MAT 425: Problem Set 2

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Problem 1.4

Solution

Proof of (a). Suppose that the l_k's are chosen such that $\sum_{k=1}^{\infty} 2^{k-1} l_k < 1$. Let $\widehat{C}_0 = [0,1]$, and let $\widehat{C}_k$ denote the construction after the k^{th} iteration; then, $\widehat{C} = \cap_{k=0}^{\infty} \widehat{C}_k$. Consider the k^{th} iteration: from $\widehat{C}_{k-1}$ we remove 2^{k-1} disjoint open intervals, say $\{E_i\}_{i=1}^{2^{k-1}}$, with each $|E_i| = l_k$. Then, the collection

$$\{\widehat{C}_k, E_1, ..., E_{2^{k-1}}\}$$

is pairwise disjoint and unions to the entirety of $\widehat{C}_{k-1}$, and each element is measurable (they are elements of the Borel σ-algebra). So, by the additivity of measure,

$$m(\widehat{C}_k) + \sum_{i=1}^{2^{k-1}} l_k = m(\widehat{C}_{k-1}) \implies m(\widehat{C}_k) = m(\widehat{C}_{k-1}) - 2^{k-1} l_k$$

Since this holds for all $k \geq 1$ and $m(\widehat{C}_0) = m([0,1]) = 1$, induction gives that $m(\widehat{C}_k) = 1 - \sum_{i=1}^{k} 2^{i-1} l_i$. Since $\widehat{C} = \cap_{i=1}^{\infty} \widehat{C}_k$ and $\widehat{C}_k \subset \widehat{C}_{k+1}$ ($\{\widehat{C}_k\}$ decreases to $\widehat{C}$), Corollary 3.3(ii) gives

$$m(\widehat{C}) = \lim_{k \to \infty} m(\widehat{C}_k) = 1 - \sum_{k=1}^{\infty} 2^{k-1} l_k > 0$$

■

Proof of (b). Let $x \in \widehat{C}$ be arbitrary. Consider the construction of $\widehat{C}$ as a binary tree where the k^{th} level contains the centers of the 2^{k-1} open intervals removed in this step of the construction. It is a binary tree because for each open interval removed in the k^{th} step, there are two open intervals removed in the $k+1^{th}$ step - one to the left and one to the right - that are closer to it than to any other open intervals removed in the k^{th} step. Consider performing binary search on these nodes to find x: there will be no node with value x, but we always get closer to x as each of the open intervals being centrally situated means that the sequence of distances $|x - x_k|$ for x_k's in the binary search is strictly decreasing. Since this sequence of distances $|x - x_k|$ is nonnegative and strictly monotonically decreasing, we see that it must converge to 0. Equivalently, we see that the sequence of nodes visited in the binary search $(x_k)_{k \in \mathbb{N}}$ must converge $x_k \to x$. However, since each x_k is in an open interval removed during the k^{th} step of construction, none of the x_k's lie in $\widehat{C}$. Furthermore, since the l_k's are decreasing to 0 (otherwise $\sum_k 2^{k-1} l_k$ wouldn't converge), we have that the widths of the open intervals removed during the construction decreases to 0 as k increases. So, this means that we have a sequence (x_k) such that $x_k \to x$, all the $x_k \notin \widehat{C}$, and each $x_k \in I_k$ for some open interval $I_k \subset \widehat{C}^C$ such that $|I_k| \to 0$ as $k \to \infty$. ■

Proof of (c). Note that since we construct each $\widehat{C}_k$ by removing open intervals, we have the statement $\widehat{C}_k$ closed $\implies$ $\widehat{C}_{k+1}$ closed. Since $\widehat{C}_0 = [0,1]$ is closed, induction tells us that all $\widehat{C}_k$ are closed, and thus that $\widehat{C}$ is also closed and contains no open interval. So, since part (b) reveals that $\widehat{C}$ contains all its limit points (the $|I_k| \to 0$ criterion means there are elements of $\widehat{C}$ infinitely close to any element of $\widehat{C}$), we know that $\widehat{C}$ is a perfect set. So, $\widehat{C}$ is a perfect set containing no open interval. ■

Proof of (d). We have shown already that $m(\widehat{C}) > 0$. So, suppose by way of contradiction that $\widehat{C}$ is countable, i.e. that $\widehat{C} = \{a_k\}_{k=1}^{\infty}$. Let $\epsilon > 0$ be arbitrary. Then, we can form a covering of $\widehat{C}$ as the union of open balls

$$\widehat{C} \subset \bigcup_{k=1}^{\infty} B_{\epsilon/2^k}(a_k),$$

Problem 1.4 continued on next page. . .

since each a_k is contained in its own ball. Subadditivity of measure (we don't care if the balls are disjoint) yields

$$m(\widehat{C}) \leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$$

Since this holds for all ϵ, we see that $m(\widehat{C}) = 0$. This is a contradiction, and so we find that $\widehat{C}$ must be uncountable. ■

Problem 1.10

Solution

Proof of (a). Note first that for all $x \in [0, 1]$, we have that $0 \leq F_n(x) \leq 1$ for all $n \geq 1$. So, the product $f_n(x) = F_1(x)F_2(x)...F_n(x)$ must clearly also lie in $[0, 1]$ for all $x \in [0, 1]$ and all $n \geq 1$. Also,

$$f_{n+1}(x) = F_{n+1}(x) \cdot f_n(x) \leq f_n(x)$$

for all $x \in [0, 1]$ and all $n \geq 1$ since $F_{n+1} \leq 1$. So, we see that for all $x \in [0, 1]$, $\{f_n(x)\}_n$ is a bounded, monotonically decreasing sequence. Therefore, it must converge to a limit, say $f(x)$. We can then conclude that there is some function f such that $f_n \to f$ pointwise. ■

Proof of (b). Note first that for all $x \in \widehat{C}$, we have by definition that $f(x) = 1$, since $x \in \widehat{C}$ means that $x \notin \widehat{C}_k$ $\forall k$, which means $F_k(x) = 1$ $\forall k$. Fix any $x \in \widehat{C}$, and consider the construction of $\widehat{C}$ as a binary tree where the k^{th} level contains the centers of the 2^{k-1} open intervals removed in this step of the construction. It is a binary tree because for each open interval removed in the k^{th} step, there are two open intervals removed in the $k+1^{th}$ step - one to the left and one to the right - that are closer to it than to any other open intervals removed in the k^{th} step. Consider performing binary search on these nodes to find x: there will be no node with value x, but we always get closer (the sequence of distances $|x - x_k|$ for x_k's in the binary search is bounded and decreasing). Therefore, for any $\delta > 0$, we can find some node x_k for some large enough k such that $|x - x_k| < \delta$. However, since this x_k is a node, it is the center of an open interval removed during the k^{th} step of the construction. By construction, we then have that $f(x_k) = 0$, since $F_k(x_k) = 0 \implies f_n(x_k) = 0$ $\forall n \geq k$. So, we have found that for any fixed $x \in \widehat{C}$, for any $\delta > 0$, we can find an element $x_k \in [0, 1]$ such that $|x - x_k| < \delta$ but $|f(x) - f(x_k)| = 1$. So, f is discontinuous at x for all $x \in \widehat{C}$. The logic following the problem statement shows that f is not Riemann integrable! ■

Problem 1.21

Solution

Proof. Let $\mathcal{N} \subset [0,1]$ be the non-measurable subset of the unit interval that we constructed in class. Let $F : \mathcal{C} \to [0,1]$ be the surjective, continuous, (weakly) monotonically increasing function that we constructed in Exercise 1.2(c) from Stein, where $\mathcal{C}$ is the Cantor Set. Then, surjectivity yields that there is some set $E = F^{-1}(\mathcal{N})$ that maps exactly to $\mathcal{N}$ under F (that is, E is the preimage of $\mathcal{N}$ under F). However, we have that $E \subset \mathcal{C}$, since E lies within the domain of F. Recall that the Cantor Set has exterior measure 0. Therefore, monotonicity of the exterior measure yields that

$$E \subset \mathcal{C} \implies m_*(E) \leq m_*(\mathcal{C}) = 0 \implies m_*(E) = 0$$

By Property 2 of the Lebesgue measure, we then have that E is measurable and $m(E) = 0$. We have then shown that there is a continuous function F that maps a Lebesgue measurable set $E = F^{-1}(\mathcal{N})$ to a non-measurable set $\mathcal{N}$, as desired. ∎

Problem 1.32

Solution

Proof of (a). Let $\mathcal{N} \subset [0,1]$ be the non-measurable set constructed in class. Let $E \subset \mathcal{N}$ be measurable. Enumerate the rationals in $\mathbb{Q} \cap [-1,1]$ by $(r_n)_{n\in\mathbb{N}}$, and let $E_n = E + r_n$ be the translates of E by all rationals in $\mathbb{Q} \cap [-1,1]$. Then, by the translation invariance of $m(\cdot)$, we have that each E_n is also measurable and $m(E_n) = m(E)$ for all n. Also, the collection $\{E_n\}$ is pairwise disjoint, since if any $E_n \cap E_m$ were nonempty for $nneqm$, we would have $x_1 + r_n = x_2 + r_m$ for two $x_1, x_2 \in E \subset \mathcal{N}$ that are not equal (they are not equal since $r_n \neq r_m$ for $n \neq m$). If this were possible, then x_1 and x_2 would differ by a rational and $x_1 \sim x_2$, which cannot be the case since the construction of $\mathcal{N}$ only allowed for one element of each equivalence class. So, we have infinitely many disjoint sets $E_n \subset [-1,2]$ (because $E \subset [0,1]$ and the translations were in $[-1,1]$) that are all pairwise disjoint. Additivity and monotonicity of $m(\cdot)$ gives

$$\bigcup_{n=1}^{\infty} \subset [-1,2] \implies \sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} m(E) \leq 3 \implies m(E) = 0$$

■

Proof of (b). Let $G \subset \mathbb{R}$ be a set with exterior measure $m_*(G) > 0$. Since G has nonzero measure, there must be an interval of unit length $[N, N+1]$ for which $m_*(G \cap [N, N+1]) > 0$ some $N \in \mathbb{Z}$. If there weren't, we could write G as

$$G = \bigcup_{n\in\mathbb{Z}} (G \cap [n, n+1]) = \bigcup_{n\in\mathbb{Z}} (G \cap [n, n+1))$$

and the additivity of measure to see that $m_*(G) = 0$. So, let $I_N = [N, N+1]$ be an interval of unit length for the N such that $m_*(G \cap I_N) > 0$. Let $\widetilde{G} = (G \cap I_N) - N$ be the translate of $G \cap I_N$ such that $\widetilde{G} \subset [0,1]$ (this means that $m_*(\widetilde{G}) > 0$). We can now mimic the non-measurable subset $\mathcal{N}$ constructed in class, instead constructing a $\widetilde{\mathcal{N}} \subset \widetilde{G}$. To do this, create an equivalence relation $\sim$ such that any two elements of $\widetilde{G}$ are equivalent if and only if they differ by a rational, and let $\widetilde{\mathcal{N}}$ be a set containing exactly one element from each equivalence class of $\widetilde{G}$. Let $(r_n)_{n\in\mathbb{N}}$ be an enumeration of the rationals in $\mathbb{Q} \cap [-1,1]$, and define $\widetilde{\mathcal{N}}_n = \widetilde{\mathcal{N}} + r_n$ to be the translates of the set $\widetilde{\mathcal{N}}$ by the rationals r_n. We can say that every element $x \in \widetilde{G}$ must be part of some equivalence class, which means by construction of $\widetilde{\mathcal{N}}$ that it must differ from some element of $\widetilde{\mathcal{N}}$ by a rational (furthermore, the rational must be in $[-1,1]$ since $\widetilde{G} \subset [0,1]$). This, in turn, means that x must be contained in one of the $\widetilde{\mathcal{N}}_n = \widetilde{\mathcal{N}} + r_n$ for some n. Since this logic holds for all $x \in \widetilde{G}$, each $\widetilde{\mathcal{N}}_n \subset [-1,2]$, and the $\widetilde{\mathcal{N}}$'s are pairwise disjoint by the logic in part (a), we get

$$\widetilde{G} \subset \bigcup_{n=1}^{\infty} \widetilde{\mathcal{N}}_n \subset [-1,2],$$

which allows us to say via the monotonicity of the exterior measure and additivity of $m(\cdot)$ that if $\widetilde{\mathcal{N}}$ were measurable, we would have to have

$$0 < m_*(\widetilde{G}) \leq \sum_{n=1}^{\infty} m(\widetilde{\mathcal{N}}_n) = \sum_{n=1}^{\infty} m(\widetilde{\mathcal{N}}) \leq m_*([-1,2]) = 3,$$

where the measurablility of $\widetilde{\mathcal{N}}_n$ and the middle equality comes from the translation invariance of $m(\cdot)$. This cannot happen, since $\widetilde{\mathcal{N}}$'s measure cannot be 0 and it cannot be nonzero to satisfy this inequality. So, we find that $\widetilde{\mathcal{N}} \subset \widetilde{G}$ is non-measurable. Then, translating back by N, we find that $\widetilde{\mathcal{N}} + N \subset \widetilde{G} + N = G \cap I_N \subset G$ is a non-measurable subset of G. ■

Problem 1.33

Solution

Proof. Suppose, by way of contradiction, that $m_*(\mathcal{N}^C) < 1$. Then, for any $\epsilon > 0$, we can find a measurable set $U \subset [0,1]$ such that $\mathcal{N}^C \subset U \subset [0,1]$ and $m(U) < 1 - \epsilon$. Now, consider the complement U^C; we must have that U^C is also measurable, and that $U \cup U^C = [0,1]$ while $U \cap U^C = \emptyset$. The additivity of measure (Theorem 3.2) guarantees that

$$m(U) + m(U^C) = m([0,1]) = 1 \implies 1 - m(U^C) < 1 - \epsilon \implies m(U^C) > \epsilon$$

However, $\mathcal{N}^C \subset U \implies U^C \subset \mathcal{N}$, meaning that there is a measurable subset of $\mathcal{N}$ with positive measure. This is a clear contradiction of the previous exercise, and so we must have that $m_*(\mathcal{N}^C) = 1$. We also know that $m_*(\mathcal{N}) > 0$, since otherwise it would be measurable. So, we have that $\mathcal{N} \cup \mathcal{N}^C = [0,1]$, yet

$$m_*(\mathcal{N}) + m_*(\mathcal{N}^C) > m_*([0,1]),$$

as desired. ■

Problem 1.37

Solution

Proof. Pick any arbitrary interval $I = [a, b]$. Let $\epsilon > 0$ be arbitrary. Since f is continuous on the compact interval I, the Heine-Cantor Theorem grants that $f|_I$ is uniformly continuous (let us refer to $f|_I$ as f for now). Now, let $\delta > 0$ be such that for all $x, y \in I$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$ (such a δ exists by uniform continuity). Suppose, without loss of generality, that δ divides $b - a$, since we can always decrease δ to do so. Now, we can partition $[a, b]$ into intervals $\{[a, a+\delta), [a+\delta, a+2\delta), ...\}$ of width δ (call them $\{E_i\}_{i=1}^n$ with $n = \frac{b-a}{\delta}$). On each interval E_i, we know that f varies by less than $\frac{\epsilon}{2(b-a)}$ by the uniform continuity guarantee. So, if we consider the rectangle $E_i \times \left[f(a + (i-1)\delta) - \frac{\epsilon}{2(b-a)}, f(a + (i-1)\delta) + \frac{\epsilon}{2(b-a)}\right]$, we see that this rectangle simply must cover the curve f over the region E_i. However, this rectangle has volume $\delta \cdot \frac{\epsilon}{b-a}$ since each E_i has width δ. There are also n such rectangles necessary to fully cover the graph of $f|_I$ over I, yielding a combined measure of at most $n \cdot \delta \cdot \frac{\epsilon}{b-a}$ (we do not care if the rectangles aren't disjoint, as we only need to upper bound). Since $n = \frac{b-a}{\delta}$, we find that the combined measure of all the rectangles in this covering of $f|_I$ is $\frac{b-a}{\delta} \cdot \delta \cdot \frac{\epsilon}{b-a} = \epsilon$, meaning that $m_*(\Gamma|_I) < \epsilon$. Since this holds for all ϵ, we find that Γ_I is measurable and has measure 0 for any closed interval I. Writing $\mathbb{R}$ as a (not necessarily disjoint) union of countably many closed regions and using monotonicity and subadditivity allows the result to follow. So, $m(\Gamma) = 0$. ■