MAT 425: Problem Set 6

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Professor Paul Minter

Evan Dogariu
Collaborators: David Shustin

Problem 1

Solution

Proof. Let $f \in L^1(\mathbb{R}^d)$ and let $\{K_\delta\}_{\delta>0}$ be an approximation for the identity; then, $K_\delta(x)$ is integrable on $\mathbb{R}^d$ for all $\delta > 0$. By Problem 4(d) on Problem Set 5, we already know that since $f(x), K_\delta(x)$ are both integrable on $\mathbb{R}^d$, then so is $(f * K_\delta)(x)$.

Now, let us denote the difference between f and $f * K_\delta$ as $g(x) := (f * K_\delta)(x) - f(x)$. Then, since $\int_{\mathbb{R}^d} K_\delta(y)dy = 1$, we can say

$$f(x) = \int_{\mathbb{R}^d} f(x)K_\delta(y)dy \implies g(x) = \int_{\mathbb{R}^d} (f(x-y) - f(x))K_\delta(y)dy$$

So,

$$|g(x)| \leq \int_{\mathbb{R}^d} |f(x-y) - f(x)| \cdot |K_\delta(y)|dy,$$

and therefore

$$||g||_{L^1} \leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)| \cdot |K_\delta(y)|dy \right) dx$$

Note that by Corollary 3.7 and Proposition 3.9 of Chapter 2, we know that both $f(x)$ and $f(x-y)$ are measurable on $\mathbb{R}^d \times \mathbb{R}^d$, which certainly means that $|f(x-y) - f(x)|$ is. In addition, $K_\delta(y) \in L^1(\mathbb{R}^d)$ by definition of an approximation to the identity, and so $|K_\delta(y)|$ must be measurable on $\mathbb{R}^d$; this, in turn, means by Corollary 3.7 that it is measurable on $\mathbb{R}^d \times \mathbb{R}^d$. Then, the function $|f(x-y)-f(x)| \cdot |K_\delta(y)|$ is nonnegative and measurable on $\mathbb{R}^d \times \mathbb{R}^d$, which means we can apply Tonelli's Theorem to switch the integrals. Denoting $f_y(x) \equiv f(x-y)$ for each $y \in \mathbb{R}^d$, we get

$$\begin{aligned}
||g||_{L^1} &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)| \cdot |K_\delta(y)|dy \right) dx \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)| \cdot |K_\delta(y)|dx \right) dy \\
&= \int_{\mathbb{R}^d} |K_\delta(y)| \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|dx \right) dy \\
&= \int_{\mathbb{R}^d} ||f_y - f||_{L^1} |K_\delta(y)|dy
\end{aligned}$$

Let $\epsilon > 0$. By Proposition 2.5, f_y converges to f in L^1 as $y \to 0$; so, there exists an $\eta > 0$ such that $||f_y - f||_{L^1} < \epsilon$ whenever $|y| < \eta$. This allows us to split the integral and see

$$\begin{aligned}
||g||_{L^1} &\leq \int_{|y|<\eta} ||f_y - f||_{L^1} |K_\delta(y)|dy + \int_{|y|\geq\eta} ||f_y - f||_{L^1} |K_\delta(y)|dy \\
&< \epsilon \int_{|y|<\eta} |K_\delta(y)|dy + \int_{|y|\geq\eta} ||f_y - f||_{L^1} |K_\delta(y)|dy
\end{aligned}$$

By property (ii) of approximations to the identity, $\int_{|y|<\eta} |K_\delta(y)|dy \leq \int_{\mathbb{R}^d} |K_\delta(y)|dy \leq A$ for some constant A independent of δ. By property (iii), there exists some $\delta > 0$ such that for all $\delta' < \delta$, we have $\int_{|y|\geq\eta} |K_{\delta'}(y)|dy < \epsilon$. Lastly, by the triangle inequality and translation invariance of the integral, $||f_y - f||_{L^1} \leq ||f_y||_{L^1} + ||f||_{L^1} = 2||f||_{L^1}$. Putting this all together, we find that for all $\delta' < \delta$

$$\begin{aligned}
||g||_{L^1} &< \epsilon \int_{|y|<\eta} |K_\delta(y)|dy + \int_{|y|\geq\eta} ||f_y - f||_{L^1} |K_\delta(y)|dy \\
&\leq A\epsilon + 2||f||_{L^1} \int_{|y|\geq\eta} |K_{\delta'}(y)|dy < A\epsilon + 2||f||_{L^1}\epsilon
\end{aligned}$$

Problem 1 continued on next page. . .

Since such a δ exists for all ϵ, we see that $||g||_{L^1} \to 0$ as $\delta \to 0$. Therefore, $f * K_\delta$ converges in L^1 to f as $\delta \to 0$. ■

Problem 2

Solution

Proof. Let $f \in L^1(\mathbb{R}^n)$ be not identically 0. Then, $||f||_{L^1} = D$ for some $D > 0$. By Proposition 1.12(i), there exists some ball B such that $\int_{B^C} |f| < \frac{D}{2} \implies \int_B |f| = ||f||_{L^1} - \int_{B^C} |f| > D - \frac{D}{2} = \frac{D}{2} > 0$. Let $a := \sup_{x \in B} |x|$ be the maximal distance from the origin to a point in this ball, and let $A := \max\{a, 2\} > 1$. Now, for each $x \in \mathbb{R}^n$, let B_x be the ball centered at the origin of radius $A|x|$. Then, for all x with $|x| \geq 1$ we have that

$$A|x| > |x| \implies x \in B_x \qquad \text{and} \qquad A|x| \geq a \implies B \subset B_x$$

So, we get that since $|f|$ is nonnegative and $B \subset B_x$,

$$\frac{1}{m(B_x)} \int_{B_x} |f| \geq \frac{1}{m(B_x)} \int_B |f| > \frac{D/2}{m(B_x)} = \frac{c}{|x|^n}$$

for some constant $c > 0$ (here, $c = \frac{D}{2A^n \cdot m(B_1(O))}$ where $B_1(O)$ is the unit ball). Therefore, for $|x| \geq 1$ we have

$$f^*(x) := \sup_{B' \ni x} \frac{1}{m(B')} \int_{B'} |f| \geq \frac{1}{m(B_x)} \int_{B_x} |f| > \frac{c}{|x|^n}$$

because of the definition of a supremum and the fact that $B_x \ni x$. This fact shows that f^* is not integrable on $\mathbb{R}^n$ since it is larger than $c/|x|^n$, which is itself not integrable. To see that $c/|x|^n$ is not integrable, it suffices to note that by Riemann integration,

$$\frac{c}{|x|^n} = \int_{|x|}^{\infty} \frac{cn}{t^{n+1}} dt$$

So, we get that by Tonelli's theorem,

$$\int_{\mathbb{R}^n} \frac{c}{|x|^n} dx = \int_{\mathbb{R}^n} \int_{|x|}^{\infty} \frac{cn}{t^{n+1}} dt dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}} \mathbb{1}_{\{|x| \leq t\}} \frac{cn}{t^{n+1}} dt dx = \int_0^{\infty} \int_{\mathbb{R}^n} \mathbb{1}_{\{|x| \leq t\}} \frac{cn}{t^{n+1}} dx dt$$

$$= \int_0^{\infty} \frac{cn}{t^{n+1}} m(B_t(O)) dt = cn \cdot m(B_1(O)) \int_0^{\infty} \frac{t^n}{t^{n+1}} dt = cn \cdot m(B_1(O)) \int_0^{\infty} \frac{1}{t} dt,$$

where $B_t(O)$ is the ball of radius t about the origin, which we know has measure $t^n \cdot m(B_1(O))$ with $B_1(O)$ the unit ball. Since the integral $\int_0^{\infty} \frac{1}{t} dt$ diverges by the p-test (it is greater than its right Riemann sums, which diverge by the p-series test), we can confirm that $\frac{c}{|x|^n}$ is not integrable on $\mathbb{R}^n$; therefore, neither is f^*.

Suppose now that f is supported in the unit ball with $||f||_{L^1} = 1$. Let $E_\alpha := \{x \in \mathbb{R}^n : f^*(x) > \alpha\}$. For each x with $|x| \geq 1$, let $B_{|x|}(O)$ be the ball of radius $|x|$ about the origin. Then, by virtue of the supremum in the definition of f^* we have

$$\left\{ x : |x| \geq 1 \text{ and } \frac{1}{m(B_{|x|}(O))} \int_{B_{|x|}(O)} |f| > \alpha \right\} \subset E_\alpha$$

Note that since $B_{|x|}(O) \supset B_1(O)$ for such x and f is supported in the unit ball, $\int_{B_{|x|}(O)} |f| = \int_{B_1(O)} |f| = ||f||_{L^1} = 1$. Therefore,

$$\left\{ x : |x| \geq 1 \text{ and } \frac{1}{m(B_{|x|}(O))} > \alpha \right\} \subset E_\alpha$$

If $\alpha < \frac{1}{m(B_1(O))}$, this set is not empty. In these cases, the set $\left\{ x : |x| \geq 1 \text{ and } \frac{1}{m(B_{|x|}(O))} > \alpha \right\}$ contains the set $B_a(O) \setminus B_1(O)$ for the a such that $\frac{1}{m(B_a(O))} = \alpha$ (we selected α small enough that $a > 1$). To see this note that for all $x \in B_a(O) \setminus B_1(O)$ we have $B_{|x|}(O) \subset B_a(O) \implies \frac{1}{m(B_{|x|}(O))} > \frac{1}{m(B_a(O))} = \alpha$.

So, $B_a(O) \setminus B_1(O) \subset E_\alpha$. However, we can also show that $B_1(O) \subset E_\alpha$; indeed, if $x \in B_1(O)$ then $\frac{1}{m(B_1(O))} \int_{B_1(O)} |f| = \frac{1}{m(B_1(O))} > \alpha \implies f^*(x) > \alpha \implies x \in E_\alpha$. Therefore,

$$B_a(O) \subset E_\alpha \implies m(E_\alpha) \geq m(B_a(O)) = \frac{1}{\alpha},$$

as desired. ■

Problem 3

Solution

Proof of (a). Fix $\alpha > 0$. Suppose that $f \in L^2(\mathbb{R}^n)$. Let $f_1(x) := \mathbb{1}_{\{x' : |f(x')| > \alpha/2\}} \cdot f(x)$. Then,

$$\int_{\mathbb{R}^n} |f_1| = \int_{\{|f| > \alpha/2\}} |f| = \int_{\{\alpha/2 < |f| < 1\}} |f| + \int_{|f| \geq 1} |f|$$

Note that over the region $\{|f| \geq 1\}$ we must have that $|f| \leq |f|^2$. So,

$$\int_{\mathbb{R}^n} |f_1| \leq \int_{\{\alpha/2 < |f| < 1\}} |f| + \int_{|f| \geq 1} |f|^2 \leq \int_{\{\alpha/2 < |f| < 1\}} |f| + \int_{\mathbb{R}^n} |f|^2$$

Since $f \in L^2(\mathbb{R}^n)$, we know that $\int_{\mathbb{R}^n} |f|^2$ is finite. So, to prove $f_1 \in L^1(\mathbb{R}^n)$, all we must do is show that $\int_{\{\alpha/2 < |f| < 1\}} |f|$ is finite. However,

$$\int_{\{\alpha/2 < |f| < 1\}} |f| \leq \int_{\{\alpha/2 < |f| < 1\}} 1 = m(\{\alpha/2 < |f| < 1\})$$

Since

$$m(\{\alpha/2 < |f| < 1\}) = m(\{\alpha^2/4 < |f|^2 < 1\}) \leq \frac{4}{\alpha^2} \int_{\{\alpha^2/4 < |f|^2 < 1\}} |f|^2 \leq \frac{4}{\alpha^2} ||f||_{L^2} < \infty,$$

we know that $m(\{\alpha/2 < |f| < 1\})$ is finite, and therefore that $\int_{\{\alpha/2 < |f| < 1\}} |f|$ is as well. So, we get that $\int_{\mathbb{R}^n} |f_1| < \infty$, and thus that $f_1 \in L^1(\mathbb{R}^n)$.

Now, suppose that $f^*(x) > \alpha$ for some x. Then, there exists a ball B such that $\int_B |f| > \alpha \cdot m(B)$ by definition of f^*. Therefore,

$$\alpha \cdot m(B) < \int_B |f| = \int_{B \cap \{|f| > \alpha/2\}} |f| + \int_{B \cap \{|f| \leq \alpha/2\}} |f| \leq \int_{B \cap \{|f| > \alpha/2\}} |f| + \frac{\alpha}{2} \cdot m(B),$$

where the last step is since $B \cap \{|f| \leq \alpha/2\} \subset B$ and $|f| \leq \alpha/2$ over the region of interest. However, the result above reads that

$$\frac{\alpha}{2} \cdot m(B) < \int_{B \cap \{|f| > \alpha/2\}} |f| = \int_B |f_1| \implies \frac{1}{m(B)} \int_B |f_1| > \frac{\alpha}{2}$$

Since this value is attained by the given B, it certainly holds for all balls containing x, and so $f_1^*(x) > \alpha/2$. Since this held for all x s.t. $f^*(x) > \alpha$, we then find that

$$\{x : f^*(x) > \alpha\} \subset \left\{x : f_1^*(x) > \frac{\alpha}{2}\right\}$$

With these two above results, we can prove the claim. Note that by Theorem 1.1(iii), we have that

$$m\left(\left\{x : f_1^*(x) > \frac{\alpha}{2}\right\}\right) \leq \frac{2 \cdot 3^n}{\alpha} \int_{\mathbb{R}^n} |f_1| = \frac{2 \cdot 3^n}{\alpha} \int_{\{|f| > \alpha/2\}} |f|$$

by definition of f_1. Finally, monotonicity of measure grants that

$$m(\{x : f^*(x) > \alpha\}) \leq m\left(\left\{x : f_1^*(x) > \frac{\alpha}{2}\right\}\right) \leq \frac{2 \cdot 3^n}{\alpha} \int_{\{|f| > \alpha/2\}} |f|$$

as desired. ∎

Problem 3 continued on next page. . .

Proof of (b). Let us begin by noting that the function $F(x, \alpha) := \mathbb{1}_{\{f^* > \alpha\}}(x)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^+$ by measurability of f^*, which comes from Theorem 1.1(i) (I proved this result in Problem 3 on Problem Set 5). So, the function $\alpha \mathbb{1}_{\{f^* > \alpha\}}(x)$ is as well, which means that we can apply Tonelli's Theorem. In particular,

$$2\int_0^\infty \alpha m(\{f^* > \alpha\})d\alpha = 2\int_0^\infty \alpha \left(\int_{\mathbb{R}^n} \mathbb{1}_{\{f^* > \alpha\}}(x)dx\right) d\alpha = 2\int_{\mathbb{R}^n} \left(\int_0^\infty \alpha \mathbb{1}_{\{f^*(x) > \alpha\}} d\alpha\right) dx$$

Since $\alpha \mathbb{1}_{\{f^*(x) > \alpha\}} = \begin{cases} \alpha & \alpha < f^*(x) \\ 0 & else \end{cases}$, we get that

$$2\int_0^\infty \alpha m(\{f^* > \alpha\})d\alpha = 2\int_{\mathbb{R}^n} \left(\int_0^{f^*(x)} \alpha d\alpha\right) dx = 2\int_{\mathbb{R}^n} \left[\frac{\alpha^2}{2}\right]_{\alpha=0}^{f^*(x)} dx = \int_{\mathbb{R}^n} |f^*(x)|^2 dx$$

■

Proof of (c). We have from part (a) that $\alpha m(E_\alpha) \leq 2 \cdot 3^n \int_{\{|f| > \alpha/2\}} |f|$. Plugging this into the result from (b), we get

$$||f^*||_{L^2}^2 \leq 2\int_0^\infty \left(2 \cdot 3^n \int_{\{|f| > \alpha/2\}} |f(x)|dx\right) d\alpha = 4 \cdot 3^n \int_0^\infty \left(\int_{\{|f| > \alpha/2\}} |f(x)|dx\right) d\alpha$$

Since $\mathbb{1}_{\{|f| > \alpha/2\}}|f|$ is nonnegative, we can apply Tonelli's Theorem to switch the integrals and get that

$$||f^*||_{L^2}^2 \leq 4 \cdot 3^n \int_{\mathbb{R}^n} \left(\int_0^\infty \mathbb{1}_{\{|f(x)| > \alpha/2\}} |f(x)| d\alpha\right) dx = 4 \cdot 3^n \int_{\mathbb{R}^n} |f(x)| \left(\int_0^\infty \mathbb{1}_{\{|f(x)| > \alpha/2\}} d\alpha\right) dx$$

(Note that in the above, we use the fact that the function $F(x, \alpha) := \mathbb{1}_{\{|f| > \alpha/2\}}(x)$ is measurable on $\mathbb{R}^n \times \mathbb{R}$, which I proved on Problem 3 of Problem Set 5). We can note by the relative scale invariance of the integral that

$$\int_0^\infty \mathbb{1}_{\{|f(x)| > \alpha/2\}} d\alpha = 2\int_0^\infty \mathbb{1}_{\{|f(x)| > \alpha\}} d\alpha = 2\int_0^{|f(x)|} d\alpha = 2|f(x)|$$

So,

$$||f^*||_{L^2}^2 \leq 8 \cdot 3^n \int_{\mathbb{R}^n} |f(x)| \cdot |f(x)|dx = 8 \cdot 3^n ||f||_{L^2}^2$$

Since the square root is monotonic,

$$||f^*||_{L^2} \leq \sqrt{8 \cdot 3^n} ||f||_{L^2}$$

as desired. ■

Problem 4

Solution

Proof. Let $(r_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ be an enumeration of the rationals. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{1}_{[r_n,\infty)}(x)$$

Firstly, we note that this function is bounded and increasing. To see boundedness, note that for any $x \in \mathbb{R}$ we have

$$|f(x)| = \sum_{\substack{n\in\mathbb{N} \\ r_n \leq x}} \frac{1}{2^n} \leq \sum_{n\in\mathbb{N}} \frac{1}{2^n} = 1,$$

where the inequality holds since each element of the sum is nonnegative. To see that f is increasing, let $x < y$ be arbitrary. Then, there exist some rationals lying in between x and y that contribute something positive to the sum. In particular,

$$f(y) - f(x) = \sum_{\substack{n\in\mathbb{N} \\ r_n \leq y}} \frac{1}{2^n} - \sum_{\substack{n\in\mathbb{N} \\ r_n \leq x}} \frac{1}{2^n} = \sum_{\substack{n\in\mathbb{N} \\ x < r_n \leq y}} \frac{1}{2^n} > 0,$$

and so f is actually strictly increasing (there is always a r_n in between x and y). Also, clearly f is discontinuous at each rational, since there is a jump of size 2^{-n} at each r_n. Now, note that the convergence of the partial sums is uniform. To see this, let $\epsilon > 0$. Let N' be such that $\sum_{n=N'}^{\infty} 2^{-n} < \epsilon$. Then, for all $N > N'$ and all $x \in \mathbb{R}$ we have

$$\left| f(x) - \sum_{n=1}^{N} \frac{1}{2^n} \mathbb{1}_{[r_n,\infty)}(x) \right| = \sum_{n=N+1}^{\infty} \frac{1}{2^n} \mathbb{1}_{[r_n,\infty)}(x) = \sum_{\substack{n>N \\ r_n \leq x}} \frac{1}{2^n} \leq \sum_{n>N} \frac{1}{2^n} \leq \sum_{n=N'}^{\infty} \frac{1}{2^n} < \epsilon$$

So, the convergence is uniform. Therefore, to see that f is continuous at all irrationals, it suffices to show that every partial sum is continuous at all irrationals since continuity is inherited by uniform convergence. To this end, let $N \in \mathbb{N}$ be arbitrary. We want to show that the function

$$f_N(x) := \sum_{n=1}^{N} \frac{1}{2^n} \mathbb{1}_{[r_n,\infty)}(x)$$

is continuous at each irrational. However, each $\mathbb{1}_{[r_n,\infty)}$ is certainly continuous at every irrational z, since for each $z \in \mathbb{R} \setminus \mathbb{Q}$ there exists a ball around z with radius $< |z - r_n|$ such that $\mathbb{1}_{[r_n,\infty)}$ is constant on this ball. Since f_N is a finite sum of functions continuous on $\mathbb{R} \setminus \mathbb{Q}$, we therefore have that f_N is also continuous on $\mathbb{R} \setminus \mathbb{Q}$. Lastly, since $f_N \to f$ uniformly as $N \to \infty$, we get that f is also continuous at every point in $\mathbb{R} \setminus \mathbb{Q}$. So, f is bounded and strictly increasing, and its set of discontinuities is precisely $\mathbb{Q}$, as desired. ■

Problem 5

Solution

Proof. If $a, b > 0$, let

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

($\Longrightarrow$) Suppose that f is of bounded variation in $[0,1]$. Consider the family of partitions $\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ given for each N by

$$t_k = \left(k\pi + \frac{\pi}{2}\right)^{-1/b}$$

for $k \leq N$, with $x_0 = 0$ and $x_N = 1$. Note that for such t_k's, we always have

$$\sin(t_k^{-b}) = \sin\left(k\pi + \frac{\pi}{2}\right) = (-1)^k \implies f(t_k) = t_k^a \cdot (-1)^k$$

Therefore, we can sum the variation over these partitions $\mathcal{P}_N$ and get

$$\sum_{k=1}^{N} |f(t_k) - f(t_{k-1})| = \sum_{k=1}^{N} |t_k^a \cdot (-1)^k - t_{k-1}^a \cdot (-1)^{k-1}|$$

Note that $t_k^a \cdot (-1)^k$ and $t_{k-1}^a \cdot (-1)^{k-1}$ will always be of the opposite sign, and so this sum equals

$$= \sum_{k=1}^{N} t_k^a + t_{k-1}^a = t_N^a + t_0^a + 2\sum_{k=1}^{N-1} t_k^a = 1 + 2\sum_{k=1}^{N-1} t_k^a \geq \sum_{k=1}^{N-1} t_k^a$$

We can plug in our t_k's to get that our variation is larger than the series

$$\sum_{k=1}^{N} |f(t_k) - f(t_{k-1})| \geq \sum_{k=1}^{N-1} \left(k\pi + \frac{\pi}{2}\right)^{-a/b}$$

Suppose by way of contradiction that $a \leq b \implies a/b \leq 1 \implies -a/b \geq -1$. Then, this sum is divergent for $N \to \infty$ by the p-series test. In particular, that means that we can never bound the variation $\sum_{k=1}^{N} |f(t_k) - f(t_{k-1})|$ uniformly over all partitions, because we can always select a partition $\mathcal{P}_N$ with a large enough N to overcome this bound. Therefore, f cannot be of bounded variation. This is a contradiction, and so we see that $a > b$.

($\Longleftarrow$) Suppose now that $a > b$. Note that f is differentiable over $(0, 1]$, since it is the product of a differentiable function and a composition of two differentiable functions. We readily compute for $x > 0$ that

$$f'(x) = ax^{a-1}\sin(x^{-b}) + x^a \cos(x^{-b}) \cdot (-bx^{-b-1}) = ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b})$$

Note that for every $\epsilon > 0$,

$$\int_{[\epsilon,1]} |ax^{a-1}\sin(x^{-b})|dx \leq \int_{[\epsilon,1]} ax^{a-1}dx = [x^a]_{x=\epsilon}^1 \leq 1$$

Now, we can also compute

$$\int_{[\epsilon,1]} |bx^{a-b-1}\cos(x^{-b})|dx \leq b\int_{[\epsilon,1]} |x^{a-b-1}|dx = \left[\frac{b}{(a-b)}x^{a-b}\right]_{x=\epsilon}^1 \leq \frac{b}{a-b},$$

where we were able to bound $\left[\frac{b}{(a-b)}x^{a-b}\right]_{x=\epsilon}^1$ since $a - b > 0 \implies \epsilon^{a-b}$ is small. What this tells us is that $f' \in L^1([\epsilon, 1])$ for every ϵ, and therefore that $f' \in L^1([0, 1])$. Since f' is integrable, continuous, and bounded

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(and therefore Riemann integrable) over this interval, we can use the properties of Riemann integration to recover that

$$f(x) = f(\epsilon) + \int_{[\epsilon,x]} f'(t)dt$$

Note that f is continuous at 0, as $|f(x)| \leq x^a \implies |\lim_{x\to 0} f(x)| \leq \lim_{x\to 0} x^a = 0 = f(0)$. So, we can take the limit as $\epsilon \to 0$ to see that

$$f(x) = f(0) + \lim_{\epsilon\to 0}\int_{[\epsilon,x]} f'(t)dt$$

However, note that $\int_{[\epsilon,x]} f'(t)dt \to \int_{[0,x]} f'(t)dt$ since

$$\left|\int_{[0,x]} f'(t)dt - \int_{[\epsilon,x]} f'(t)dt\right| = \left|\int_{[0,\epsilon]} f'(t)dt\right|,$$

and the term on the right can be made arbitrarily small by Proposition 1.12(ii) of Chapter 2 (since $f' \in L^1([0,1])$). This means that $f(x) = f(0) + \int_{[0,x]} f'(t)dt$, and so by the remarks in Section 3.2, f is absolutely continuous on $[0,1]$. Therefore, $f \in BV([0,1])$.

Now, let $\alpha \in (0,1)$ be arbitrary. Select a such that $\alpha = \frac{a}{a+1} \implies a = \frac{\alpha}{1-\alpha}$, and set $b = a$; this already yields that f is not of bounded variation on $[0,1]$. We would like to show that there is some $A > 0$ such that $|f(x+h) - f(x)| \leq Ah^\alpha$ for all $h \geq 0$. Firstly, note that since $|f(x)| = |x^a \sin(x^{-a})| \leq x^a$, we have by the triangle inequality

$$|f(x+h) - f(x)| \leq |f(x+h)| + |f(x)| \leq (x+h)^a + x^a \leq 2(x+h)^a$$

since x^a is monotonically increasing and $h \geq 0 \implies x + h \geq x$. However, we can also bound this variation a different way. Note first that f' exists everywhere, except at 0. So, we can apply the mean value theorem to say that for every $x \geq 0$ and every $h > 0$ (with $h \leq 1 - x$ of course), there exists some $c \in (x, x+h)$ such that

$$f'(c) = \frac{f(x+h) - f(x)}{h} \implies |f(x+h) - f(x)| = h|f'(c)|$$

Using the functional form of f' from earlier,

$$\begin{aligned}|f(x+h) - f(x)| &= h|f'(c)| = h\left|ac^{a-1}\sin(c^{-a}) - ac^{a-a-1}\cos(c^{-a})\right| \\ &\leq h(|ac^{a-1}\sin(c^{-a})| + |ac^{-1}\cos(c^{-a})|) \leq h(ac^{a-1} + ac^{-1})\end{aligned}$$

Since $c \leq 1$, we know that $c^{a-1} = c^a/c \leq 1/c$. Also, $c > x \implies \frac{1}{c} < \frac{1}{x}$, and so

$$|f(x+h) - f(x)| \leq \frac{2ha}{c} < \frac{2ha}{x}$$

With these two bounds, we can now show that f is α-Holder continuous. Let $x, y \in [0,1]$ be arbitrary, and suppose without loss of generality that $x < y$; define $h = y - x$. Then,

$$|f(y) - f(x)| = |f(x+h) - f(x)|$$

If it is the case that $x^{a+1} \leq h \implies x \leq h^{1/(a+1)}$, then we can use the first bound to see

$$|f(x+h) - f(x)| \leq 2(x+h)^a \leq 2(h^{1/(a+1)} + h)^a \leq 2(2h^{1/(a+1)})^a = 2 \cdot 2^a \cdot h^\alpha,$$

where we used that $h \leq 1 \implies h \leq h^{1/(a+1)}$. If instead it is the case that $x^{a+1} > h \implies x > h^{1/(a+1)} \implies 1/x < h^{-1/(a+1)}$, then we can use the second bound to see that

$$|f(x+h) - f(x)| \leq \frac{2ha}{x} \leq 2hah^{-1/(a+1)} = 2ah^{1-\frac{1}{a+1}} = 2ah^\alpha$$

So, in either case, we have $|f(x+h) - f(x)| \leq \max\{2a, 2\cdot 2^a\}h^\alpha$, and so

$$|f(y) - f(x)| \leq \max\{2a, 2\cdot 2^a\} \cdot |y - x|^\alpha$$

■

Problem 6

Solution

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be defined as

$$F(x) = \begin{cases} x^2 \sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Note that over the set $[-1, 1] \setminus \{0\}$, F' exists since the function is the product of a differentiable function and a composition of two differentiable functions over this region. In fact, we can compute via the product rule and the chain rule that for $x \neq 0$,

$$F'(x) = x \cdot \sin(x^{-2}) + x^2 \cos(x^{-2}) \cdot -2x^{-3} = x \cdot \sin(x^{-2}) + \frac{\cos(x^{-2})}{x}$$

Now, to show that $F'(0)$ exists, we need to show that the limit

$$A := \lim_{h \to 0} \frac{F(0 + h) - F(0)}{h}$$

exists. Since $F(0) \equiv 0$, we have

$$A = \lim_{h \to 0} \frac{F(h)}{h} = \frac{h^2 \sin(h^{-2})}{h} = \lim_{h \to 0} h \sin(h^{-2})$$

Note that, since

$$0 \leq |h \sin(h^{-2})| \leq |h|$$

by boundedness of sin, we can apply the Squeeze Theorem to see that

$$0 \leq A = \lim_{h \to 0} h \sin(h^{-2}) \leq \lim_{h \to 0} |h| = 0$$

So, $A = 0$, and the limit defining $F'(0)$ therefore exists. So, F' exists everywhere.

To show that F' is not integrable over $[-1, 1]$, we must show that $|F'|$ dominates a function that is not integrable. To this end, define for each $k \in \mathbb{N}$ the value

$$t_k := \frac{1}{\sqrt{k\pi}}$$

Then, we have that $|\sin(t_k^{-2})| = |\sin(k\pi)| = 0$ and $|\cos(t_k^{-2})| = |\cos(k\pi)| = 1 \implies F'(t_k) = \sqrt{k\pi}$ for each k. Also, we have that $t_k \leq t_1 = \frac{1}{\sqrt{\pi}} < 1$ for all k. Note that over each period, the function

$$|F'(x)| = \left| x \cdot \sin(x^{-2}) + \frac{\cos(x^{-2})}{x} \right|$$

is concave, and so any straight line drawn between two points coming from the same period lies underneath the curve. We will construct a function G made from triangles whose peaks lie at $(t_k, F'(t_k))$ for each k, and whose widths are such that they decay to 0 within the same period of F' that they peak in. In particular, we require the width of each triangle to be

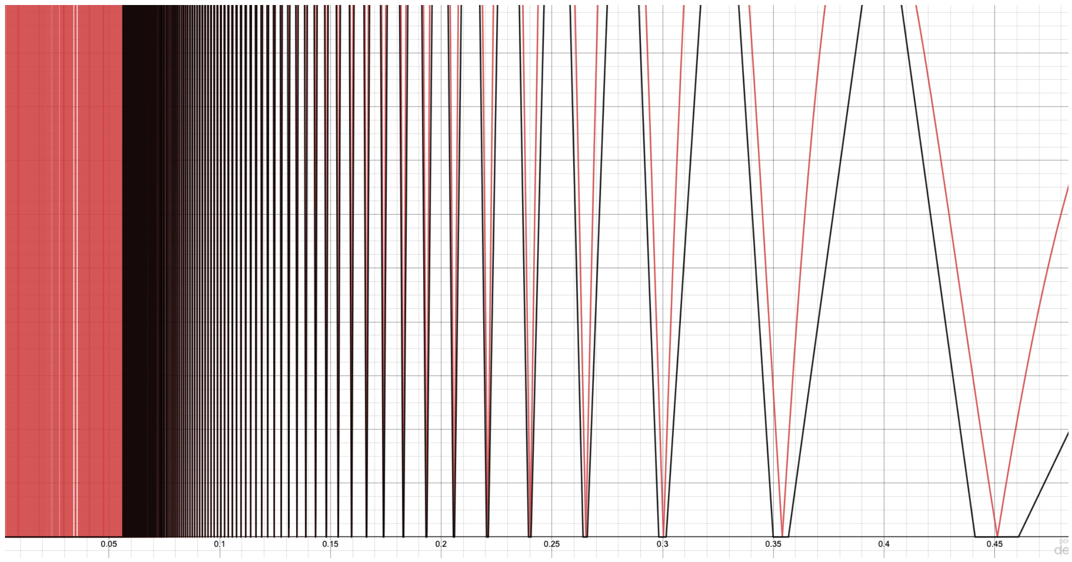
$$2 \cdot \left(\frac{1}{\sqrt{k\pi}} - \frac{1}{\sqrt{k\pi + \frac{\pi}{2}}} \right) := 2\Delta_k$$

Writing it out explicitly, we can define a function $G : [-1, 1] \to \mathbb{R}$ by

$$G(x) := \sum_{k=1}^{\infty} F(t_k) \cdot \max \left\{ 0, 1 - \left| \frac{x - t_k}{\Delta_k} \right| \right\}$$

This function is graphed below in black to verify that it indeed lies below F', which is graphed in red.

Problem 6 continued on next page...



We know by our concavity argument (and a lovely proof by Desmos) that $0 \le G(x) \le |F'(x)|$ for all $x \in [-1, 1]$ (note that $G \equiv 0$ on $[-1, 0)$). Furthermore, because we selected each width of the triangles to ensure that they lie within the same period as the peaks, all of the triangles are disjoint. So, we get that

$$\int_{[-1,1]} G = \sum_{k=1}^{\infty} F(t_k) \cdot \Delta_k = \sum_{k=1}^{\infty} \sqrt{k\pi}\left(\frac{1}{\sqrt{k\pi}} - \frac{1}{\sqrt{k\pi + \frac{\pi}{2}}}\right) = \sum_{k=1}^{\infty}\left(1 - \frac{1}{\sqrt{1 + \frac{1}{2k}}}\right)$$

Note that the function $1 - \frac{1}{\sqrt{1+\frac{1}{2x}}}$ is continuous, and therefore Riemann integrable in the extended sense; so, we can simply compute its integral and apply the integral test to prove that this sum diverges. We have

$$\int_1^{\infty} \frac{1}{\sqrt{1 + \frac{1}{2x}}} dx = \int_1^{\infty} \sqrt{\frac{2x}{2x+1}} dx$$

With the substitution $x = \tan^2(u)/2$, we get

$$= \int_{arctan(\sqrt{2})}^{\pi/2} \frac{\tan(u)}{\sqrt{1 + \tan^2(u)}} \tan(u)\sec^2(u) du = \int_{arctan(\sqrt{2})}^{\pi/2} \tan^2(u)\sec(u) du$$

$$= \int_{arctan(\sqrt{2})}^{\pi/2} \sec^3(u) - \sec(u) du$$

Using the reduction formula for the integral of powers of sec (derived via integration by parts), we get

$$= \left[\frac{\sec(u)\tan(u)}{2}\right]_{arctan(\sqrt{2})}^{\pi/2} - \frac{1}{2}\int_{arctan(\sqrt{2})}^{\pi/2} \sec(u) du$$

$$= \left[\frac{\sec(u)\tan(u)}{2} - \ln(\tan(u) + \sec(u))\right]_{arctan(\sqrt{2})}^{\pi/2}$$

Since $\sec(\arctan(a)) = \sqrt{1 + a^2}$, we get

$$= \lim_{u \to \pi/2} \frac{\sec(u)\tan(u)}{2} - \ln(\tan(u) + \sec(u)) - \frac{\sqrt{6}}{2} + \ln(\sqrt{2} + \sqrt{3})$$

This limit certainly diverges: to see this, let us substitute $a = cos(u)$

$$= \lim_{a \to 0} \frac{\sqrt{1 - a^2}}{2a^2} - \ln(\sqrt{1 - a^2} + 1) + \ln(a) + C = \infty$$

So, the sum diverges, which means $\int_{[-1,1]} F' \ge \int_{[-1,1]} G = \infty$, and so F' is not integrable on $[-1, 1]$. ■

Problem 7

Solution

Proof of (a). Firstly, F must be measurable since it can be written as the difference of two increasing functions; so, since an increasing function has countably many discontinuities, F is continuous a.e. and is therefore measurable. This means that $|F(x+h)-F(x)|$ is also measurable. Note that we can suppose without loss of generality that $h>0$; indeed, we are certainly done if $h=0$ and if $h<0$ we have

$$\int_{\mathbb{R}} |F(x+h)-F(x)|dx = \int_{\mathbb{R}} |F(x)-F(x+h)|dx = \int_{\mathbb{R}} |F(x)-F(x-|h|)|dx = \int_{\mathbb{R}} |F(x+|h|)-F(x)|dx,$$

where the last equality is the translation invariance of the integral. So, let $h>0$ be arbitrary. Let us note that we can partition the integral via

$$\begin{aligned}\int_{\mathbb{R}} |F(x+h)-F(x)|dx &= \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} \mathbb{1}_{[kh,(k+1)h)}(x)\right) |F(x+h)-F(x)|dx \\ &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \mathbb{1}_{[kh,(k+1)h)}(x)\cdot|F(x+h)-F(x)|dx \\ &= \sum_{k\in\mathbb{Z}} \int_{kh}^{(k+1)h} |F(x+h)-F(x)|dx,\end{aligned}$$

where the first equality is since $\sum_{k\in\mathbb{Z}} \mathbb{1}_{[kh,(k+1)h)}(x)$ equals the identity function and the second equality is an application of Corollary 1.10 of Chapter 2 since the elements of our sum are positive and measurable (the product of an indicator function and a measurable function $|F(x+h)-F(x)|$). By the translation invariance of the integral,

$$\int_{kh}^{(k+1)h} |F(x+h)-F(x)|dx = \int_0^h |F(x+(k+1)h)-F(x+kh)|\,dx$$

Pluggin this in, we get

$$\begin{aligned}\int_{\mathbb{R}} |F(x+h)-F(x)|dx &= \sum_{k\in\mathbb{Z}} \int_0^h |F(x+(k+1)h)-F(x+kh)|\,dx \\ &= \int_0^h \sum_{k\in\mathbb{Z}} |F(x+(k+1)h)-F(x+kh)|\,dx\end{aligned}$$

Now, let us note that for every $n\in\mathbb{N}$, by the definition of T_F we clearly have

$$\sum_{k=-n}^{n} |F(x+(k+1)h)-F(x+kh)| \le T_F(x-nh, x+(n+1)h) \le \sup_{a,b} T_F(a,b)$$

Taking the limit as $n\to\infty$, this inequality must still hold: in particular,

$$\sum_{k\in\mathbb{Z}} |F(x+(k+1)h)-F(x+kh)| = \lim_{n\to\infty} \sum_{k=-n}^{n} |F(x+(k+1)h)-F(x+kh)| \le \sup_{a,b} T_F(a,b)$$

Thus, letting $A := \sup_{[a,b]\subset\mathbb{R}} T_F(a,b) < \infty$ we get

$$\int_{\mathbb{R}} |F(x+h)-F(x)|dx = \int_0^h \sum_{k\in\mathbb{Z}} |F(x+(k+1)h)-F(x+kh)|\,dx \le \int_0^h A dx = Ah,$$

Problem 7 continued on next page. . .

completing the proof. ■

Proof of (b). Let us write $\varphi_n(x) := \frac{\varphi(x+1/n)-\varphi(x)}{1/n}$ as a sequence of functions such that $\varphi_n \to \varphi'$ pointwise. Then, clearly $F\varphi_n \to F\varphi'$ pointwise as $n \to \infty$ as well. Let B be a ball of finite radius that φ is supported on, and let $M_F > 0$ be such that $|F| \leq M_F$ since F is bounded. Furthermore, since φ' is continuous and supported on a compact set B, then it is bounded; say, $|\varphi'(x)| \leq M_{\varphi'}$ for all $x \in B$ for some $M_{\varphi'} > 0$. Then, we can note that for all x, the mean value theorem gives us that since φ' is continuous, there is some $c_x \in B$ such that $\varphi'(c_x) = \varphi_n(x)$. This means that

$$|\varphi_n(x)| = |\varphi'(c_x)| \leq M_{\varphi'}$$

So, we get that

$$|F(x)\varphi_n(x)| = |F(x)\varphi_n(x)| \cdot \mathbb{1}_B(x) \leq M_F \cdot M_{\varphi'} \cdot \mathbb{1}_B(x)$$

Note that since B has finite measure, the function $M_F \cdot M_{\varphi'} \cdot \mathbb{1}_B(x)$ is integrable. Since it dominates $|F\varphi_n|$ for all n, we can apply dominated convergence. In particular,

$$\begin{aligned}
\left|\int_{\mathbb{R}} F(x)\varphi'(x)dx\right| &= \left|\lim_{n\to\infty}\int_{\mathbb{R}} F(x)\varphi_n(x)dx\right| = \lim_{n\to\infty}\left|\int_{\mathbb{R}} F(x)\varphi_n(x)dx\right| \\
&= \lim_{n\to\infty}\left|\int_{\mathbb{R}} F(x)\cdot\frac{\varphi(x+1/n)-\varphi(x)}{1/n}dx\right| \\
&= \lim_{n\to\infty} n\cdot\left|\int_{\mathbb{R}} F(x)\varphi(x+1/n)dx - \int_{\mathbb{R}} F(x)\varphi(x)dx\right|
\end{aligned}$$

By the translation invariance of the integral, $\int_{\mathbb{R}} F(x)\varphi(x+1/n)dx = \int_{\mathbb{R}} F(x-1/n)\varphi(x)dx$, and so

$$\begin{aligned}
\left|\int_{\mathbb{R}} F(x)\varphi'(x)dx\right| &\leq \lim_{n\to\infty} n\cdot\left|\int_{\mathbb{R}} F(x-1/n)\varphi(x)dx - \int_{\mathbb{R}} F(x)\varphi(x)dx\right| \\
&= \lim_{n\to\infty} n\cdot\left|\int_{\mathbb{R}} \varphi(x)(F(x-1/n)-F(x))dx\right| \\
&= \lim_{n\to\infty}\left|\int_{\mathbb{R}} \varphi(x)\frac{F(x-1/n)-F(x)}{1/n}dx\right| \\
&\leq \lim_{n\to\infty}\int_{\mathbb{R}}\left|\varphi(x)\frac{F(x-1/n)-F(x)}{1/n}\right|dx \\
&\leq \lim_{n\to\infty}\int_{\mathbb{R}}\left|\frac{F(x-1/n)-F(x)}{1/n}\right|dx,
\end{aligned}$$

where the last inequality is because $\sup_{\mathbb{R}}|\varphi| \leq 1$. Note, however, that applying part (a) with $h = -1/n$ yields that for every $n \in \mathbb{N}$, we have

$$\int_{\mathbb{R}}\left|\frac{F(x-1/n)-F(x)}{1/n}\right|dx \leq \frac{A\cdot|-1/n|}{1/n} = A$$

Since this holds for each element of the sequence, it certainly holds for the limit as well. This grants

$$\left|\int_{\mathbb{R}} F(x)\varphi'(x)dx\right| \leq \lim_{n\to\infty}\int_{\mathbb{R}}\left|\frac{F(x-1/n)-F(x)}{1/n}\right|dx \leq A,$$

and we are done. ■

Problem 8

Solution

Proof. ($\Longrightarrow$) Suppose that f is M-Lipschitz. Then, let $\epsilon > 0$ be arbitrary. Let $\delta := \epsilon/M$. Therefore, for any disjoint intervals $(a_1, b_1), ..., (a_N, b_N)$ with $\sum_{j=1}^N (b_j - a_j) < \delta$, we have

$$\sum_{j=1}^N |f(b_j) - f(a_j)| \leq \sum_{j=1}^N M \cdot |b_j - a_j| = M \sum_{j=1}^N (b_j - a_j) < M\delta = \epsilon,$$

where the first inequality is just an application of the Lipschitz condition. Note that this is precisely the definition of absolute continuity, as δ doesn't depend on the intervals we selected or on N. From the remark after the definition of absolute continuity, we know that f is of bounded variation on any bounded interval. Let

$$E := \{x \in \mathbb{R} : f'(x) \text{ doesn't exist}\}$$

Since f is of bounded variation on any bounded interval, then on each interval $[k, k+1)$ we know that f is differentiable a.e. by Theorem 3.4. So, this means that $m_*(E \cap [k, k+1)) = 0$ for every $k \in \mathbb{Z}$. So, by subadditivity of exterior measure,

$$m_*(E) \leq \sum_{k \in \mathbb{Z}} m_*(E \cap [k, k+1)) = 0 \implies m(E) = 0$$

Now, for any $x \notin E$, we know that $f'(x)$ exists. For such x,

$$|f'(x)| = \left| \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right| = \lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leq \lim_{h \to 0} \left| \frac{M(h)}{h} \right| = M,$$

where the inequality is an application of the Lipschitz condition. So, for every $x \notin E$ (i.e. for a.e. $x \in \mathbb{R}$) we have $|f'(x)| \leq M$.

($\Longleftarrow$) Suppose now that f is absolutely continuous and $|f'(x)| \leq M$ for a.e. $x \in \mathbb{R}$. Let $x, y\mathbb{R}$ be arbitrary, and suppose without loss of generality that $x < y$. Theorem 3.11 grants that

$$f(y) - f(x) = \int_x^y f'(t)dt \implies |f(y) - f(x)| = \left| \int_x^y f'(t)dt \right| \leq \int_x^y |f'(t)|dt \leq M \int_x^y dt = M \cdot |y - x|$$

Since this holds for every pair $x, y \in \mathbb{R}$, we have that f is M-Lipschitz, as desired. ■

Problem 9

Solution

Proof. Let $E \subset \mathbb{R}^n$ be covered in the Vitali sense by $\mathcal{B}$ with $0 < m_*(E) < \infty$. Let $\eta > 0$. Fix a $\delta > 0$ such that the following two conditions hold:

$$\delta < 3^{-n} \quad \text{and} \quad 2\delta + \delta^2 < \eta$$

Note that such a selection certainly can be made since $\lim_{\delta \to 0} 2\delta + \delta^2 = 0$. We can now begin our construction.

First, select $C_1 \supset E$ measurable such that $m(C_1) \leq (1+\delta)m_*(E)$ and such that $\mathcal{B}$ is still a Vitali cover for C_1 (we can do this by Observation 3 of the exterior measure and the properties of a Vitali cover). Next, select a compact $K_1 \subset C_1$ such that $m(C_1 \setminus K_1) \leq \epsilon/2$. Next, select an open $O_1 \supset K_1$ such that $m(O_1) \leq (1+\delta)m(K_1)$. Now, restrict $\mathcal{B}$ such that it only contains balls in O_1 and is still a Vitali cover of K_1; we can do this by restricting the radii of the balls to simply be less than the distance from the boundary of K_1 to the boundary of O_1 (this distance will always be nonzero since no point on the boundary of K_1 can also be on the boundary of O_1 since K_1 is closed and O_1 is open). Note that this preserves the Vitali covering property, since there are still balls of arbitrarily small measure covering all points of K_1. Now, since K_1 is compact, there is a finite collection of balls covering K_1. From here, we can apply the elementary Vitali covering lemma (Lemma 1.2) to find a *disjoint* finite collection of balls $\{B_j^{(1)}\}_{j=1}^{N_1} \subset \mathcal{B}$ such that

$$m(K_1) \leq 3^n \sum_{j=1}^{N_1} \left|B_j^{(1)}\right|$$

From here, define

$$C_2 := K_1 \setminus \left(\bigcup_{j=1}^{N_1} \overline{B_j^{(1)}} \right)$$

and remove from $\mathcal{B}$ all the balls that are not disjoint with $\bigcup_{j=1}^{N_1} \overline{B_j^{(1)}}$, and it will still be a Vitali cover of C_2 (note that C_2 is measurable because K_1 is and the balls are as well). This completes one iteration of the construction. In the next iteration, we repeat the steps, listed more briefly and with arbitrary step indices $i > 1$ below:

1. Select compact $K_i \subset C_i$ such that $m(C_i \setminus K_i) \leq \epsilon/2^i$.
2. Select an open $O_i \supset K_i$ s.t. $m(O_i) \leq (1+\delta)m(K_i)$
3. Restrict $\mathcal{B}$ such that it only contains balls in O_i and is still a Vitali cover of K_i.
4. Apply Lemma 1.2 to a finite subcover to find a disjoint finite collection of balls $\{B_j^{(i)}\}_{j=1}^{N_i} \subset \mathcal{B}$ with

$$m(K_i) \leq 3^n \sum_{j=1}^{N_i} \left|B_j^{(i)}\right|$$

5. Define

$$C_{i+1} := K_i \setminus \left(\bigcup_{j=1}^{N_i} \overline{B_j^{(i)}} \right)$$

and remove from $\mathcal{B}$ all the balls that are not disjoint with $\bigcup_{j=1}^{N_i} \overline{B_j^{(i)}}$, such that it will still be a Vitali cover of C_{i+1} (which is measurable).

Problem 9 continued on next page...

After T steps of the construction, we will have a finite collection

$$\widetilde{\mathcal{B}}_T := \bigcup_{i=1}^{T} \bigcup_{j=1}^{N_i} B_j^{(i)}$$

We know that all of these selected balls must be disjoint from each other because in step 5 of the construction we ensure that our current Vitali cover $\mathcal{B}$ is disjoint from the previously selected balls; an induction argument shows that $\widetilde{\mathcal{B}}_T$ is indeed a disjoint collection. Now, we can note that our selections ensure the following chain of inequalities: for each step i,

$$m(C_{i+1}) = m\left(K_i \setminus \left(\bigcup_{j=1}^{N_i} \overline{B_j^{(i)}}\right)\right) \leq m\left(O_i \setminus \left(\bigcup_{j=1}^{N_i} \overline{B_j^{(i)}}\right)\right)$$

Since all the $B_j^{(i)}$'s are contained in O_i via our restriction of the Vitali covering in step 3, we get that

$$m(C_{i+1}) \leq m(O_i) - \sum_{j=1}^{N_i} \left|B_j^{(i)}\right|$$

By the guarantee of Lemma 1.2 in step 4, we know that $\sum_{j=1}^{N_i} \left|B_j^{(i)}\right| \geq \frac{m(K_i)}{3^n}$, and so

$$m(C_{i+1}) \leq m(O_i) - 3^{-n} m(K_i)$$

By selection of O_i in step 2, we know that $m(O_i) \leq (1+\delta)m(K_i)$, and so since $K_i \subset C_i$, we get

$$m(C_{i+1}) \leq (1 + \delta - 3^{-n})m(K_i) \leq (1 + \delta - 3^{-n})m(C_i)$$

$$\implies m(C_{i+1}) \leq (1 + \delta - 3^{-n})^i \cdot m(C_1)$$

Note that after any number of steps T,

$$C_T \cup \widetilde{\mathcal{B}}_T \cup \left(\bigcup_{i=1}^{T} C_i \setminus K_i\right) \supset C_1$$

since every point that was initially in C_1 was either removed by selecting a compact K_i in step 1, removed by selecting balls in step 4, or remains after all the steps. This means that

$$C_T \supset C_1 \setminus \widetilde{\mathcal{B}}_T \setminus \left(\bigcup_{i=1}^{T} C_i \setminus K_i\right) \implies m(C_T) \geq m(C_1 \setminus \widetilde{\mathcal{B}}_T) - \sum_{i=1}^{T} m(C_i \setminus K_i),$$

where for the inequality we used the fact that $C_i \setminus K_i \subset C_1$ for all i, and so set subtraction equates to subtracting out measure. However, because of the way we selected $K_i \subset C_i$ in step 1, we know that $\sum_{i=1}^{T} m(C_i \setminus K_i) \leq \sum_{i=1}^{\infty} \epsilon/2^i = \epsilon$, and so taking $\epsilon \to 0$ we get

$$m(C_1 \setminus \widetilde{\mathcal{B}}_T) \leq m(C_T)$$

Lastly, we note that since $E \subset C_1$, monotonicity of exterior measure yields

$$m_*(E \setminus \widetilde{\mathcal{B}}_T) \leq m(C_T) \leq (1 + \delta - 3^{-n})^T \cdot m(C_1)$$

Since the factor is < 1 and we are free to make T as large as possible, we find that if we continue the construction indefinitely we get that

$$m_*(E \setminus \widetilde{\mathcal{B}}_\infty) = 0$$

Problem 9 continued on next page...

as desired.

To verify the other condition, note that

$$\begin{aligned}
(C_{i+1}) &\leq m(O_i) - \sum_{j=1}^{N_i} |B_j^{(i)}| \leq (1+\delta)m(K_i) - \sum_{j=1}^{N_i} |B_j^{(i)}| \\
&\implies \sum_{j=1}^{N_i} |B_j^{(i)}| \leq (1+\delta)m(K_i) - m(C_{i+1}) \leq (1+\delta)m(C_i) - m(C_{i+1}) \\
&\implies \sum_{i=1}^{T}\sum_{j=1}^{N_i} |B_j^i| \leq \delta \sum_{i=1}^{T} m(C_i) + m(C_1) - m(C_T) \\
&\leq \delta m(C_1) \sum_{i=2}^{T} (1+\delta-3^{-n})^i + (1+\delta)m(C_1) - m(C_T) \\
&\leq \delta \cdot \left(\frac{1}{1-(1+\delta-3^{-n})} \right) \cdot m(C_1) + (1+\delta)m(C_1) \\
&= \left(1 + \delta\left(1 + \frac{1}{3^{-n}-\delta}\right)\right) m(C_1) = \left(\frac{3^{-n}-\delta}{3^{-n}-\delta} + \frac{3^{-n}\delta - \delta^2 + \delta}{3^{-n}-\delta} \right) m(C_1) \\
&= \frac{3^{-n} + 3^{-n}\delta - \delta^2}{3^{-n}-\delta} \cdot m(C_1) \leq (1+\delta) \frac{3^{-n} + 3^{-n}\delta - \delta^2}{3^{-n}-\delta} \cdot m_*(E) \\
&= \frac{3^{-n} + 3^{-n}\delta - \delta^2 + 3^{-n}\delta + 3^{-n}\delta^2 - \delta^3}{3^{-n}-\delta} m_*(E) = \left(\delta + \delta^2 + \frac{3^{-n} + 3^{-n}\delta}{3^{-n}-\delta} \right) m_*(E) \\
&\leq \left(\delta + \delta^2 + \frac{3^{-n} + 3^{-n}\delta}{3^{-n}} \right) m_*(E) = \left(1 + 2\delta + \delta^2\right) m_*(E) \\
&\leq (1+\eta) m_*(E),
\end{aligned}$$

where the first line was already derived earlier, the second line makes use of the fact that $K_i \subset C_i$, the third line computes the telescoping sum of the second line over i's, the fourth line makes use of the decaying form of $m(C_i)$ from earlier, the fifth line uses the geometric series and the fact that $m(C_T) \geq 0$, the seventh line uses that $m(C_1) \leq (1+\delta)m_*(E)$, and the last line uses our selection of δ from the beginning of the proof. Since the bound

$$\sum_{B \in \widetilde{\mathcal{B}}_T} |B| = \sum_{i=1}^{T}\sum_{j=1}^{N_i} |B_j^i| \leq (1+\eta)m_*(E)$$

holds for all T, it certainly holds in the limit; i.e.

$$\sum_{B \in \widetilde{\mathcal{B}}_\infty} |B| \leq (1+\eta)m_*(E)$$

as desired. ∎

Problem 10

Solution

Proof of (a). Let B be the unit ball and define $\varphi(x) := \frac{1}{m(B)}\mathbb{1}_B(X)$. Also, for $\delta > 0$ define

$$\varphi_\delta(x) := \frac{1}{\delta^2} \cdot \varphi(x/\delta)$$

Let $x \in \mathbb{R}^2$ be arbitrary with $x_1x_2 \neq 0$. We can compute that, since $\varphi(-x) = \varphi(x)$,

$$\begin{aligned}
(\varphi_\delta)^*_\mathcal{R}(x) &= \sup_{R\in\mathcal{R}} \frac{1}{m(R)} \int_R |\varphi_\delta(x-y)|dy = \sup_{R\in\mathcal{R}} \frac{1}{m(R)} \int_R |\varphi_\delta(y-x)|dy \\
&= \sup_{R\in\mathcal{R}} \frac{1}{m(R)} \int_{\mathbb{R}^2} \mathbb{1}_R(y)|\varphi_\delta(y-x)|dy \\
&= \sup_{R\in\mathcal{R}} \frac{1}{m(R)} \int_{\mathbb{R}^2} \mathbb{1}_R(y+x)|\varphi_\delta(y)|dy \\
&= \sup_{R\in\mathcal{R}} \frac{1}{m(R)\cdot m(B)\cdot\delta^2} \int_{\mathbb{R}^2} \mathbb{1}_R(y+x)\mathbb{1}_B(y/\delta)dy
\end{aligned}$$

Note that $y \in B_\delta(O) \iff y/\delta \in B$ by scaling, and so $\mathbb{1}_B(y/\delta) = \mathbb{1}_{B_\delta(O)}(y)$. This means that

$$\begin{aligned}
(\varphi_\delta)^*_\mathcal{R}(x) &= \sup_{R\in\mathcal{R}} \frac{1}{m(R)\cdot m(B)\cdot\delta^2} \int_{\mathbb{R}^2} \mathbb{1}_R(y+x)\mathbb{1}_{B_\delta(O)}(y)dy \\
&= \sup_{R\in\mathcal{R}} \frac{1}{m(R)\cdot m(B)\cdot\delta^2} \int_{\mathbb{R}^2} \mathbb{1}_R(y)\mathbb{1}_{B_\delta(O)}(y-x)dy
\end{aligned}$$

Now, note that $y - x \in B_\delta(O) \iff y \in B_\delta(x)$ clearly, and so $\mathbb{1}_{B_\delta(O)}(y-x) = \mathbb{1}_{B_\delta(x)}(y)$. This means

$$\begin{aligned}
(\varphi_\delta)^*_\mathcal{R}(x) &= \sup_{R\in\mathcal{R}} \frac{1}{m(R)\cdot m(B)\cdot\delta^2} \int_{\mathbb{R}^2} \mathbb{1}_R(y)\mathbb{1}_{B_\delta(x)}(y)dy \\
&= \sup_{R\in\mathcal{R}} \frac{m(R\cap B_\delta(x))}{m(R)\cdot m(B)\cdot\delta^2}
\end{aligned}$$

Since $m(B)\cdot\delta^2 = m(B_\delta(O)) = m(B_\delta(x))$, we get

$$(\varphi_\delta)^*_\mathcal{R}(x) = \sup_{R\in\mathcal{R}} \frac{1}{m(R)} \frac{m(R\cap B_\delta(x))}{m(B_\delta(x))}$$

Suppose, without loss of generality, that $\delta < |x|$; this means that $O \notin B_\delta(x)$. Clearly, we would like to take the supremum over rectangles for which one of the vertices is at the origin; if this is not the case, we can always shrink the rectangle so that the vertex in the opposite quadrant as x goes to the origin and improve the value of $\frac{1}{m(R)}\frac{m(R\cap B_\delta(x))}{m(B_\delta(x))}$ by decreasing $m(R)$ without changing $m(R\cap B_\delta(x))$. Also, certainly the opposite vertex must lie within the square of side length δ around x, since if it undershoots this square we have $m(R\cap B_\delta(x)) = 0$, and if it overshoots we can shrink R without changing $m(R\cap B_\delta(x))$. In any case, we find that as $\delta \to 0$, we must have the opposite vertex lie precisely at x to maximize the function; this can also be found by Lebesgue differentiation, since

$$\lim_{\delta\to 0} \frac{m(R\cap B_\delta(x))}{m(B_\delta(x))} = \lim_{\delta\to 0} \frac{1}{m(B_\delta(x))} \int_{B_\delta(x)} \mathbb{1}_R = \mathbb{1}_R(x)$$

for almost every x, and so the smallest such R has the opposite vertex at x. In either case, we find that for a.e. x,

$$(\varphi_\delta)^*_\mathcal{R}(x) \to \sup_{r\in\mathcal{R}} \frac{1}{m(R)}\mathbb{1}_R(x) = \frac{1}{|x_1x_2|} \quad \text{as } \delta \to 0$$

Problem 10 continued on next page...

Suppose by way of contradiction that the weak-type inequality held. Then, we would have that

$$m(\{|x| \leq 1 : (\varphi_\delta)^*_{\mathcal{R}}(x) > \alpha\}) \leq m(\{x : (\varphi_\delta)^*_{\mathcal{R}}(x) > \alpha\}) \leq \frac{A}{\alpha}$$

Taking $\delta \to 0$, this would imply that for all $\alpha > 0$,

$$m(\{|x| \leq 1 : |x_1 x_2|^{-1} > \alpha\}) \leq \frac{A}{\alpha}$$

Note that the set $\{|x| \leq 1 : |x_1x_2|^{-1} > \alpha\} = \{|x| \leq 1 : |x_1x_2| < 1/\alpha\}$ is the region of the plane contained in the disk that lies between the hyperbolas $x_1x_2 < 1/\alpha$ and $-x_1x_2 < 1/\alpha$, which will equal 4 times the area of the region of the disk under the hyperbola $x_1x_2 < 1/\alpha$ in the first quadrant. We will do a routine integration for values of α large enough that the hyperbola intersects the disk to find this area. To this end, let $x_\pm = \sqrt{\frac{1 \pm \sqrt{1-4/\alpha^2}}{2}}$ be the roots of the expression $\sqrt{1-x^2} = 1/\alpha x$; i.e. these are the points where the hyperbola and disk intersect. We then have that the area of the set $V := m(\{|x| \leq 1 : |x_1x_2| < 1/\alpha\})$ is

$$V = 4\int_0^{x_-} \sqrt{1-x^2}dx + 4\int_{x_-}^{x_+} \frac{1}{\alpha x}dx + 4\int_{x_+}^1 \sqrt{1-x^2}dx \geq \int_{x_-}^{x_+} \frac{1}{\alpha x}dx = \frac{1}{\alpha}\ln(x_+/x_-)$$

$$= \frac{1}{2\alpha}\ln\left(\frac{1+\sqrt{1-4/\alpha^2}}{1-\sqrt{1-4/\alpha^2}}\right) = \frac{1}{2\alpha}\ln\left(\frac{1+1-4/\alpha^2+2\sqrt{1-4/\alpha^2}}{1-(1-4/\alpha^2)}\right) = \frac{1}{2\alpha}\ln\left(\frac{2\alpha^2-4+2\alpha^2\sqrt{1-4/\alpha^2}}{4}\right)$$

For α large enough that $2\alpha^2\sqrt{1-4/\alpha^2} > 4$, we get that

$$V \geq \frac{1}{2\alpha}\ln(2\alpha^2/4) \sim \frac{\ln\alpha}{\alpha}$$

Note that this contradicts the weak-type inequality for large enough α. So, the weak-type inequality cannot hold in generality. ■

Proof of (b). From the result of part (a), we know that for all $\alpha > 0$, there exists some function $f_\alpha \in L^1(\mathbb{R}^2)$ and some A_α such that

$$m(\{x : (f_\alpha)^*_{\mathcal{R}}(x) > \alpha\}) \geq \frac{A_\alpha}{\alpha}||f_\alpha||_{L^1}$$

Using this, we can select a sequence of functions $(f_n)_{n\in\mathbb{N}}$ by setting $\alpha = n$. Define the function

$$f := \sum_{n=1}^{\infty} \frac{1}{2^n \cdot ||f_n||_{L_1}}|f_n|$$

Then, this function is also in L^1; indeed, it is bounded above by 1 a.e. since each constituent in the sum is bounded above by $\frac{1}{2^n}$ a.e.. Furthermore, we know that there will always be points for which the maximal function $f^*_{\mathcal{R}}(x)$ takes the value ∞, since for the constituent f_n's we had a lower bound on the measure of the set of points for which their maximal function took a value $> n$. This means that there are points x for which $f^*_{\mathcal{R}}(x)$ is unbounded, which in particular means that for a.e. x' we can take a sequence of rectangles containing those points and attain unbounded averages. Put differently, for a.e. x' we have

$$\limsup_{diam(R)\to 0} \frac{1}{m(R)}\int_R |f(x'-y)|dy = \limsup_{diam(R)\to 0} \frac{1}{m(R)}\int_R f(x'-y)dy = \infty$$

■