_1 v = T_2 v$ for every unit vector v, which means that T_1 and T_2 must agree on an every element of an orthonormal basis of $\mathcal{H}$. Therefore, T_1 and T_2 must agree on *all* elements of $\mathcal{H}$ by linearity, and so $T_1 = T_2$. For the other direction, suppose that $T_1 = T_2$. Then, $||T_1 v - T_2 v|| = 0$ for all $v \in \mathcal{H}$, and clearly $||T_1 - T_2||_{op} = \sup\{||T_1 v - T_2 v|| : ||v|| = 1\} = 0$. So, d is a metric on $\mathcal{L}(\mathcal{H})$. ■

Proof of (d). To show that $\mathcal{L}(\mathcal{H})$ is complete in the metric d, we must show that every Cauchy sequence converges in $\mathcal{L}(\mathcal{H})$. So, let $(T_n)_{n=1}^{\infty} \subset \mathcal{L}(\mathcal{H})$ be a Cauchy sequence. We can always find a subsequence $(T_{n_k})_{k=1}^{\infty}$ with the property that

$$\left\|T_{n_{k+1}} - T_{n_k}\right\|_{op} \leq 2^{-k} \quad \forall k \geq 1$$

by repeated application of the Cauchy criterion (take $\epsilon_k = 2^{-k}$). Consider the operators defined by

$$S_N := T_{n_1} + \sum_{k=1}^{N} \left(T_{n_{k+1}} - T_{n_k}\right)$$

and

$$T := T_{n_1} + \sum_{k=1}^{\infty} \left(T_{n_{k+1}} - T_{n_k}\right)$$

Now, note that T is certainly linear, and so

$$\sup_{||v||=1} ||Tv|| = \sup_{||v||=1} \left\| T_{n_1} v + \sum_{k=1}^{\infty} \left(T_{n_{k+1}} - T_{n_k}\right) v \right\|$$

The triangle inequality (which can be applied countably many times because of the continuity of the norm $||\cdot||_{\mathcal{H}}$) and simple reasoning about suprema yield

$$\begin{aligned}
||T||_{op} = \sup_{||v||=1} ||Tv|| &\leq \sup_{||v||=1} \left\{ ||T_{n_1} v|| + \sum_{k=1}^{\infty} \left\| \left(T_{n_{k+1}} - T_{n_k}\right) v \right\| \right\} \\
&\leq \sup_{||v||=1} ||T_{n_1} v|| + \sum_{k=1}^{\infty} \left(\sup_{||v||=1} \left\| \left(T_{n_{k+1}} - T_{n_k}\right) v \right\| \right) \\
&= ||T_{n_1}||_{op} + \sum_{k=1}^{\infty} \left\|T_{n_{k+1}} - T_{n_k}\right\|_{op} \\
&\leq ||T_{n_1}||_{op} + \sum_{k=1}^{\infty} 2^{-k} \\
&= ||T_{n_1}||_{op} + 1 < \infty
\end{aligned}$$

So, T is bounded, and is therefore in $\mathcal{L}(\mathcal{H})$. Now, note that by the properties of telescoping sums, $S_N = T_{n_{N+1}}$ for all N. So, to show $T_{n_k} \to T$ in $\mathcal{L}(\mathcal{H})$ as $k \to \infty$,

$$||T - T_{n_N}||_{op} = ||T - S_{N-1}||_{op} = \left\| \sum_{k=N}^{\infty} \left(T_{n_{k+1}} - T_{n_k}\right) \right\|_{op} \leq \sum_{k=N}^{\infty} ||T_{n_{k+1}} - T_{n_k}||_{op} \leq \sum_{k=N}^{\infty} 2^{-k}$$

Problem 8 continued on next page...

Note that above, we implicitly used that the triangle inequality on $||\cdot||_{op}$ can be applied countably many times; we basically proved this in the first three lines of the earlier proof that $||T||_{op} < \infty$. So, for every $\epsilon > 0$ we can select the N large enough that $\sum_{k=m}^{\infty} 2^{-k} < \epsilon$ for every $m > N$, which would mean that $||T - T_{n_m}||_{op} < \epsilon$ for all such $m > N$. This proves that $T_{n_k} \to T$ in $||\cdot||_{op}$ as $k \to \infty$. Lastly, recall that $(T_n)_n$ is Cauchy. Let $\epsilon > 0$. Then, there is some $M > 0$ such that for all $n, m > M$,

$$||T_n - T_m||_{op} < \frac{\epsilon}{2}$$

Choose any new N large enough that for all $n_k > N$, we have $||T_{n_k} - T||_{op} < \frac{\epsilon}{2}$ (such an N exists since $T_{n_k} \to T$ in $||\cdot||_{op}$). Then, for all such $n, n_k > \max\{M, N\}$, the Cauchy criterion and the triangle inequality (part (b)) grant

$$||T_n - T||_{op} \leq ||T_n - T_{n_k}||_{op} + ||T_{n_k} - T||_{op} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since this holds for all $\epsilon > 0$, we see that $(T_n)_n$ converges to T in the $||\cdot||_{op}$ norm. ■

Problem 9

Solution

Proof. First, we prove that the general problem reduces to the case with separable $\mathcal{H}$. To see this, note that for general $\mathcal{H}$, we can write $S \subset \mathcal{H}$ to be $S := \overline{span\{f_n\}_{n=1}^{\infty}}$, which is a closed subspace and therefore is a Hilbert space. In particular, S is separable, as it certainly has a countable dense subset. Such a subset is the set of every possible linear combination of $\{f_n\}_n$ using coefficients in $\mathbb{Q}[i]$, which are the complex numbers $a + bi$ with $a, b \in \mathbb{Q}$; this approximates every sequence of linear combinations of $\{f_n\}_n$ arbitrarily well (you can select elements of $\mathbb{Q}[i]$ that are $\epsilon/2^n$ close to the coefficient of f_n for each n), and it is also countable. The proof of this very closely follows the proof of separability in Problem 1, except instead of vectors in $l^2(\mathbb{Z})$ we have vectors of coeffecients in the expansion of elements of the span. Suppose that we can prove the separable case of the claim, i.e. that there exists an $f \in S$ and a subsequence $\{f_{n_k}\}_k \subset \{f_n\}_n \subset S$ such that

$$\lim_{k\to\infty} \langle f_{n_k}, g\rangle = \langle f, g\rangle \quad \forall g \in S$$

Via the decomposition $\mathcal{H} = S \oplus S^{\perp}$, we see that for all $g \in \mathcal{H}$ we can write $g = g_S + g_{S^\perp}$ with $g_S \in S$ and $g_{S^\perp} \in S^{\perp}$. Then, for the constructed f and subsequence $\{f_{n_k}\}_k \subset \{f_n\}_n$, we know $\langle f_{n_k}, g_{S^\perp}\rangle = 0$ for each k and also $\langle f, g_{S^\perp}\rangle = 0$ by definition of $S^\perp$. Therefore, for each $g \in \mathcal{H}$,

$$\begin{aligned}\lim_{k\to\infty} \langle f_{n_k}, g\rangle &= \lim_{k\to\infty} (\langle f_{n_k}, g_S\rangle + \langle f_{n_k}, g_{S^\perp}\rangle) = \lim_{k\to\infty} \langle f_{n_k}, g_S\rangle \\ &= \langle f, g_S\rangle = \langle f, g_S\rangle + \langle f, g_{S^\perp}\rangle = \langle f, g\rangle \quad \forall g \in \mathcal{H},\end{aligned}$$

where to get from the first line to the second line we use the result for separable S. So, we see that if we can prove the claim for separable Hilbert spaces $\mathcal{H}$, then we can prove the general case via the above logic. Therefore, suppose without loss of generality that $\mathcal{H}$ is separable from here on out.

Now, as $\mathcal{H}$ is separable, we can find a countable orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of $\mathcal{H}$. We will apply a diagonalization argument to inductively find "good" subsequences of $\{f_n\}_n$, take the diagonal, and construct f using these diagonal entries. First, we require the following lemma to prove that there always is a "good" subsequence.

Lemma 3. *Fix $h \in \mathcal{H}$ to be an arbitrary element of of $\mathcal{H}$. If $\{f_n\}_n \subset \mathcal{H}$ is a bounded sequence, then we can find a subsequence $\{f_{n_k}\}_k \subset \{f_n\}_n$ such that $\lim_{k\to\infty}\langle f_{n_k}, h\rangle$ converges.*

Proof of Lemma 3. Note that the claim holds if and only if the sequence $\{\langle f_n, h\rangle\}_{n=1}^{\infty} \subset \mathbb{C}$ has a convergent subsequence. We know that $\{f_n\}_n$ is bounded, say by $M > 0$. By Cauchy-Schwarz,

$$|\langle f_n, h\rangle| \le ||f_n|| \cdot ||h|| < M||h|| \quad \forall f_n,$$

which means that the sequence $\{\langle f_n, h\rangle\}_{n=1}^{\infty}$ must also be bounded by $M||h||$. This means that $\{\langle f_n, h\rangle\}_{n=1}^{\infty}$ is actually contained in the set $\{z \in \mathbb{C} : |z| \le M||h||\}$, which is compact (it's the closed ball of radius $M||h||$ in $\mathbb{C}$). So, $\{\langle f_n, h\rangle\}_{n=1}^{\infty}$ is a sequence in a compact set of scalars, which means that it must have a convergent subsequence (by definition of sequential compactness, which is equivalent to compactness for subsets of $\mathbb{C}$). So, $\{\langle f_n, h\rangle\}_{n=1}^{\infty}$ has a convergent subsequence, from which the claim follows. ■

Armed with this lemma, we can proceed. The procedure reads:

1. For $j = 1$, we can apply Lemma 3 with $h = \varphi_1$ to select a subsequence $\{f_{n_{(k,1)}}\}_k \subset \{f_n\}_n$ such that

$$\lim_{k\to\infty} \langle f_{n_{(k,1)}}, \varphi_1\rangle =: a_1$$

Problem 9 continued on next page...

for some $a_1 \in \mathbb{C}$.

2. Now, we perform the following inductive step: suppose by way of induction that we have a subsequence $\{f_{n_{(k,j)}}\}_k \subset \{f_{n_{(k,j-1)}}\}_k \subset ... \subset \{f_{n_{(k,1)}}\}_k \subset \{f_n\}_n$ for some j such that for all $i \leq j$, the limits

$$\lim_{k\to\infty} \langle f_{n_{(k,i)}}, \varphi_i \rangle = a_i$$

converge to some a_i's in $\mathbb{C}$. Then, we can apply Lemma 3 on the sequence $\{f_{n_{(k,j)}}\}_k$ with $h = \varphi_{j+1}$ to find a subsequence $\{f_{n_{(k,j+1)}}\}_k \subset \{f_{n_{(k,j)}}\}_k$ such that

$$\lim_{k\to\infty} \langle f_{n_{(k,j+1)}}, \varphi_{j+1} \rangle =: a_{j+1}$$

for some $a_{j+1} \in \mathbb{C}$. Since it is a subsequence of $\{f_{n_{(k,j)}}\}_k$, it inherits the limit results for all $i \leq j$ as well (a subsequence of a convergent sequence in $\mathbb{C}$ also converges). So, the inductive step is proven.

3. Now, we have infinitely many subsequences $\{f_{n_{(k,j)}}\}_k \subset \{f_n\}_n$ such that for each j, we know

$$\{f_{n_{(k,j+1)}}\}_k \subset \{f_{n_{(k,j)}}\}_k \quad \text{and} \quad \lim_{k\to\infty} \langle f_{n_{(k,i)}}, \varphi_i \rangle = a_i \in \mathbb{C} \quad \forall i \leq j$$

Applying a diagonalization argument, we can define a subsequence $\{f_{n_l}\}_l \subset \{f_n\}_n$ by $f_{n_l} := f_{n_{(l,l)}}$ for every l. Note that for every j, we have the property that $f_{n_l} \in \{f_{n_{(k,l)}}\}_k \subset \{f_{n_{(k,j)}}\}_k$ for all $l \geq j$ because of the monotonic nature of the subsequences we created; put differently, the sequence $\{f_{n_l}\}_l$ is eventually a subsequence of $\{f_{n_{(k,j)}}\}_k$ for every j. This means that for each j, eventually $\{f_{n_l}\}_l$ inherits the desired limit property. In other words, for every j,

$$\lim_{l\to\infty} \langle f_{n_l}, \varphi_i \rangle = a_i \in \mathbb{C} \quad \forall i \leq j$$

Now, we have constructed a "good" subsequence $\{f_{n_k}\}_k \subset \{f_n\}_n$ (I relabeled l to k so that we can forget all the messy notation from above) such that for every j,

$$\lim_{k\to\infty} \langle f_{n_k}, \varphi_i \rangle = a_i \in \mathbb{C} \quad \forall i \leq j,$$

which means that

$$\lim_{k\to\infty} \langle f_{n_k}, \varphi_j \rangle = a_j \in \mathbb{C} \quad \forall j$$

We construct the vector

$$f := \sum_{j=1}^{\infty} a_j \varphi_j$$

To see that this sum converges, note that

$$||f||^2 = \left\| \sum_{j=1}^{\infty} a_j \varphi_j \right\|^2 = \sum_{j=1}^{\infty} |a_j|^2 = \sum_{j=1}^{\infty} \left| \lim_{k\to\infty} \langle f_{n_k}, \varphi_j \rangle \right|^2 = \sum_{j=1}^{\infty} \lim_{k\to\infty} |\langle f_{n_k}, \varphi_j \rangle|^2,$$

where the last equality uses the continutity of the norm $|\cdot|$ on $\mathbb{C}$. Now, note that we can switch the sum and the limit by monotone convergence, as the partial sums of nonnegative elements are monotonically increasing. This grants

$$||f||^2 = \lim_{k\to\infty} \left(\sum_{j=1}^{\infty} |\langle f_{n_k}, \varphi_j \rangle|^2 \right) = \lim_{k\to\infty} ||f_{n_k}||^2 = 1 < \infty,$$

where we used Parseval's identity and the fact that $\{\varphi_j\}_j$ is an orthonormal basis in the second equality. Thus, the sum defining f converges, and so $f \in \mathcal{H}$. Next, we want to show that

$$\lim_{k\to\infty} \langle f_{n_k}, g \rangle = \langle f, g \rangle \quad \forall g \in \mathcal{H}$$

To this end, fix $g \in \mathcal{H}$ arbitrary and let $\epsilon > 0$. We can write in the orthonormal basis that $g = \sum_{j=1}^{\infty} \langle g, \varphi_j \rangle \varphi$. Therefore, since $\langle f, \varphi_j \rangle = a_j = \lim_{k\to\infty} \langle f_{n_k}, \varphi_j \rangle$ for all j, we have by the continuity and (antilinearity) of the inner product $\langle f, \cdot \rangle_{\mathcal{H}}$ that

$$\langle f, g \rangle = \sum_{j=1}^{\infty} \overline{\langle g, \varphi_j \rangle} \cdot \langle f, \varphi_j \rangle = \sum_{j=1}^{\infty} a_j \overline{\langle g, \varphi_j \rangle} = \sum_{j=1}^{\infty} \lim_{k\to\infty} \langle f_{n_k}, \varphi_j \rangle \cdot \overline{\langle g, \varphi_j \rangle}$$

Similarly, for each l we get that

$$\langle f_{n_l}, g \rangle = \sum_{j=1}^{\infty} \langle f_{n_l}, \varphi_j \rangle \cdot \overline{\langle g, \varphi_j \rangle}$$

(This is basically implicitly constructing the unitary map from $\mathcal{H}$ to $l^2(\mathbb{N})$ via $h \mapsto (\langle h, \varphi_1 \rangle, \langle h, \varphi_2 \rangle, ...)$ and using the fact that this map preserves inner products, as is done in the proof of Corollary 2.5). Combining these two, we have by the continuity of $|\cdot|$ that for every n_l,

$$|\langle f - f_{n_l}, g \rangle| = \left| \sum_{j=1}^{\infty} \lim_{k\to\infty} \langle f_{n_k} - f_{n_l}, \varphi_j \rangle \cdot \overline{\langle g, \varphi_j \rangle} \right| \leq \sum_{j=1}^{\infty} |\overline{\langle g, \varphi_j \rangle}| \cdot \lim_{k\to\infty} |\langle f_{n_k} - f_{n_l}, \varphi_j \rangle|$$

Now, we know that the sequence $\{\langle f_{n_k}, \varphi_j \rangle\}_k$ is convergent in $\mathbb{C}$ and therefore Cauchy, for every j. This means that, if we hold j fixed, we can find an N large enough that for all $n_k, n_l > N$, we get

$$|\langle f_{n_k} - f_{n_l}, \varphi_j \rangle| < \frac{\epsilon}{||g||^2} |\langle g, \varphi_j \rangle| \implies \lim_{k\to\infty} |\langle f_{n_k} - f_{n_l}, \varphi_j \rangle| \leq \frac{\epsilon}{||g||^2} |\langle g, \varphi_j \rangle| \quad \forall n_l > N$$

Thus, for all $n_l > N$ we get

$$|\langle f - f_{n_l}, g \rangle| \leq \sum_{j=1}^{\infty} |\overline{\langle g, \varphi_j \rangle}| \cdot \frac{\epsilon}{||g||^2} |\langle g, \varphi_j \rangle| = \frac{\epsilon}{||g||^2} \sum_{j=1}^{\infty} |\langle g, \varphi_j \rangle|^2 = \frac{\epsilon}{||g||^2} \cdot ||g||^2 = \epsilon,$$

where the second to last equality uses Parseval's Identity. Since such an N exists for all ϵ, we see that

$$\lim_{l\to\infty} \langle f - f_{n_l}, g \rangle = 0 \implies \lim_{l\to\infty} \langle f_{n_l}, g \rangle = \langle f, g \rangle$$

Since this holds for all $g \in \mathcal{H}$, we are done. ■

Problem 10

Solution

Proof of (a). Suppose that T is an isometry, and let $f, g \in \mathcal{H}$. Since we are in a Hilbert space, we have the Parallelogram Law, which allows us to make use of the polarization identity reads. It reads

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{4}\left[\|f+g\|^2 - ||f-g||^2 + i||f+ig||^2 - i||f-ig||^2\right] \\ &= \frac{1}{4}\left[\|T(f+g)\|^2 - ||T(f-g)||^2 + i||T(f+ig)||^2 - i||T(f-ig)||^2\right] \\ &= \frac{1}{4}\left[\|Tf+Tg\|^2 - ||Tf-Tg||^2 + i||Tf+iTg||^2 - i||Tf-iTg||^2\right] \\ &= \langle Tf, Tg\rangle,\end{aligned}$$

where the second equality uses the fact that T is an isometry, the third equality uses linearity of T, and the fourth equality applies the polarization identity again. Then, for all $f, g \in \mathcal{H}$, we have that since $(T^*T)^* = T^*T$,

$$\langle T^*Tf, g\rangle = \overline{\langle g, T^*Tf\rangle} = \overline{\langle Tg, Tf\rangle} = \langle Tf, Tg\rangle = \langle f, g\rangle \implies \langle T^*Tf - f, g\rangle = 0$$

So, for every $f \in \mathcal{H}$, we have that $\langle T^*Tf - f, g\rangle = 0 \quad \forall g \in \mathcal{H}$, which means that for every $f \in \mathcal{H}$, $T^*Tf - f = 0 \implies T^*Tf = f$. So, $T^*T = I$. ■

Proof of (b). Suppose now that T is a *surjective* isometry. To show that T is injective, we want to show that $Tf = 0 \implies f = 0$ (i.e. T has a trivial kernel). So, suppose that $Tf = 0$ for some $f \in \mathcal{H}$; then, $||Tf|| = 0 \implies ||f|| = 0 \implies f = 0$, and so T is injective. Therefore, T is a bijective linear map that preserves norm, which means that T is unitary. Then, for all $f, g \in \mathcal{H}$, we have that

$$\langle T^*f, g\rangle = \overline{\langle g, T^*f\rangle} = \overline{\langle Tg, f\rangle} = \langle f, Tg\rangle \quad \text{and} \quad \langle T^{-1}f, g\rangle = \langle TT^{-1}f, Tg\rangle = \langle f, Tg\rangle,$$

where the last equality uses the result from part (a). So, for every $f \in \mathcal{H}$, we have that $\langle T^*f - T^{-1}f, g\rangle = 0$ for all $g \in \mathcal{H}$, which means that for every $f \in \mathcal{H}$, $T^*f - T^{-1}f = 0 \implies T^*f = T^{-1}f$. So, $T^* = T^{-1} \implies TT^* = I$. ■

Proof of (c). Consider the space $l^2(\mathbb{N})$ with a basis $\{e_k\}_{k=1}^\infty$ where e_k has a 1 in the k^{th} index and 0's everywhere else. Clearly, this is an orthonormal basis. Now, define T to be the map that sends $e_k \mapsto e_{2k}$ for every k, and extend it linearly. In other words, for every $f \in l^2(\mathbb{N})$, we can write $f = \sum_{k=1}^\infty a_k e_k$ and define $Tf := \sum_{k=1}^\infty a_k e_{2k}$. This is certainly a linear map; to see that it's an isometry, note that for $f = \sum_{k=1}^\infty a_k e_k$ we have $||f||^2 = \sum_{k=1}^\infty |a_k|^2$ and

$$||Tf||^2 = \left\|\sum_{k=1}^\infty a_k e_{2k}\right\|^2 = \left\langle \sum_{k=1}^\infty a_k e_{2k}, \sum_{j=1}^\infty a_j e_{2j}\right\rangle = \sum_{k=1}^\infty a_k \left\langle e_{2k}, \sum_{j=1}^\infty a_j e_{2j}\right\rangle$$

Since $\langle e_k, e_j\rangle = \delta_{k,j}$, we get that

$$||Tf||^2 = \sum_{k=1}^\infty a_k \left\langle e_{2k}, \sum_{j=1}^\infty a_j e_{2j}\right\rangle = \sum_{k=1}^\infty a_k \langle e_{2k}, a_k e_{2k}\rangle = \sum_{k=1}^\infty a_k \overline{a_k} = \sum_{k=1}^\infty |a_k|^2 = ||f||^2$$

Since this holds for every f, we know that T is an isometry. However, it is certainly not surjective, since there is no element of $l^2(\mathbb{N})$ that gets mapped to e_1, or more generally to e_i for any odd i. So, since T is not surjective, it cannot be bijective, and therefore can't be unitary. ■

Problem 10 continued on next page...

Proof of (d). Suppose that T^*T is unitary. Then, if T^*T has any nontrivial eigenvalues, they must have unit norm (this is a well-known property of unitary operators). However, since we have that $(T^*T)^* = T^*T^{**} = T^*T$, then T^*T is self-adjoint, which means it must have real eigenvalues, if any (for any normalized eigenvector v, we have $\lambda_v = \langle T^*Tv, v\rangle = \langle v, T^*Tv\rangle = \overline{\langle T^*Tv, v\rangle}$). So, this means that T^*T must have eigenvalues of either -1 or 1. However, suppose by way of contradiction that v is a (normalized) eigenvector of T^*T with eigenvalue $\lambda_v = -1$. Then,

$$-1 = \langle T^*Tv, v\rangle = \langle v, T^*Tv\rangle = \langle Tv, Tv\rangle = ||Tv||^2 \geq 0$$

This is a contradiction, and so T^*T can only have an eigenvalue of value 1. The only unitary operator with all nontrivial eigenvalues equalling 1 is the identity operator, and so $T^*T = I$. From here, we can conclude that for all $v \in \mathcal{H}$, we have

$$||v||^2 = \langle v, v\rangle = \langle v, Iv\rangle = \langle v, T^*Tv\rangle = \langle Tv, Tv\rangle = ||Tv||^2$$

Therefore, T preserves norms, and is thus an isometry. ■