MAT 425: Problem Set 4

Due on March 3, 2023

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Problem 1

Solution

Proof of (a). Let

$$f_n(x) = \frac{sin(e^x)}{1+nx^2}$$

Then, we can see pretty clearly that $f_n \to 0$ pointwise for a.e. $x \in [0,\infty]$. Using composition rules and continuity, we can use Property 2 of measurable functions to see that all the f_n are measurable. Now, we can find a dominating function by observing that for all $x \in [0,\infty]$,

$$|f_n(x)| \leq \frac{1}{1+nx^2} \leq \frac{1}{1+x^2} \leq \sum_{k=0}^{\infty} \frac{1}{1+k^2}\mathbb{1}_{[k,k+1]}(x) := g(x)$$

The first inequality comes from the fact that $|sin(a)| \leq 1$. The second inequality comes from the fact that $n \geq 1$ and $x^2 \geq 0 \implies 1+nx^2 \geq 1+x^2$. The third inequality can be seen in the following way: fix x and let $k' \geq 0$ be the integer such that $x \in [k', k'+1)$; then, $k' \leq x \implies 1+k'^2 \leq 1+x^2$, and so $\frac{1}{1+k'^2} \geq \frac{1}{1+x^2}$. Since the indicator functions are 0 for all other $k \neq k'$, this is the value of $g(x)$, and the last inequality holds. So, $|f_n| \leq g$ for this nonnegative function g. We want to show that g is integrable. To see this, note that

$$\int_0^{\infty} g = \sum_{k=0}^{\infty} \frac{1}{1+k^2} m([k,k+1]) = \sum_{k=0}^{\infty} \frac{1}{1+k^2} \leq 1 + \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

So, we have that $(f_n)_n$ are measurable, converge to 0 pointwise, and $|f_n| \leq g$ for a single integrable g. So, dominated convergence yields that

$$\lim_{n\to\infty} \int_0^{\infty} f_n(x)dx = \int_0^{\infty} \lim_{n\to\infty} f_n(x)dx = \int_0^{\infty} 0 = 0$$

■

Proof of (b). Let

$$f_n(x) = \frac{ncos(x)}{1+n^2x^{3/2}}$$

We can use simple properties to see that

$$\left|\int_0^1 f_n(x)dx\right| \leq \int_0^1 |f_n(x)|dx \leq \int_0^1 \frac{n}{1+n^2x^{3/2}}dx,$$

where the first inequality is the triangle inequality for integrals and the second inequality comes from $|cos(a)| \leq 1$. Now, use the substitution $x = n^{-4/3}t$ for some change of variables t. Then, the rightmost integral equals

$$\int_0^{n^{4/3}} n^{-4/3} \cdot \frac{n}{1+t^{3/2}}dt = n^{-1/3} \int_0^{n^{4/3}} \frac{1}{1+t^{3/2}}dt$$

Since $n \geq 1$, we can split this integral into a part from 0 to 1 and a remainder, yielding

$$= n^{-1/3} \int_0^1 \frac{1}{1+t^{3/2}}dt + n^{-1/3} \int_1^{n^{4/3}} \frac{1}{1+t^{3/2}}dt$$

Since $t^{3/2} \geq t^2$ on $[0,1]$ and $t^{3/2} \geq t$ on $[1,\infty)$, this grants that the above expression is

$$\leq n^{-1/3} \int_0^1 \frac{1}{1+t^2}dt + n^{-1/3} \int_1^{n^{4/3}} \frac{1}{1+t}dt$$
$$= n^{-1/3} \cdot \frac{\pi}{4} + n^{-1/3} \cdot \Big(\ln(1+n^{4/3}) - \ln(1+1)\Big) = \frac{C + \ln(1+n^{4/3})}{n^{1/3}}$$

for some constant $C = \frac{\pi}{4} - \ln(2)$ that doesn't depend on n. In other words, $\left|\int_0^1 f_n\right| \leq \frac{C+\ln(1+n^{4/3})}{n^{1/3}}$ for all $n \geq 1$. Taking the limit, we get

$$\lim_{n\to\infty}\left|\int_0^1 \frac{ncos(x)}{1+n^2x^{3/2}}dx\right| \leq \lim_{n\to\infty}\frac{C+\ln(1+n^{4/3})}{n^{1/3}} = 0,$$

where the right limit equals 0 because $\ln(1+a^4)$ grows asymptotically slower than a. So, $\int_0^1 f_n \to 0$ as desired. ■

Problem 2

Solution

Proof of (a). Let $f(x) = \frac{sin(x)}{x}$. To show that f is not integrable over $[1, \infty)$, we want to show that $\int_1^\infty |f|$ is not $< \infty$. Now, by definition we have that

$$\int_1^\infty |f| = \sup_{0 \leq g \leq |f|} \int_1^\infty g,$$

where the sup is taken over functions g that lie below $|f|$ over the interval $[1, \infty)$. So, it suffices to construct a single function g lying below $|f|$ such that $\int g$ is not finite. Consider the following construction of g: over each period $[k\pi, (k+1)\pi)$, we select an interval $I_k = [(k + \frac{1}{4})\pi, (k + \frac{3}{4})\pi]$. Over this interval, we want g to have the value of

$$\frac{|sin((k + \frac{3}{4})\pi)|}{(k + \frac{3}{4})\pi} = \frac{\frac{\sqrt{2}}{2}}{(k + \frac{3}{4})\pi}$$

Note that this will have a value smaller than $\left|\frac{sin(x)}{x}\right|$ over the interval I_k (it equals the value of $|sin(x)/x|$ on the right endpoint of I_k). Then, for every $n \in \mathbb{N}$ we can construct a simple function given by

$$g_n(x) = \sum_{k=1}^n \frac{\frac{\sqrt{2}}{2}}{(k + \frac{3}{4})\pi} \cdot \mathbb{1}_{I_k}(x)$$

For any selection of n, we have that $0 \leq g_n \leq |f|$ over the interval in question. Also, the integral of this simple function g_n over $[1, \infty)$ can be computed to be

$$\int_1^\infty g_n = \sum_{k=1}^n \frac{\frac{\sqrt{2}}{2}}{(k + \frac{3}{4})\pi} m(I_k) = \sum_{k=1}^n \frac{\frac{\sqrt{2}}{2}}{(k + \frac{3}{4})\pi} \cdot \frac{\pi}{2} = \sum_{k=1}^n \frac{\sqrt{2}}{4k + 3}$$

So, the integrals of each g_n are the partial sums of a harmonic sum, and they can therefore be made to be as large as desired for large enough n. This means that the supremum in the definition of $\int_1^\infty |f|$ is unbounded (for any big R there exists a N such that for all $n \geq N$, $\int g_n > R$ where $0 \leq g_n \leq |f|$), and we see that f is not integrable over $[1, \infty)$. ■

Proof of (b). Consider the sequence

$$\left(\int_1^n \frac{sin(x)}{x}\right)_n$$

of real numbers. We want to show that it is Cauchy, since then we will show that it converges to something finite (after all, each element of the sequence is certainly finite). So, let $\epsilon > 0$ be arbitrary. Let $n, m > \frac{2}{\epsilon}$ be arbitrary, and suppose WOLOG that $m \leq n$. We are interested in the difference

$$\left|\int_1^n \frac{sin(x)}{x} dx - \int_1^m \frac{sin(x)}{x} dx\right| = \left|\int_m^n \frac{sin(x)}{x} dx\right|$$

Observe using the quotient rule and the fundamental theorem of calculus, we see

$$\frac{d}{dx}\left[\frac{cos(x)}{x}\right] = \frac{-xsin(x) - cos(x)}{x^2} = -\frac{sin(x)}{x} - \frac{cos(x)}{x^2}$$

$$\implies \int_m^n \frac{sin(x)}{x} dx = -\left.\frac{cos(x)}{x}\right|_m^n - \int_m^n \frac{cos(x)}{x^2} dx$$

So,

$$\begin{aligned}
\left|\int_m^n \frac{sin(x)}{x}dx\right| &= \left|\frac{cos(m)}{m} - \frac{cos(n)}{n} - \int_m^n \frac{cos(x)}{x^2}dx\right| \\
&\leq \left|\frac{cos(m)}{m}\right| + \left|\frac{cos(n)}{n}\right| + \left|\int_m^n \frac{cos(x)}{x^2}dx\right| \\
&\leq \frac{1}{m} + \frac{1}{n} + \int_m^n \left|\frac{cos(x)}{x^2}\right| dx \\
&\leq \frac{1}{m} + \frac{1}{n} + \int_m^n \frac{1}{x^2}dx \\
&= \frac{1}{m} + \frac{1}{n} + \left[-\frac{1}{x}\right]_m^n \\
&= \frac{1}{m} + \frac{1}{n} + \frac{1}{m} - \frac{1}{n} = \frac{2}{m} < \epsilon
\end{aligned}$$

This shows us that for all $m, n \geq \frac{2}{\epsilon}$,

$$\left|\int_1^n \frac{sin(x)}{x}dx - \int_1^m \frac{sin(x)}{x}dx\right| < \epsilon$$

This proves that the sequence is Cauchy, and therefore that it converges. ∎

Problem 3

Solution

Proof. We define the sequence of nonnegative continuous functions as follows. For notational convenience, denote each function by $f_{k,n}$ where $n \in \mathbb{N}$ and $k \in \{0, ..., n\}$. The sequence is then $(f_{k,n})_{k,n}$ and looks like

$$f_{0,1},\ f_{1,1},\ f_{0,2},\ f_{1,2},\ f_{2,2},\ f_{0,3},\ f_{1,3},\ f_{2,3},\ f_{3,3},\ f_{0,4}, ...$$

Now, for a value of n and a value $k \in \{0, ..., n\}$, define $f_{k,n} : [0,1] \to [0, \infty)$ by

$$f_{k,n}(x) = \begin{cases} 1 - n \cdot \left|x - \frac{k}{n}\right| & x \in \left[\frac{k-1}{n}, \frac{k+1}{n}\right] \\ 0 & else \end{cases}$$

In words, $f_{k,n}$ is a triangle of height 1 and width $2/n$ that is centered at k/n. It is clear that each $f_{k,n}$ is continuous, as plugging in $x = \frac{k \pm 1}{n}$ will give 0. Also, it is also clear that the integral $\int_0^1 f_{k,n} \to 0$, since we always have that

$$\left|\int_0^1 f_{k,n}\right| \leq \text{area of triangle of width } 2/n \text{ and height } 1 = \frac{1}{n}$$

So, as $n \to \infty$ (which happens as we traverse the sequence), the integral converges to 0. Now, all that is left is to show that for all points $x \in [0,1]$, $f_{n,k}(x)$ doesn't converge. To see this, we can show that it isn't Cauchy. Fix $x \in [0,1]$ to be arbitrary. The negation of the Cauchy criterion in this setting states that there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist two $m, n \geq N$ for which there are two $k_m \in \{0, ..., m\}$ and $k_n \in \{0, ..., n\}$ such that

$$|f_{k_m,m}(x) - f_{k_n,n}(x)| \geq \epsilon$$

Let $\epsilon = \frac{1}{2}$. Let N be arbitrary. Using properties of the density of rationals in $[0,1]$, there exists a rational $\frac{p}{q} \in [0,1]$ s.t. $|x - p/q| < \frac{1}{2qN}$. In particular, the property we use is Dirichlet's Approximation Theorem, which states that for all real α and any integer k, there exists a rational p/q such that $|\alpha - \frac{p}{q}| < \frac{1}{q \cdot k}$; we apply this theorem with $\alpha = x$ and $k = 2N$ to get our rational $\frac{p}{q} \in [0,1]$. We see that $qN \geq N$ and

$$f_{pN,qN}(p/q) = \max\left(0, 1 - qN \cdot \left|\frac{p}{q} - \frac{pN}{qN}\right|\right) = \max(0, 1 - 0) = 1$$

Since $f_{pN,qN}$ is linear around the tip with slope $\pm qN$ depending on which side we descend, we see that $|x - p/q| < \frac{1}{2qN} \implies f_{pN,qN}(x) > 1 - qN \cdot \frac{1}{2qN} = 1 - \frac{1}{2} = \frac{1}{2}$. However, there are very clearly two m, k_m for which $m \geq N$ and $f_{k_m,m}(x) = 0$. This gives us that $m, qN \geq N$ and

$$|f_{k_m,m}(x) - f_{pN,qN}(x)| = f_{pN,qN}(x) > \frac{1}{2} = \epsilon$$

This means that for all $x \in [0,1]$, the sequence is not Cauchy and therefore doesn't converge. ∎

Problem 4

Solution

Proof of (a). Let $(f_k)_k$ be a sequence of nonnegative measurable functions converging pointwise a.e. to f with $\int f_k \leq \int f < \infty$ for all k. Fatou's Lemma gives us for free that

$$\int f \leq \liminf_{k\to\infty} \int f_k$$

Also, the fact that $\int f_k \leq \int f$ for all k gives

$$\limsup_{k\to\infty} \int f_k \leq \int f$$

Combining these two inequalities,

$$\limsup_{k\to\infty} \int f_k \leq \int f \leq \liminf_{k\to\infty} \int f_k$$

However, since $\liminf \leq \limsup$ always, we see that

$$\liminf_{k\to\infty} \int f_k = \limsup_{k\to\infty} \int f_k = \int f \implies \int f_k \to \int f$$

Namely, the limit of $\int f_k$ exists and equals $\int f$. ■

Proof of (b). To begin, split $\mathbb{R}^n$ into countably many almost disjoint sets of finite measure $\{I_n\}_n$. For each n define the function

$$g_{n,k} := |f_k - f| \cdot \mathbb{1}_{I_n}$$

Now, we know that f_k and f are integrable, and therefore bounded a.e., which means us that $g_{n,k}$ is as well. In fact, the conditions given tell us that $g_{n,k}$ is uniformly bounded for fixed n, say $g_{n,k} \leq M_n$. We want to show that $\lim_{k\to\infty} \int g_{n,k} = 0$ for each n. To this end, fix an arbitrary n. Let $\epsilon > 0$. Applying Egorov's Theorem to I_n, we get a set $A_{n,\epsilon}$ on which $f_k \to f$ uniformly and s.t. $m(I_n \setminus A_{n,\epsilon}) < \epsilon$. So, there exists some K_n such that for all $k \geq K_n$, we have that $|f_k - f| < \epsilon \implies g_{n,k} < \epsilon$ on $A_{n,\epsilon}$. This allows us to say that for $k \geq K_n$,

$$\int g_{n,k} = \int_{I_n} g_{n,k} = \int_{A_{n,\epsilon}} g_{n,k} + \int_{I_n \setminus A_{n,\epsilon}} g_{n,k}$$

Over $A_{n,\epsilon}$, we know that $g_{n,k} < \epsilon$. Elsewhere, we know that it is $\leq M_n$. So, for $k \geq K_n$,

$$\int g_{n,k} \leq \epsilon \cdot m(A_{n,\epsilon}) + M_n m(I_n \setminus A_{n,\epsilon}) \leq \epsilon \cdot m(I_n) + \epsilon \cdot M_n$$

Since neither $m(I_n)$ nor M_n depend on ϵ and this holds for all ϵ, this tells us that $\int g_{n,k} \to 0$ for fixed n and $k \to \infty$. All that is left to do is to observe that because the I_n's partition $\mathbb{R}^n$, we have that

$$|f_k - f| = \sum_{n=1}^{\infty} |f_k - f| \cdot \mathbb{1}_{I_n} = \sum_{n=1}^{\infty} g_{n,k}$$

So, since each $g_{n,k} \geq 0$, Corollary 1.10 gives that

$$\int |f_k - f| = \int \sum_{n=1}^{\infty} g_{n,k} = \sum_{n=1}^{\infty} \int g_{n,k}$$

Taking the limit as $k \to \infty$ and applying our result that $\int g_{n,k} \to 0$ for each fixed n yields the result. ■

Problem 5

Solution

Proof of (a). Note that for any $r < s$, we have that $[a,b] \cap I(c,r) \subset [a,b] \cap I(c,s)$, and so

$$\sup_{x,y \in [a,b] \cap I(c,r)} |f(x) - f(y)| \leq \sup_{x,y \in [a,b] \cap I(c,s)} |f(x) - f(y)|$$

since the supremum is taken over a subset. This means that $osc_f(c,r) \leq osc_f(c,s)$ whenever $r < s$; in other words, $osc_f(c,\cdot)$ is a monotonically decreasing function. Since it is bounded below by 0 ($|f(x) - f(y)| \geq 0$) and decreasing, we know that $\lim_{r \to 0} osc_f(c,r)$ exists for all f, c. So, $osc_f(c)$ is well defined.

($\implies$) Suppose first that f is continuous at c. Let $\epsilon > 0$. Then, $\exists \delta > 0$ such that $\forall x, y \in (c - \delta, c + \delta)$, we must have that $|f(x) - f(y)| < \epsilon$ by the definition of continuity. Taking the supremum,

$$\sup_{x,y \in [a,b] \cap I(c,\delta)} |f(x) - f(y)| \leq \epsilon \implies osc_f(c,\delta) \leq \epsilon$$

Since $osc_f(c) \leq osc_f(c,\delta)$ (osc is decreasing), we know that $osc_f(c) \leq \epsilon$. Since this holds for all ϵ, it must be that $osc_f(c) = 0$.

($\impliedby$) Suppose now that $osc_f(c) = 0$. Let $\epsilon > 0$. Then, since $\lim_{r \to 0} osc_f(c,r) = 0$, there must be some $r > 0$ such that $osc_f(c,r) < \epsilon$. Therefore,

$$\sup_{x,y \in [a,b] \cap I(c,r)} |f(x) - f(y)| < \epsilon$$

Since this holds for the supremum, it must hold for all pairs $x, y \in [a,b] \cap I(c,r)$. In other words, we have that for all $x, y \in [a,b] \cap (c - r, c + r)$, it holds that $|f(x) - f(y)| < \epsilon$. This is precisely the definition of continuity at c. ■

Proof of (b). Let $\epsilon > 0$. Define $A_\epsilon := \{c \in [a,b] : osc_f(c) \geq \epsilon\} \subset \mathbb{R}$. We want to show that A_ϵ is closed and bounded, as this would show that it is compact. Clearly it is bounded as $A_\epsilon \subset [a,b]$. To show closure, consider any sequence $(c_k)_k \subset A_\epsilon$ that converges to a point c. We want to show that $c \in A_\epsilon$; so, suppose by way of contradiction that $c \notin A_\epsilon$. Then, $osc_f(c) = \lim_{r \to 0} osc_f(c,r) < \epsilon$ by definition of A_ϵ. So, there must exist some $r > 0$ such that $osc_f(c,r) < \epsilon$ by the properties of a limit. Now, since $c_k \to c$, there must exist some $c_N \in E$ such that $|c_N - c| < r$. There must then also exist some $\delta > 0$ such that $|c_N - c| < r - \delta$, which means that $I(c_N, \delta) \subset I(c,r)$. Then, we have that

$$osc_f(c_N, \delta) \leq osc_f(c,r) < \epsilon,$$

where the first inequality is true because the supremum is being taken over a subset $I(c_N, \delta) \subset I(c,r)$, and we already had the second inequality from earlier. So, since $osc_f(c_N, \cdot)$ is decreasing,

$$osc_f(c_N) \leq osc_f(c_N, \delta) < \epsilon \implies c_N \notin A_\epsilon$$

by definition of A_ϵ. This is a contradiction, and so we must have that $c \in A_\epsilon$. Therefore, A_ϵ contains its limit points, and so is closed. A_ϵ is then compact, as desired. ■

Proof of (c). Let $\epsilon > 0$. Define $A_\epsilon := \{c \in [a,b] : osc_f(c) \geq \epsilon\} \subset \mathbb{R}$ as before. By part (a), we know that f is discontinuous at every point of A_ϵ. So, since the set of discontinuities of f has measure 0, then so does A_ϵ. By Theorem 3.4 of Chapter 1, there must exist an open set U s.t. $A_\epsilon \subset U$ and $m(U \setminus A_\epsilon) \leq \epsilon \implies m(U) \leq \epsilon$. Since U is an open set on $\mathbb{R}$, it is a countable union of disjoint open intervals; this is an open cover of A_ϵ by disjoint open intervals. Since A_ϵ is compact by part (b), this open cover must have a finite subcover. In other words, there exists a finite set of disjoint open intervals $\{U_1, ..., U_n\} = \{(a_1, b_1), ..., (a_n, b_n)\}$ that covers A_ϵ. Since $\bigcup_i U_i \subset U$ and $m(U) \leq \epsilon$, the total length of these intervals must also be $\leq \epsilon$.

Now, note that since $[a, b]$ is closed and $\bigcup_i U_i$ is open, the set $[a, b] \setminus \bigcup_i U_i$ over which $osc_f < \epsilon$ is closed, and therefore compact. Since f is ϵ-continuous over the set $[a, b] \setminus \bigcup_i U_i$, which is compact, f is uniformly ϵ-continuous over $[a, b] \setminus \bigcup_i U_i$. So, there exists some $\delta > 0$ such that for all $x, y \in [a, b] \setminus \bigcup_i U_i$, we have $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. We construct our partition P in the following way: split the set $[a, b] \setminus \bigcup_i U_i$ into finitely many disjoint intervals $\{I_k\}_{k=1}^N$ such that each I_k is of length less than δ. Then, the partition is

$$P = \left(\bigcup_{k=1}^N I_k\right) \cup \left(\bigcup_{i=1}^n U_i\right),$$

where f is ϵ-continuous over all the I_k's and each I_k has length $< \delta$, and all of the U_i's together have total length $\leq \epsilon$. Then, we can write

$$\begin{aligned} U(f, P) - L(f, P) &= \left(\sum_{k=1}^N |I_k| \left(\sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x)\right)\right) + \left(\sum_{i=1}^n |U_i| \left(\sup_{x \in U_i} f(x) - \inf_{x \in U_i} f(x)\right)\right) \\ &= \left(\sum_{k=1}^N |I_k| \left(\sup_{x,y \in I_k} |f(x) - f(y)|\right)\right) + \left(\sum_{i=1}^n |U_i| \left(\sup_{x,y \in U_i} |f(x) - f(y)|\right)\right) \end{aligned}$$

Since f is ϵ-continuous with parameter δ over each I_k and they are all of width $< \delta$, the supremum of the variation along each I_k is $< \epsilon$. Also, since f is bounded (say $f \leq M$), we know that over each U_i the supremum of the variation along each U_i is $\leq 2M$. This gives us that

$$U(f, P) - L(f, P) \leq \epsilon \sum_{k=1}^N |I_k| + 2M \sum_{i=1}^N |U_i|$$

Since the I_k's are disjoint and subsets of $[a, b]$, their total length sums to $\leq b - a$. Also, we already know that the total length of the U_i's sums to $\leq \epsilon$. We then get that

$$U(f, P) - L(f, P) \leq (b - a)\epsilon + 2M\epsilon$$

Since this holds for all ϵ, we find that f is Riemann integrable by the Riemann integrability condition. ■

Proof of (d). Note that the set of discontinuities of f is contained in $\bigcup_n A_{1/n}$. To see this, suppose that f is discontinuous at c; part (a) then gives that $osc_f(c) > 0$. So, there exists some $m \geq \frac{1}{osc_f(c)} \implies osc_f(c) \geq 1/m \implies c \in A_{1/m} \implies c \in \bigcup_n A_{1/n}$.

Now, suppose that f is Riemann integrable on $[a, b]$. Define $A_\epsilon := \{c \in [a, b] \ : \ osc_f(c) \geq \epsilon\} \subset \mathbb{R}$ as before. Fix some arbitrary n, and let $\epsilon > 0$; we want to show that $m(A_{1/n}) < \epsilon$. Since f is Riemann integrable, the Riemann integrability condition says that there exists some partition P such that

$$U(f, P) - L(f, P) < \frac{\epsilon}{n}$$

Let $P = \{x_1, ..., x_N\}$ such that $a = x_1 < ... < x_N = b$ denote the endpoints of the intervals of this partition.

Then, by selection of P, we have that

$$\begin{aligned}\frac{\epsilon}{n} &> U(f,P) - L(f,P) \\ &= \sum_{k=1}^{N-1} \left(\sup_{x\in[x_k,x_{k+1}]} f(x) - \inf_{x\in[x_k,x_{k+1}]} f(x) \right)(x_{k+1}-x_k) \\ &= \sum_{k=1}^{N-1} \left(\sup_{x,y\in[x_k,x_{k+1}]} |f(x)-f(y)| \right)(x_{k+1}-x_k)\end{aligned}$$

Let $K := \{k : (x_k, x_{k+1}) \cap A_{1/n} \neq \emptyset\}$ be the set of indices of intervals of P whose interiors intersect $A_{1/n}$. Note that for all $k \in K$ we have that there is some $c \in (x_k, x_{k+1})$ such that $osc_f(c) \geq 1/n$. So, for such c and k we can say that $\exists r > 0$ such that $osc_f(c, r) \geq 1/n$ and $I(c, r) \subset [x_k, x_{k+1}]$. Then, there exists a $y \in I(c, r) \implies y \in [x_k, x_{k+1}]$ such that $|f(y) - f(c)| \geq 1/n$. This immediately yields that for $k \in K$,

$$\sup_{x,y\in[x_k,x_{k+1}]} |f(x)-f(y)| \geq \frac{1}{n}$$

So, going back to our earlier sum,

$$\begin{aligned}\frac{\epsilon}{n} &> \sum_{k=1}^{N-1} \left(\sup_{x,y\in[x_k,x_{k+1}]} |f(x)-f(y)| \right)(x_{k+1}-x_k) \\ &\geq \sum_{k\in K} \left(\sup_{x,y\in[x_k,x_{k+1}]} |f(x)-f(y)| \right)(x_{k+1}-x_k) \\ &\geq \sum_{k\in K} \frac{1}{n}(x_{k+1}-x_k) = \frac{1}{n}\sum_{k\in K}(x_{k+1}-x_k),\end{aligned}$$

where the second inequality comes from the fact that each element of the top line's sum is nonnegative. This tells us that $\sum_{k\in K}(x_{k+1} - x_k) < \epsilon$; that is, the total length of all the intervals in P whose interior intersect $A_{1/n}$ is $< \epsilon$. Since $A_{1/n} \subset \bigcup_{k\in K}[x_k, x_{k+1}]$ by construction of K, we find that

$$m_*(A_{1/n}) \leq \sum_{k\in K} m_*([x_k, x_{k+1}]) = \sum_{k\in K}(x_{k+1}-x_k) < \epsilon$$

Since $m_*(A_{1/n}) < \epsilon$ for all ϵ, it must be measurable with measure 0! This logic holds for all $n \implies m(A_{1/n}) = 0\ \forall n$. So,

$$\{\text{discontinuities of } f\} \subset \bigcup_{n=1}^{\infty} A_{1/n} \implies m_*(\{\text{discontinuities of } f\}) \leq \sum_{n=1}^{\infty} m(A_{1/n}) = 0$$

So, the set of discontinuities of f has measure 0. ■