MAT 425: Problem Set 5

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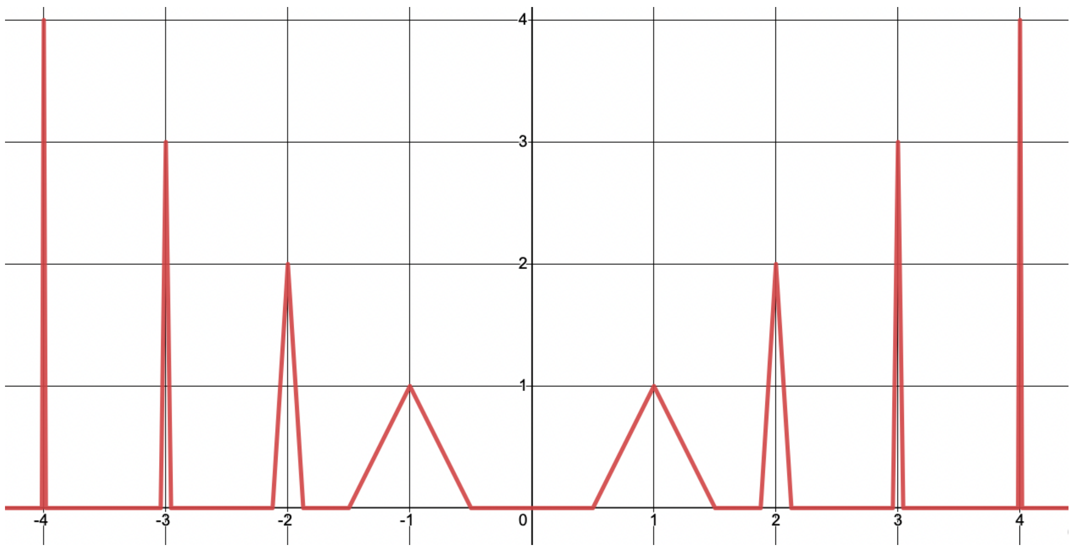
Problem 1

Solution

Proof of (a). For all $x \in \mathbb{R}$, let $r(x)$ denote the integer closest to x. Consider the function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by

$$f(x) = |r(x)| \cdot \max\left\{1 - |r(x)| \cdot 2^{|r(x)|} \cdot |x - r(x)|, \quad 0\right\}$$

This opaque definition is better explained in words: in short, at each integer value $k \in \mathbb{Z} \setminus \{0\}$ is centered a triangle with width $2^{-|k|+1}/|k|$ and height $|k|$. Each triangle linearly decreases from its peak of $|k|$ to 0, and the function f is 0 everywhere outside of these triangles. Since no triangle has a width of $> 1/2$, each triangle decreases to a y-value of 0 before the next triangle begins; this ensures continuity. Also, the function is by definition nonnegative, since it is always at least 0. A plot of the function is given below for clarity:



The function is clearly measurable since it is continuous; to see that it is integrable, note that we can write its integral as the sum of the areas of each triangle:

$$\int_{\mathbb{R}} f = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{2} \cdot |k| \cdot \frac{2^{-|k|+1}}{|k|} = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} 2^{-|k|} = 2 \cdot \sum_{k=1}^{\infty} 2^{-k} = 2 < \infty$$

So, f is integrable. However, $\limsup_{x \to \infty} f(x) = \infty$ as the sequence $(f(k))_k$ grows unboundedly as $k \to \infty$ (this is because $f(k) = k$ for each $k \in \mathbb{N}$). ■

Proof of (b). Suppose that $f : \mathbb{R} \to \mathbb{R}$ is positive, uniformly continuous, and integrable. Suppose by way of contradiction that $\lim_{|x| \to \infty} f(x)$ exists, but is not equal to 0 (say it is equal to a for some $a > 0$). Then, by the definition of a limit, there is some $R > 0$ such that for all x with $|x| > R$, we have $|f(x) - a| < a/2 \implies f(x) > a/2$. This would contradict Proposition 1.12(i), which states that for any $\epsilon > 0$ there is some R' such that

$$\int_{|x| > R'} f < \epsilon$$

Clearly, if f maintained a nonzero value $a/2$ at infinity, its integral over the region $|x| > R'$ could not be arbitrarily small; this is a contradiction. So, we see that if $\lim_{|x| \to \infty} f(x)$ exists, it *must* equal 0. Now, all that is left to do is to show that the limit exists.

Let $\epsilon > 0$. Since f is uniformly continuous, let $\delta > 0$ be such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon/2$. Suppose by way of contradiction that $\lim_{|x|\to\infty} f(x)$ doesn't exist. Then, for any $R > 0$ (unrelated to the earlier R), there must exist a y with $|y| > R$ such that $f(y) > \epsilon$ (if this weren't the case, then $f(y) \leq \epsilon$ for all $|y| > R$ for all R, and the limit therefore exists and equals 0). This means that for all $x \in (y - \delta, y + \delta)$, we have $|f(x) - f(y)| < \epsilon/2 \implies f(x) > \epsilon/2$. Therefore, for every $R > 0$ we can find a y with $|y| > R$ such that

$$\int_{(y-\delta,y+\delta)} f > \int_{(y-\delta,y+\delta)} \frac{\epsilon}{2} = \epsilon\delta$$

I will refer by $(*)$ to the statement that for every $R > 0$ there exists some $y \in \mathbb{R}$ with $|y| > R$ such that $\int_{(y-\delta,y+\delta)} f > \epsilon\delta$. We know by Proposition 1.12(i) that there exists some $R_0 > 0$ with $\int_{|y|>R_0} f < \epsilon$. Now, let $n > 1/\delta$ be some big natural number. Construct a sequence as follows:

1. From $(*)$, we know that there exists some y_1 with $|y_1| > R_1 := R_0 + \delta$ and $\int_{(y_1-\delta,y_1+\delta)} f > \epsilon\delta$
2. For each $i \in \{2, ..., n\}$, select R_i such that $R_i > |y_{i-1}| + 2\delta$. Then, apply $(*)$ to find a y_i such that $|y_i| > R_i$ and $\int_{(y_i-\delta,y_i+\delta)} f > \epsilon\delta$

This construction has the following properties:

- $R_0 + \delta < R_1 < ... < R_n$. This comes trivially from the construction of each R_i.
- The interval $(y_i - \delta, y_i + \delta)$ is disjoint from $(y_j - \delta, y_j + \delta)$ for all $1 \leq i < j$. To see this, note that $|y_j| > R_j \geq R_{i+1} > |y_i| + 2\delta$.
- For all $i \geq 1$ and all $x \in (y_i - \delta, y_i + \delta)$, we have $|x| > R_i - \delta > R_0$. The first inequality comes from the way we selected y_i, and the second inequality comes from the way we selected R_1 and the fact that $R_i \geq R_1$.

We can use these properties to say that

$$\bigcup_{i=1}^{n} (y_i - \delta, y_i + \delta) \subset \{y : |y| > R_0\}$$

Since f is nonnegative, this reveals that

$$\int_{|y|>R_0} f \geq \sum_{i=1}^{n} \int_{(y_i-\delta,y_i+\delta)} f > \sum_{i=1}^{n} \delta\epsilon > n \cdot \delta\epsilon > \epsilon,$$

where for the second inequality we used the property granted by $(*)$ and for the fourth inequality we used that $n > 1/\delta$. This, however, is a contradiction since we selected R_0 specifically so that $\int_{|y|>R_0} f < \epsilon$. Therefore, the limit $\lim_{|x|\to\infty} f(x)$ must exist. Thus, it must also equal 0. ■

Problem 2

Solution

Proof. Suppose that f is integrable and that $\int_E f \geq 0$ for all measurable E. Let x be an arbitrary point in the Lebesgue set of f. Then, for all balls B such that $x \in B$, we have that $\int_B f \geq 0$ by the assumed property. Taking the limit as the balls B shrink about x,

$$\lim_{\substack{m(B)\to 0 \\ B \text{ ball} \\ B \ni x}} \int_B f \geq 0 \implies \lim_{\substack{m(B)\to 0 \\ B \text{ ball} \\ B \ni x}} \frac{1}{m(B)} \int_B f \geq 0 \implies f(x) \geq 0,$$

where the first implication simply comes from the fact that $m(B) \geq 0$, and the second implication uses the definition of Lebesgue points of functions. So, $f \geq 0$ over its Lebesgue set. Corollary 1.6 grants that since $f \in L^1(\mathbb{R}^d)$, almost every $x \in \mathbb{R}^d$ is in the Lebesgue set of f. So, since $f \geq 0$ for all points in its Lebesgue set, we find that $f \geq 0$ a.e..

Suppose now that $\int_E f = 0$ for every measurable E. This means that $\int_E f \geq 0$ and $\int_E (-f) \geq 0$ for every measurable E. Using the above result, this tells us that $f \geq 0$ a.e. and also that $(-f) \geq 0 \implies f \leq 0$ a.e.. Let A be the set over which f is not ≥ 0, and let B be the set over which f is not ≤ 0; the previous statement says that $m(A) = m(B) = 0$. Also, $f(x) = 0$ for all $x \notin A \cup B$. From this, we see that $f = 0$ a.e.. ■

Problem 3

Solution

Proof. Let $f \in L^1(\mathbb{R}^n)$ and define $E_\alpha := \{x \in \mathbb{R}^n : |f(x)| > \alpha\}$. Define the function $F : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \{0, 1\}$ by

$$F(x, \alpha) := \mathbb{1}_{E_\alpha}(x)$$

To see that F is measurable on $\mathbb{R}^{n+1}$, consider the set $\{F < a\} \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ for various values of a . If $a \leq 0$, this set is simply the empty set (and is therefore measurable) since $F \geq 0$ always. If $a \geq 1$, this set is simply $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ (and is therefore measurable) since $F \leq 1$ always. Lastly, for $a \in (0, 1)$ we have that

$$\{F < a\} = \{F = 0\} = \{F = 1\}^C = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} : 0 \leq \alpha < |f(x)|\}^C,$$

where the last equality is by definition of E_α. Corollary 3.8(i) states that because f is measurable, so is $\{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} : 0 \leq \alpha < |f(x)|\}$. Since the space of measurable sets is closed under complements, this tells us that $\{F < a\}$ is measurable for $a \in (0, 1)$. Therefore, $\{F < a\}$ is measurable for all a, and F is then a measurable function over $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Since F is nonnegative and measurable, we can apply Theorem 3.2(iii) to see that

$$\int_{\mathbb{R}^n} \left(\int_{[0,\infty)} F(x, \alpha) d\alpha \right) dx = \int_{[0,\infty)} \left(\int_{\mathbb{R}^n} F(x, \alpha) dx \right) d\alpha$$

$$\implies \int_{\mathbb{R}^n} \left(\int_{[0,\infty)} \mathbb{1}_{E_\alpha}(x) d\alpha \right) dx = \int_{[0,\infty)} \left(\int_{\mathbb{R}^n} \mathbb{1}_{E_\alpha}(x) dx \right) d\alpha$$

Now, note that for all $x \in \mathbb{R}^n$ we have $x \in E_\alpha \iff \alpha < |f(x)|$, and so

$$\int_{[0,\infty)} \mathbb{1}_{E_\alpha}(x) d\alpha = \int_{[0,|f(x)|)} 1 d\alpha = |f(x)|$$

Also, we know that

$$\int_{\mathbb{R}^n} \mathbb{1}_{E_\alpha}(x) dx = m(E_\alpha)$$

Plugging these results in, we get that

$$\int_{\mathbb{R}^n} |f(x)| dx = \int_{[0,\infty)} m(E_\alpha) d\alpha$$

as desired. ■

Problem 4

Solution

Proof of (a). The fact that f, g are integrable means that they are certainly measurable on $\mathbb{R}^n$. By Proposition 3.9, we get that f being measurable on $\mathbb{R}^n$ implies that $\widetilde{f}(x, y) = f(x - y)$ is measurable on $\mathbb{R}^{2n}$. Also, Corollary 3.7 gives that $\widetilde{g}(x, y) = g(y)$ is measurable on $\mathbb{R}^{2n}$ as well. Since f, g are integrable, we know that $\widetilde{f}, \widetilde{g}$ are finite a.e.. This means that $\widetilde{f} \cdot \widetilde{g}$ is measurable on $\mathbb{R}^{2n}$ (we can redefine them on a set of measure 0 so that they are finite everywhere and then apply property 5 of measurable functions; since we only change a set of measure 0, which is measurable, the measurability of $\widetilde{f}, \widetilde{g}$, and $\widetilde{f} \cdot \widetilde{g}$ are not affected). So, $f(x - y)g(y)$ is measurable on $\mathbb{R}^{2n}$. ∎

Proof of (b). By Theorem 3.2(iii), since $|f(x - y)g(y)| \geq 0$, we can write

$$\int_{\mathbb{R}^{2n}} |f(x-y)g(y)|dxdy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)| \cdot |g(y)|dx \right) dy$$

Since $|g(y)|$ is constant with respect to x, we can move it out and get

$$= \int_{\mathbb{R}^n} |g(y)| \left(\int_{\mathbb{R}^n} |f(x-y)|dx \right) dy$$

By translation invariance of the integral, $\int_{\mathbb{R}^n} |f(x - y)|dx = ||f||_{L^1}$, which is constant with respect to the variable y. So,

$$= ||f||_{L^1} \int_{\mathbb{R}^n} |g(y|dy = ||f||_{L^1} \cdot ||g||_{L^1} < \infty,$$

where we know that this is finite since f, g are both integrable. We have therefore proven that $f(x - y)g(y)$ is integrable on $\mathbb{R}^{2n}$. ∎

Proof of (c). From part (b), we know that $h(x, y) := f(x - y)g(y)$ is integrable on $\mathbb{R}^{2n}$. Then, Fubini's Theorem (i) grants that the slice h^x is integrable on $\mathbb{R}^n$ for almost every x. In other words, for almost every x we have that $f(x - y)g(y)$ is integrable on $\mathbb{R}^n$. So, $(f * g)(x)$ is well defined for a.e. x. ∎

Proof of (d). Fubini's Theorem (ii) gives that $f * g$ is integrable on $\mathbb{R}^n$ as well. Now,

$$||f * g||_{L^1} = \int_{\mathbb{R}^n} |(f * g)(x)| \, dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)dy \right| dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)| \, dydx,$$

with equality holding if and only if f and g are nonnegative. From here we proceed exactly as we did in part (b): nonnegativity of $|f(x - y)g(y)|$ over $\mathbb{R}^{2n}$ allows us to use Theorem 3.2(iii) to switch the integrals and get that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)| \, dydx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)| \, dxdy$$

$|g(y)|$ is constant with respect to x, and so

$$= \int_{\mathbb{R}^n} |g(y)| \left(\int_{\mathbb{R}^n} |f(x-y)| \, dx \right) dy$$

Translation invariance of the integral gives that $\int_{\mathbb{R}^n} |f(x - y)| \, dx = ||f||_{L^1}$, which is constant with respect to y. So,

$$= ||f||_{L^1} \int_{\mathbb{R}^n} |g(y)|dy = ||f||_{L^1} \cdot ||g||_{L^1}$$

Equality holds if and only if both f and g are nonnegative. ■

Proof of (e). Define

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$$

and provide an analogous definition for $\widehat{g}$. Note that

$$|\widehat{f}(\xi)| = \left|\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx\right| \leq \int_{\mathbb{R}^n} |f(x)|\cdot|e^{-2\pi i x\cdot\xi}|dx = \int_{\mathbb{R}^n} |f(x)|dx = ||f||_{L^1},$$

where the second equality holds since $|e^{ia}| = 1$ for all $a \in \mathbb{R}$. So, $\widehat{f}$ is bounded in magnitude by $||f||_{L^1} < \infty$.

Now, we would like to show that $\widehat{f}$ is continuous. Let $\epsilon > 0$. By Proposition 1.12(i), since f is integrable there exists a $R > 0$ such that

$$\int_{|x|>R} |f(x)|dx < \epsilon$$

Fix $\xi \in \mathbb{R}^n$. Then, for all $h \in \mathbb{R}^n$ with $|h| < \delta := \frac{\epsilon}{||f||_{L^1}\cdot R}$, we can say that

$$\begin{aligned}
|\widehat{f}(\xi+h) - \widehat{f}(\xi)| &= \left|\int_{\mathbb{R}^n} f(x)\left(e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right)dx\right| \\
&\leq \int_{\mathbb{R}^n} \left|f(x)\left(e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right)\right|dx \\
&= \int_{|x|>R} \left|f(x)\left(e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right)\right|dx + \int_{|x|\leq R} \left|f(x)\left(e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right)\right|dx
\end{aligned}$$

We can bound these two terms seperately. Firstly, since $\int_{|x|>R}|f(x)|dx < \epsilon$, we have that

$$\begin{aligned}
\int_{|x|>R} \left|f(x)\left(e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right)\right|dx &\leq \int_{|x|>R} |f(x)|\left(\left|e^{-2\pi i x\cdot(\xi+h)}\right| + \left|e^{-2\pi i x\cdot\xi}\right|\right)dx \\
&= 2\int_{|x|>R} |f(x)|dx < 2\epsilon
\end{aligned}$$

For the second term, let us first derive a cute fact about exponentials. For notation, let $a := -2\pi x\cdot(\xi+h)$ and $b := -2\pi x\cdot\xi$. Then,

$$\begin{aligned}
e^{ia} - e^{ib} &= e^{ib}e^{i\left(\frac{a-b}{2}\right)}\left(e^{i\left(\frac{a-b}{2}\right)} - e^{i\left(\frac{b-a}{2}\right)}\right) \\
&= e^{ib}e^{i\left(\frac{a-b}{2}\right)}\left(e^{i\left(\frac{a-b}{2}\right)} - e^{-i\left(\frac{a-b}{2}\right)}\right) \\
&= e^{ib}e^{i\left(\frac{a-b}{2}\right)}\cdot 2i\cdot sin\left(\frac{a-b}{2}\right) \\
\implies \left|e^{ia} - e^{ib}\right| &= 2\left|sin\left(\frac{a-b}{2}\right)\right|
\end{aligned}$$

Since $-x \leq sin(x) \leq x$ for all $x \in \mathbb{R}$, this gives us that

$$\left|e^{ia} - e^{ib}\right| \leq 2\left|\frac{a-b}{2}\right| = |a-b|$$

Plugging in our values for a and b we get

$$\left|e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right| \leq 2\pi|x\cdot h| \leq 2\pi|x|\cdot|h|,$$

where the last inequality is the Cauchy-Schwartz inequality. Since $|x| \le R$ over the interval and $|h| < \delta$,

$$\left|e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right| \le 2\pi R\delta = 2\pi\frac{\epsilon}{||f||_{L^1}}$$

Going back to our earlier inequality,

$$\begin{aligned}
|\widehat{f}(\xi+h) - \widehat{f}(\xi)| &\le 2\epsilon + \int_{|x|\le R} \left|f(x)\left(e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right)\right| dx \\
&= 2\epsilon + \int_{|x|\le R} |f(x)| \cdot \left|\left(e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right)\right| dx \\
&\le 2\epsilon + 2\pi\frac{\epsilon}{||f||^{L^1}}\int_{|x|\le R} |f(x)|\, dx \le 2\epsilon + 2\pi\frac{\epsilon}{||f||^{L^1}}\int_{\mathbb{R}^n} |f(x)|\, dx \\
&= 2\epsilon + 2\pi\epsilon
\end{aligned}$$

Since this holds for all h with $|h| < \delta$, we see that $\widehat{f}$ is continuous at ξ; since it holds for all ξ, we have that $\widehat{f}$ is continuous (in fact, it is uniformly continuous since δ didn't depend on ξ).

To prove the last part, fix a ξ. We have

$$\begin{aligned}
\widehat{(f*g)}(\xi) &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(x-y)g(y)dy\right)e^{-2\pi i x\cdot\xi}dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x-y)g(y)dydx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-2\pi i(x-y)\cdot\xi}\cdot e^{-2\pi i y\cdot\xi}\cdot f(x-y)g(y)dydx
\end{aligned}$$

Note that the integrability of $f(x-y)g(y)$ over $\mathbb{R}^{2n}$ implies the integrability of $e^{-2\pi i(x-y)\cdot\xi}\cdot e^{-2\pi i y\cdot\xi}\cdot f(x-y)g(y)$ over $\mathbb{R}^{2n}$ since $\left|e^{-2\pi i(x-y)\cdot\xi}\cdot e^{-2\pi i y\cdot\xi}\cdot f(x-y)g(y)\right| = |f(x-y)g(y)|$. This means that we can apply Fubini's Theorem (iii) to switch the order of the integrals and get that

$$\widehat{(f*g)}(\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-2\pi i(x-y)\cdot\xi}\cdot e^{-2\pi i y\cdot\xi}\cdot f(x-y)g(y)dxdy$$

Since $e^{-2\pi i y\cdot\xi}\cdot g(y)$ is constant with respect to x,

$$= \int_{\mathbb{R}^n} e^{-2\pi i y\cdot\xi}\cdot g(y)\left(\int_{\mathbb{R}^n} e^{-2\pi i(x-y)\cdot\xi}\cdot f(x-y)dx\right)dy$$

By translation invariance of the integral, $\int_{\mathbb{R}^n} e^{-2\pi i(x-y)\cdot\xi}\cdot f(x-y)dx = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\cdot f(x)dx = \widehat{f}(\xi)$, and so

$$= \widehat{f}(\xi)\cdot\int_{\mathbb{R}^n} e^{-2\pi i y\cdot\xi}\cdot g(y)dy = \widehat{f}(\xi)\cdot\widehat{g}(\xi)$$

This is the desired result. ■

Problem 5

Solution

Proof. Following the hint, define

$$\xi' := \frac{1}{2} \cdot \frac{\xi}{|\xi|^2}$$

Then, we get that

$$\int_{\mathbb{R}^n} [f(x) - f(x - \xi')] e^{-2\pi i x \cdot \xi} dx = \widehat{f}(\xi) - \int_{\mathbb{R}^n} f(x - \xi') e^{-2\pi i x \cdot \xi} dx$$

By translation invariance of the integral,

$$\int_{\mathbb{R}^n} f(x - \xi') e^{-2\pi i x \cdot \xi} dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i (x + \xi') \cdot \xi} dx$$

We can note that

$$(x + \xi') \cdot \xi = x \cdot \xi + \frac{|\xi|^2}{2|\xi|^2} = x \cdot \xi + \frac{1}{2},$$

and so

$$\int_{\mathbb{R}^n} f(x) e^{-2\pi i (x + \xi') \cdot \xi} dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \cdot e^{-2\pi i \cdot 1/2} dx = e^{-i\pi} \widehat{f}(\xi) = -\widehat{f}(\xi)$$

Therefore,

$$\int_{\mathbb{R}^n} [f(x) - f(x - \xi')] e^{-2\pi i x \cdot \xi} dx = 2\widehat{f}(\xi)$$

Define the function

$$f_\xi(x) := f(x - \xi') e^{-2\pi i x \cdot \xi} = f\left(x - \frac{\xi}{2|\xi|^2}\right) e^{-2\pi i x \cdot \xi}$$

This tells us that $f_\xi(x) \to f(x) e^{-2\pi i x \cdot \xi}$ pointwise as $|\xi| \to \infty$ (by this I mean that they get arbitrarily close). So, by Proposition 2.5, since $f \in L^1$ we get that

$$\int_{\mathbb{R}^n} [f(x) - f(x - \xi')] e^{-2\pi i x \cdot \xi} dx \to 0 \qquad \text{as } |\xi| \to \infty$$

$$\implies 2\widehat{f}(\xi) \to 0 \qquad \text{as } |\xi| \to \infty$$

■

Problem 6

Solution

Proof. Define $g(x) := f(x) \cdot \mathbb{1}_{[0,2\pi]}$. Then,

$$\int_{[0,2\pi]} f(x)e^{-inx}dx = \int_{\mathbb{R}} g(x)e^{-2\pi i x(n/2\pi)}dx = \widehat{g}\left(\frac{n}{2\pi}\right)$$

From Problem 5, we know that $\widehat{g}\left(\frac{n}{2\pi}\right) \to 0$ as $n/2\pi \to \infty$, proving the first result.

Now, we can use some trigonometric identities to see that

$$cos^2(nx + u_n) = \frac{cos(2nx + 2u_n)}{2} + \frac{1}{2} = \frac{cos(2nx)cos(2u_n)}{2} - \frac{sin(2nx)sin(2u_n)}{2} + \frac{1}{2}$$

$$\implies \int_E cos^2(nx + u_n)dx = \frac{m(E)}{2} + \frac{cos(2u_n)}{2}\int_E cos(2nx)dx - \frac{sin(2u_n)}{2}\int_E sin(2nx)dx$$

By the relative scale invariance of the Lebesgue integral, $\int_E cos(2nx)dx = \frac{1}{2}\int_E cos(nx)dx$, and the same for $sin(\cdot)$. Thus,

$$\int_E cos^2(nx + u_n)dx = \frac{m(E)}{2} + \frac{cos(2u_n)}{4}\int_E cos(nx)dx - \frac{sin(2u_n)}{4}\int_E sin(nx)dx$$

Note that

$$\int_{[0,2\pi]} \mathbb{1}_E e^{-inx}dx = \int_E cos(nx)dx - i\int_E sin(nx)dx$$

So,

$$\int_E cos(nx)dx = Re\left(\int_{[0,2\pi]} \mathbb{1}_E e^{-inx}dx\right) \quad \text{and} \quad \int_E sin(nx)dx = -Im\left(\int_{[0,2\pi]} \mathbb{1}_E e^{-inx}dx\right)$$

By the first result of this problem, we know that $\int_{[0,2\pi]} \mathbb{1}_E e^{-inx}dx \to 0$ as $n \to \infty$ since $\mathbb{1}_E$ is clearly integrable on $[0, 2\pi]$. Therefore, the real and imaginary parts must also converge to 0 (a complex sequence $(z_n)_n$ converges to $z \in \mathbb{C}$ if and only if the sequences $(Re\ z_n)_n \to Re(z)$ and $(Im\ z_n)_n \to Im(z)$). So,

$$\lim_{n\to\infty}\int_E cos(nx)dx = \lim_{n\to\infty}\int_E sin(nx)dx = 0$$

This means that as $n \to \infty$, regardless of the sequence $(u_n)_n$, the triangle inequality and the boundedness of $|cos(2u_n)|, |sin(2u_n)| \le 1$ give

$$\begin{aligned}\left|\int_E cos^2(nx + u_n)dx - \frac{m(E)}{2}\right| &\le \left|\frac{cos(2u_n)}{4}\int_E cos(nx)dx\right| + \left|\frac{sin(2u_n)}{4}\int_E sin(nx)dx\right| \\ &\le \frac{1}{4}\left|\int_E cos(nx)dx\right| + \frac{1}{4}\left|\int_E sin(nx)dx\right| \\ &\to 0\end{aligned}$$

as desired. ■

Problem 7

Solution

Proof. Define $\widetilde{E}$ to be the set of $x \in [0, 2\pi)$ for which $\sum_{n=0}^{\infty} A_n(x)$ converges. Then, $A_n \to 0$ pointwise on $\widetilde{E}$. Note that $A_n(x) = A_n(x + 2\pi k)$ for all $k \in \mathbb{Z}$. So, if it were the case that $m(\widetilde{E}) = 0$, then $\sum A_n(x)$ would converge on only a set of measure 0 on all intervals $[2\pi k, 2\pi(k+1))$ with $k \in \mathbb{Z}$, which means that it would converge only on a set of measure 0 over $\mathbb{R}$; this would be a contradiction. So, $m(\widetilde{E}) > 0$. However, since $\widetilde{E} \subset [0, 2\pi]$, we know $m(\widetilde{E}) < \infty$. Let $0 < \epsilon < m(\widetilde{E})$. Since $A_n \to 0$ on $\widetilde{E}$, we can apply Egorov's theorem to find a closed set $A_\epsilon \subset \widetilde{E}$ such that $A_n \to 0$ uniformly on A_ϵ and $m(\widetilde{E} \setminus A_\epsilon) < \epsilon$. We have

$$m(A_\epsilon) + m(\widetilde{E} \setminus A_\epsilon) = m(\widetilde{E}) \implies m(A_\epsilon) = m(\widetilde{E}) - m(\widetilde{E} \setminus A_\epsilon) > m(\widetilde{E}) - \epsilon > 0$$

If we define $E := A_\epsilon$, we have therefore found a (closed) set E of positive measure on which $A_n \to 0$ uniformly.

From here, note that we can refactor the expression for $A_n(x)$ with a well known trigonometric identity. If one lets $a_n = \gamma cos(\phi)$ and $b_n = \gamma sin(\phi)$ for some γ and ϕ, we can note that

$$\gamma cos(nx - \phi) = \gamma cos(nx)cos(-\phi) - \gamma sin(nx)sin(-\phi)$$

$$= \gamma cos(\phi) \cdot cos(nx) + \gamma sin(\phi) \cdot sin(nx) = a_n cos(nx) + b_n sin(nx) = A_n(x)$$

We can solve for γ and ϕ via

$$\gamma^2 = \gamma^2(sin^2(\phi) + cos^2(\phi)) = a_n^2 + b_n^2 \implies \gamma = \sqrt{a_n^2 + b_n^2}$$

Therefore, we can also get

$$\phi = cos^{-1}\left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}}\right)$$

With this in mind, we know that $A_n(x) = \sqrt{a_n^2 + b_n^2} cos(nx - \phi)$.

Since $A_n \to 0$ uniformly on E, then we have that

$$\int_E A_n \to 0$$

by bounded convergence (for large enough n, we can always bound $|A_n|$). Note, however, that

$$\int_E A_n = \sqrt{a_n^2 + b_n^2} \int_E cos(nx - \phi)dx$$

Because E has positive measure, it must contain a ball B of some radius, say δ (this is an interval since $E \subset \mathbb{R}$). Consider the sequence $(n_k)_k$ given by $n_k = \lceil k/\delta + \frac{\pi}{2} \rceil$. Then,

$$\int_B cos(n_k x - \phi)dx$$

does not converge to 0 since we are always integrating over half a phase, which will always have a nonzero integral. So, there is a subsequence $(n_k)_k$ such that $\int_E cos(n_k x - \phi)dx$ doesn't converge to 0 as $k \to \infty$. So, the only way for $\int_E A_{nk} \to 0$ is for $\sqrt{a_{n_k}^2 + b_{n_k}^2} \to 0$, which in turn can only happen if both $a_{nk} \to 0$ and $b_{nk} \to 0$ as $k \to \infty$. Since $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ must exist (after all, $\lim_{n\to\infty} A_n(x)$ exists), they must agree with the limit along this subsequence. Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

■

Problem 8

Solution

Proof of (a). Suppose that

$$||f - f_k||_{L^1} \to 0 \qquad \text{as } k \to \infty$$

Let $\epsilon > 0$ be arbitrary. Define

$$E_k := \{x \in \mathbb{R}^n : |f(x) - f_k(x)| > \epsilon\}$$

By Chebyshev's inequality,

$$m(E_k) \leq \frac{1}{\epsilon} \int_{\mathbb{R}^n} |f - f_k| \implies 0 \leq \epsilon m(E_k) \leq ||f - f_k||_{L^1}$$

Since $||f - f_k||_{L^1} \to 0$, the above inequality tells us that $m(E_k) \to 0$ as $k \to \infty$ as well (by the Squeeze theorem). This is what we set out to prove. ■

Proof of (b). Define a sequence of functions $f_k : \mathbb{R} \to \mathbb{R}$ by

$$f_k(x) = \max\{k^2(1 - k|x|),\ 0\}$$

In words, $f_k(x)$ is a triangle centered at 0 with width $2/k$ and height k^2. f_k is clearly measurable, as it is continuous. Then, for all $\epsilon > 0$ we have

$$\{|f_k - 0| > \epsilon\} \subset \{|f_k - 0| > 0\} = \{f_k > 0\} = \left(-\frac{1}{k}, \frac{1}{k}\right),$$

which means that

$$m(\{|f_k - 0| > \epsilon\}) \leq m\left(\left(-\frac{1}{k}, \frac{1}{k}\right)\right) = \frac{2}{k},$$

and so $m(\{|f_k - 0| > \epsilon\}) \to 0$ as $k \to \infty$. So, $(f_k)_k$ converges in measure to 0.

However,

$$||f_k - 0||_{L^1} = ||f_k||_{L_1} = \frac{1}{2} \cdot k^2 \cdot \frac{2}{k} = k,$$

and so $||f_k - 0||_{L^1} \to \infty$ as $k \to \infty$. So, $f_k \to 0$ in measure, but not in L^1. ■