MAT 425: Problem Set 3

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Collaborators: None

Problem 1

Solution

Proof. Let $\alpha > 0$. We begin by noting that

$$\alpha \cdot \mathbb{1}_{E_\alpha} = \alpha \cdot \mathbb{1}_{\{x : f(x) > \alpha\}} \leq f$$

holds over all x, since if $x \notin E_\alpha$ the left hand side is $0 \leq f(x)$ by nonnegativity of f, and if $x \in E_\alpha$ then the left hand side is $\alpha < f(x)$ by definition of E_α. By Proposition 1.6(iii), we then see that

$$0 \leq \alpha \cdot \mathbb{1}_{E_\alpha} \leq f \implies \int \alpha \cdot \mathbb{1}_{E_\alpha} \leq \int f$$

However, since $\alpha \cdot \mathbb{1}_{E_\alpha}$ is a simple function, its integral is simply $\int \alpha \cdot \mathbb{1}_{E_\alpha} = \alpha \cdot m(E_\alpha)$. This gives us that

$$\alpha \cdot m(E_\alpha) \leq \int f \implies m(E_\alpha) \leq \frac{1}{\alpha} \int f$$

■

Problem 2

Solution

Proof. Fix $\alpha > 0$. Let E be the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q, with p, q relatively prime integers, such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\alpha}}$$

Fix an interval $[k, k+1]$ for some $k \in \mathbb{Z}$. Define $E_k = E \cap [k, k+1]$. Then, $m_*(E_k) = m_*(E_0)$ for all k by translation invariance. Let us then try and determine the measure of E_0. Enumerate all the irreducible rationals p/q in $[-1, 2]$ (irreducible means p, q are relatively prime) by $(r_n)_{n=1}^{\infty}$, and for each r_n define $\pi(r_n) = q_n$ to be the denominator of the irreducible rational in its irreducible representation (which is unique). Note that we only worry about rationals between -1 and 2 because an $x \in [0, 1]$ can only satisfy the desired property for rationals in this range, since the right hand side is always less than or equal to 1. Now, let

$$A_n = \left\{ x \in [0, 1] \setminus \mathbb{Q} : |x - r_n| \leq \frac{1}{\pi(r_n)^{2+\alpha}} \right\}$$

denote the set of all elements of $[0, 1]$ desirably close to r_n for each n. Since each A_n is just a closed interval around r_n of the given width, we see that each A_n is measurable with $m(A_n) = \frac{2}{\pi(r_n)^{2+\alpha}}$. Now, we may notice that for each possible denominator $\pi(r_n) = j$, there is an upper bound of $3j$ different r_n's that can have this denominator (since they must lie within -1 and 2). So, we can compute that

$$\sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} \frac{2}{\pi(r_n)^{2+\alpha}} \leq \sum_{j=1}^{\infty} 3j \cdot \frac{2}{j^{2+\alpha}} = \sum_{j=1}^{\infty} \frac{6}{j^{1+\alpha}}$$

Since $\alpha > 0$, this sum converges to something finite, and so $\sum_{n=1}^{\infty} m(A_n) < \infty$. Also, note that by construction of E_0, we have precisely that

$$E_0 = \{x \in \mathbb{R} : x \in A_n \text{ for infinitely many } n\}$$

We can then apply the Borel-Cantelli Lemma directly to see that E_0 is measurable with $m(E_0)$. This gives us for free that $m(E_k) = 0$ for all k by translation invariance. So, even though the E_k's are not disjoint, by monotonicity and subadditivity we have that

$$E \subset \bigcup_{k \in \mathbb{Z}} E \cap [k, k+1] = \bigcup_{k \in \mathbb{Z}} E_k \implies m_*(E) \leq \sum_{k=1}^{\infty} m(E_k) = 0$$

So, E is measurable with measure 0. ■

Problem 3

Solution

Proof. Let $C = \{E \subset \mathbb{R}^2 : \forall y \in \mathbb{R}, E^y \in \mathcal{B}_\mathbb{R}\}$ be as defined in the hint, where $\mathcal{B}_\mathbb{R}$ is the Borel σ-algebra on $\mathbb{R}$. We want to show first that C is a σ-algebra; that is, it is closed under complements and countable unions. So, suppose first that $E \in C$. Then, for all $y \in \mathbb{R}$, we can say that $(E^y)^C = (E^C)^y$ because

$$(E^y)^C = \{x : (x, y) \notin E\} = \{x : (x, y) \in E^C\} = (E^C)^y$$

Since $E^y \in \mathcal{B}_\mathbb{R}$ (since $E \in C$), then so is $(E^y)^C$ by the closure of the Borel σ-algebra. Then, $(E^C)^y = (E^y)^C \in \mathcal{B}_\mathbb{R}$ for all y tells us that $E^C \in C$ as well. So, C is closed under complements.

Next, consider $\{E_n\}_{n=1}^\infty$ such that $E_n \in C$ for all n. Define $E = \bigcup_n E_n$. For all $y \in \mathbb{R}$ we claim that

$$E^y = \left(\bigcup_{n=1}^\infty E_n\right)^y = \bigcup_{n=1}^\infty (E_n)^y$$

To see this, fix y and suppose that $x \in (\bigcup_{n=1}^\infty E_n)^y$. Then $(x, y) \in \bigcup_n E_n$, and so $(x, y) \in E_k$ for some k. Therefore, $x \in (E_k)^y$ for that k, and so $x \in \bigcup_n (E_n)^y$. This gives us the $\subset$ direction. Now, fix y and suppose that $x \in \bigcup_n (E_n)^y$, which means that $x \in (E_k)^y$ for some k. Then, $(x, y) \in E_k$ for that k, and so $(x, y) \in \bigcup_n E_n$. Therefore, $x \in (\bigcup_n E_n)^y$, giving us the $\supset$ direction. These together prove the claim, and show that $E^y = \bigcup_n (E_n)^y$. Then, since each $E_n \in C$ tells us that each $(E_n)^y \in \mathcal{B}_\mathbb{R}$, we have by closure of $\mathcal{B}_\mathbb{R}$ under countable union that $\bigcup_n (E_n)^y \in \mathcal{B}_\mathbb{R}$. By our earlier claim, this means that $E^y \in \mathcal{B}_\mathbb{R}$. Since this held for all fixed $y \in \mathbb{R}$, we get that $E \in C$ by construction of C. So, C is closed under countable union and must be a σ-algebra.

We now wish to show that C contains the open sets. Let $E \subset \mathbb{R}^2$ be an arbitrary open set. Then, for all $(x, y) \in E$, there exists a $\delta > 0$ such that $B_\delta((x, y)) \subset E$. Fix a y. For all x such that $(x, y) \in E$, we can select any $x' \in (x - \delta, x + \delta)$ and have that

$$\{(x', y) : x' \in (x - \delta, x + \delta) \subset B_\delta((x, y)) \subset E\}$$

This is equivalent to stating that for every $x \in E^y$, there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset E^y$, and so E^y is open. Therefore, $E^y \in \mathcal{B}_\mathbb{R}$ for all y, yielding that $E \in C$. So, C is a σ-algebra containing every open set. This means that $\mathcal{B}_{\mathbb{R}^2} \subset C$ by construction of $\mathcal{B}_{\mathbb{R}^2}$.

Now, we can prove the problem statement. Let $E \subset \mathbb{R}^2$ be Borel. Then, by the above conclusion, $E \in \mathcal{B}_{\mathbb{R}^2} \implies E \in C$. So, all slices E^y must be in $\mathcal{B}_\mathbb{R}$ by construction of C, and we are done. ∎

Problem 4

Solution

Proof of (a). Let f be M-Lipschitz. Let $\epsilon > 0$. Then, there exists a $\delta > 0$ (namely, $\delta = \frac{\epsilon}{M}$) such that for all x, y with $|x - y| \leq \delta$,

$$|f(x) - f(y)| \leq M|x - y| \leq M\delta = M\frac{\epsilon}{M} = \epsilon,$$

where the first inequality comes from the Lipschitz condition. This is precisely the continuity condition; since it holds for all such x, y, we know that f is continuous. ■

Proof of (b). Suppose that $E \subset \mathbb{R}^n$ has $m(E) = 0$. Let $1 > \epsilon > 0$. Then, there exists a countable collection of closed cubes $(Q_j)_{j=1}^\infty$ such that

$$E \subset \bigcup_{j=1}^\infty Q_j \quad \text{and} \quad \sum_{j=1}^\infty |Q_j| < \epsilon$$

Let $f(Q_j)$ be the image of each cube under f. Any two points $x, y \in Q_j$ have coordinate-wise distances of at most the side length of Q_j, which is $|Q_j|^{1/n}$. So, we can say that $|x-y| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \leq \sqrt{n|Q_j|^{1/n^2}} = |Q_j|^{1/n}\sqrt{n}$ for all $x, y \in Q_j$. The Lipschitz condition then guarantees that for all $x, y \in Q_j$,

$$|f(x) - f(y)| \leq M|x - y| \leq M|Q_j|^{1/n}\sqrt{n}$$

So, all points $f(x), f(y) \in f(Q_j)$ are at most $M|Q_j|^{1/n}\sqrt{n}$ distance apart, which means that $f(Q_j)$ lies within a cube of side length $2M|Q_j|^{1/n}\sqrt{n}$. In other words, for each closed cube $Q_j \subset \mathbb{R}^n$, we can find a closed cube $\widetilde{Q}_j \subset \mathbb{R}^m$ such that

$$f(Q_j) \subset \widetilde{Q}_j \quad \text{and} \quad |\widetilde{Q}_j| \leq (2M|Q_j|^{1/n}\sqrt{n})^m = (2M\sqrt{n})^m |Q_j|^{m/n}$$

Since $|Q_j| < 1$ (because $\epsilon < 1$) and $m \geq n$, we can say that the $|Q_j|^{m/n} \leq |Q_j|$, and so $|\widetilde{Q}_j| \leq (2M\sqrt{n})^m |Q_j|$. Now, note that because of the fact that the image of a union is the union of the images, we get

$$E \subset \bigcup_{j=1}^\infty Q_j \implies f(E) \subset \bigcup_{j=1}^\infty f(Q_j) \subset \bigcup_{j=1}^\infty \widetilde{Q}_j$$

Using monotonicity of exterior measure, and the fact that each closed cube $\widetilde{Q}_j$ has exterior measure $|\widetilde{Q}_j|$, we get

$$m_*(f(E)) \leq \sum_{j=1}^\infty |\widetilde{Q}_j| \leq \sum_{j=1}^\infty (M\sqrt{n})^m |Q_j| = (2M\sqrt{n})^m \sum_{j=1}^\infty |Q_j| < (2M\sqrt{n})^m \epsilon$$

Since this holds for all arbitrary $\epsilon > 0$ and $(2M\sqrt{n})^m$ doesn't depend on ϵ, taking $\epsilon \to 0$ yields that $m_*(f(E)) = 0$, and so $f(E)$ is measurable with measure 0. ■

Proof of (c). Let $F \subset \mathbb{R}^n$ be an F_σ set. Then, F is a countable union of closed sets $F = \bigcup_{j=1}^\infty E_j$ for closed E_j. We know that the image of a union is the union of an image, yielding that

$$f(F) = f\left(\bigcup_{j=1}^\infty E_j\right) = \bigcup_{j=1}^\infty f(E_j)$$

Note that we can write each E_j as a countable union with closed balls

$$E_j = \bigcup_{r=1}^\infty (E_j \cap \overline{B}_r(0)),$$

yielding

$$f(F) = \bigcup_{j=1}^{\infty} \bigcup_{r=1}^{\infty} f(E_j \cap \overline{B}_r(0))$$

We have that each $E_j \cap \overline{B}_r(0)$ is compact, as it is closed and bounded in $\mathbb{R}^n$. So, we find that each $f(E_j \cap \overline{B}_r(0))$ is also compact, and thus closed, since continuous functions map compact sets to compact sets. Therefore, $f(F)$ is a countable union of closed sets, and so it is a F_σ set. Since this holds for all such F, we see that f carries F_σ sets in $\mathbb{R}^n$ to F_σ sets in $\mathbb{R}^m$. ■

Proof of (d). Let $E \subset \mathbb{R}^n$ be measurable. By Corollary 3.5(ii), there exists an F_σ set $F \subset E$ such that $E \setminus F$ has measure 0. So, we write

$$f(E) = f(F) \cup f(E \setminus F)$$

By part (b), $m(f(E \setminus F)) = 0$. By part (c), $f(F)$ is F_σ. So, we arrive at the result that $f(E)$ is the union of an F_σ set with a set of measure 0, which by Corollary 3.5(ii) yields that $f(E)$ is measurable. ■

Proof of (e). It **does** hold for any Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$. To see this, let f be such a function with Lipschitz constant M. The only result we need to prove is that this f maps sets of measure 0 to sets of measure 0, since we can still apply the result from part (c) to this f (in part (c) we never cared whether $m \geq n$ held). So, following the steps in the proof of (b), suppose that $E \subset \mathbb{R}^2$ has $m(E) = 0$. Let $\epsilon > 0$. Then, there exists a countable collection of closed cubes $(Q_j)_{j=1}^{\infty}$ such that

$$E \subset \bigcup_{j=1}^{\infty} Q_j \quad \text{and} \quad \sum_{j=1}^{\infty} |Q_j| < \epsilon$$

Let $f(Q_j)$ be the image of each cube under f. Since each Q_j is compact (closed and bounded) and f is continuous, we know that each $f(Q_j) \subset \mathbb{R}$ is a compact set and is therefore measurable. Furthermore, by the Lipschitz condition, we know that the maximum distance between any two points in $f(Q_j)$ is $M\sqrt{2}|Q_j|^{1/2}$ by the same logic as was used in the proof of part (b). This means that $f(Q_j)$ must be contained within an interval of width $M\sqrt{2}|Q_j|^{1/2}$, meaning that $m(f(Q_j)) \leq M\sqrt{2}|Q_j|^{1/2}$. So, we see that

$$E \subset \bigcup_{j=1}^{\infty} f(Q_j) \implies m(f(E)) \leq \sum_{j=1}^{\infty} m(f(Q_j)) \leq \sum_{j=1}^{\infty} M\sqrt{2}|Q_j|^{1/2}$$

oops ■

Proof of (f). Suppose that f is α-Holder for some $\alpha > 1$. Let $x, y \in \mathbb{R}^n$ be arbitrary. Fix $n > 0$, and define a sequence of $n + 1$ points equally spaced and interpolating between x and y. In other words, define $(x_i)_{i=0}^n$, where

$$x_i = x + i \cdot \frac{y - x}{n}$$

So, $x_0 = x$ and $x_n = y$. We then have via a telescoping sum that

$$|f(x) - f(y)| = \left| \sum_{i=0}^{n-1} f(x_i) - f(x_{i+1}) \right| \leq \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})| \leq \sum_{i=0}^{n-1} M|x_i - x_{i+1}|^\alpha,$$

where the first inequality is the triangle inequality and the second comes from the α-Holder condition. However, we note by construction that

$$x_i - x_{i+1} = -\frac{y - x}{n},$$

and so

$$|f(x) - f(y)| \leq \sum_{i=0}^{n-1} M \left| \frac{y-x}{n} \right|^\alpha = M|y-x|^\alpha \cdot \sum_{i=0}^{n-1} \frac{1}{n^\alpha} = \frac{M|y-x|^\alpha}{n^{\alpha-1}}$$

Taking the limit as $n \to \infty$ we see that since $\alpha > 1$, the right hand side approaches 0, and so $f(x) = f(y)$. Since this holds for all x, y, we find that f must be constant. ■

Proof of (g). We know that the Cantor-Lebesgue function F is α-Holder with $\alpha = \log(2)/\log(3)$. We also know from the previous PSET that F maps a measurable set to a non-measurable set. If we rescale any closed interval to $[0, 1]$, we can apply the Cantor-Lebesgue function on that rescaled interval. With this logic, there exists an α-Holder function from $\mathbb{R} \to \mathbb{R}$ that maps a measurable set to a nonmeasurable set. This yields the claim. ■

Problem 5

Solution

Proof of (a). Let $f : [0,1] \to \mathbb{R}$ be a C^2 function. We want to show that f is Lipschitz. Fix any two arbitrary $x, y \in [0,1]$; suppose without loss of generality that $x < y$. By the Mean Value Theorem (which we can apply since f is differentiable), there exists some $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c) \cdot (y - x)$$

Since $[0,1]$ is compact and $|f'|$ is continuous (f' is differentiable), we know that $|f'|$ achieves a maximum on $[0,1]$. Let $M = \max_{a \in [0,1]} |f'(a)|$. Then we have that

$$|f(y) - f(x)| = |f'(c)| \cdot |y - x| \leq M|y - x|$$

This means that f is Lipschitz! Now, define a function $g : [0,1]^2 \to \mathbb{R}^2$ by $g(x_1, x_2) = (x_1 + x_2, f(x_1) + f(x_2))$. We see that the set $\Gamma + \Gamma$ is the image of $[0,1]^2$, a compact set, under g. We want to show that g is Lipschitz, which will tell us that $g([0,1]^2) = \Gamma + \Gamma$ is measurable by Problem 4(d). So, let $(x, y), (x', y') \in [0,1]^2$ be arbitrary. Note that if we consider the $L1$ norm (all norms are equivalent on $\mathbb{R}^2$),

$$\begin{aligned}
||g(x,y) - g(x',y')||_1 &= ||(x + y - x' - y', f(x) + f(y) - f(x') - f(y')||_1 \\
&= |x + y - x' - y'| + |f(x) + f(y) - f(x') - f(y')| \\
&\leq |x - x'| + |y - y'| + |f(x) - f(x')| + |f(y) - f(y')| \\
&\leq |x - x'| + |y - y'| + M|x - x'| + M|y - y'| \\
&= (M + 1)(|x - x'| + |y - y'|) = (M + 1)||(x, y) - (x', y')||_1
\end{aligned}$$

So, we see that g is Lipschitz with constant $M + 1$. Therefore, $\Gamma + \Gamma$, which is the image of a measurable set $[0,1]^2$ under g, is measurable. ■

Proof of (b). We want to show that the following statements are equivalent:

(i) $m(\Gamma + \Gamma) > 0$

(ii) $\Gamma + \Gamma$ contains an open set

(iii) f is not linear

((i) $\implies$ (iii)) To show this direction, we will instead show the contrapositive. That is, suppose that f is linear. Then, for all $x_1, x_2 \in [0,1]$, we see that

$$f(x_1) + f(x_2) = f(x_1 + x_2) = 2 \cdot f\left(\frac{x_1 + x_2}{2}\right)$$

by linearity. So, this means that

$$g(x_1, x_2) = g\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)$$

for all $x_1, x_2 \in [0,1]$. This means that the images $g([0,1]^2) = g(\{(x,x) : x \in [0,1]\})$ are equal, since for each pair $(x_1, x_2) \in \mathbb{R}^2$ we have just seen that there is some single value (x, x) that maps to the same thing. So, $\Gamma + \Gamma = g(\{(x,x) : x \in [0,1]\})$. Note that this is precisely the curve of the function $h : [0,2] \to \mathbb{R}$ given by $h(x) = 2 * f(x/2)$, which is continuous since f is. So, by Problem 6 on the last PSET, we see that $\Gamma + \Gamma$ has measure 0.

((iii) $\implies$ (ii)) For this direction, suppose that f is not linear. We can compute the derivative of g (the Jacobian matrix) to be

$$(Dg)((a,b)) = \begin{bmatrix} 1 & f'(a) \\ 1 & f'(b) \end{bmatrix}$$

using simple rules of differentiation. We compute the determinant of this matrix to be

$$det((Dg)((a,b))) = f'(b) - f'(a)$$

for all $(a,b) \in [0,1]^2$. Since f is nonlinear, there must be some pair $a, b \in [0,1]$ with $f'(a) \neq f'(b)$ (if not, then f' would be constant over $[0,1]$ and f would be linear). Select the point (a,b) such that this property holds; then, at this point we have that $det((Dg)((a,b))) \neq 0$. The inverse function theorem tells us that at this point (a,b) for which this determinant is nonzero, there exists an open neighborhood around (a,b), say $B_\delta((a,b)) \subset [0,1]^2$, and an open neighborhood around $g((a,b))$, say $B_\epsilon(g((a,b))) \subset \mathbb{R}^2$, such that $g(B_\delta((a,b))) \subset B_\epsilon(g((a,b)))$ and $g : B_\delta((a,b)) \to B_\epsilon(g((a,b)))$ is bijective. Note that this tells us that there is some open ball $B_\epsilon(g((a,b))) \subset g([0,1]^2) = \Gamma + \Gamma$. That is, there exists an open set in $\Gamma + \Gamma$.

((ii) $\implies$ (i)) Suppose that $\Gamma + \Gamma$ contains an open set. Then, there is some closed ball of radius δ contained in $\Gamma + \Gamma$ for some δ, by definition of openness (take any open ball and shrink it slightly to get a closed ball). Then, using the geometry of a square circumscribed by a circle of radius δ, we find that there exists a closed cube (a square in $\mathbb{R}^2$) of side length $\delta\sqrt{2}$. This means that there exists some closed square $Q \subset \Gamma + \Gamma$ with $|Q| = 2\delta^2$. Since $\Gamma + \Gamma$ is measurable by part (a), this tells us that

$$m(\Gamma + \Gamma) \geq |Q| = 2\delta^2 > 0$$

With these three implications, we see that each of the three statements implies the next, yielding that the statements are equivalent. So, we are done. ■