# MAT 425: Problem Set 1

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# Problem 1.2

## Solution

**Proof of (a).** Note that the construction of the Cantor set corresponds to deleting the open interval of each middle third, and repeating. However, splitting each interval into thirds and selecting one corresponds *precisely* to the ternary expansion; a number is in the interval $(0, 1/3)$ if and only if it has a 0 in the first digit of its ternary expansion; it is in the interval $(1/3, 2/3)$ if and only if it has a 1 in the first digit, and so on. More precisely, we can write down that any open interval corresponding to a middle third that is removed during this construction are of the form $\left(\frac{3m-2}{3^n}, \frac{3m-1}{3^n}\right)$ for some value $m \in \mathbb{N}$ and a depth $n \in \mathbb{N}$. A number is in such an open interval, however, if and only if it has a 1 in the $n^{th}$ digit of its ternary representation, which can be seen quite simply from induction (splitting an already-existing interval into thirds corresponds to adding a new digit to the ternary representation, and selecting the middle one correspondings to adding a digit of 1). Thus, an element is in the Cantor set if and only if it *doesn't* have a 1 in its ternary representation. ■

**Proof of (b).** Note that for any element $x \in \mathcal{C}$, we select a unique representation (there can only be one representation with all $a_k \in \{0, 2\}$ since any change to a coefficient cannot be mitigated by changing lower order coefficients). Such a representation exists by $(a)$. Furthermore, $F$ outputs a deterministic function $F(x) = \sum_{k=1}^{\infty} \frac{a_k}{2^{k+1}}$ of the coefficients. Lastly, since each $a_k \in \{0, 2\} \implies 0 \leq a_k \leq 2$, we have that

$$0 \leq F(x) \leq \sum_{k=1}^{\infty} \frac{2}{2^{k+1}} = 1$$

for all $x \in \mathcal{C}$. The above logic verifies that $F$ has domain $\mathcal{C}$ and range $[0, 1]$, and that $F(x)$ can only output one value. So, $F$ is well defined.

To see surjectivity, note that each number $y \in [0, 1]$ has a binary representation given by $y = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$ with $b_k \in \{0, 1\}$ for all $k$. Therefore, the number $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ with each $a_k = 2b_k$ will be in $\mathcal{C}$ by part $(a)$ (since each $a_k$ is in $\{0, 2\}$), and will clearly map $F(x) = y$ by the construction of $F$. So, $F$ is surjective.

Next, 0 has the unique ternary representation $a_k = 0$ for all $k$, and so $F(0) = 0$. Similarly, it is clear that $1 = \sum_{k=1}^{\infty} \frac{2}{3^k}$ by the infinite geometric series, and so 1 has the unique ternary representation where $a_k = 2$ for all $k$. This yields

$$F(1) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

So, $F(0) = 0$ and $F(1) = 1$.

To see continuity, we can prove that $F$ is (weakly) monotone increasing from 0 to 1. Then, since surjective and (weakly) monotone functions are continuous, we are done. So, we want to show that $F$ is (weakly) monotone increasing on $\mathcal{C}$. Suppose that $x, y \in \mathcal{C}$ are such that $x < y$. Let $\{a_k^{(x)}\}_{k \in \mathbb{N}}$ and $\{a_k^{(y)}\}_{k \in \mathbb{N}}$ be their unique Cantor ternary representations respectively (by this I mean that $a_k^{(x)}, a_k^{(y)} \in \{0, 2\} \; \forall k$). Let $N$ be the index of the first element at which they disagree. Then, we must have that $a_N^{(y)} = 2$ and $a_N^{(x)} = 0$ in order for $y$ to be larger. So,

$$F(y) - F(x) = \sum_{k=1}^{\infty} \frac{a_k^{(y)} - a_k^{(x)}}{2^{k+1}} = \sum_{k=N}^{\infty} \frac{a_k^{(y)} - a_k^{(x)}}{2^{k+1}} = \frac{2}{2^{N+1}} + \sum_{k=N+1}^{\infty} \frac{a_k^{(y)} - a_k^{(x)}}{2^{k+1}}$$

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We know that each difference $a_k^{(y)} - a_k^{(x)} \in \{-2, 0, 2\} \implies a_k^{(y)} - a_k^{(x)} \geq -2$. Therefore,

$$F(y) - F(x) \geq \frac{1}{2^N} + \sum_{k=N+1}^{\infty} \frac{-2}{2^{k+1}} = \frac{1}{2^N}\left(1 - \sum_{k=1}^{\infty} \frac{1}{2^k}\right) = 0$$

So, we see that for any two $x, y \in \mathcal{C}$ with $y > x$, we have $F(y) \geq F(x)$. This means that $F$ is monotone increasing on $\mathcal{C}$ and is surjective onto $[0, 1]$; this means that it must be continuous. ■

**Proof of (c).** We saw the surjectivity of $F$ in part $(b)$. ■

**Proof of (d).** Consider two different endpoints of an open interval $(a, b)$ in the complement of the Cantor set. Since we construct the Cantor set by removing middle thirds, we know that $(a, b)$ must be equal to $\left(\frac{3m-2}{3^n}, \frac{3m-1}{3^n}\right)$ for some depth $n \in \mathbb{N}$ and value $m \in \mathbb{N}$. So, we can compute that the ternary expansion for $b$ will be of the form $a_n^{(b)} = 2$ and $a_k^{(b)} = 0$ for all $k > n$, while the ternary expansion for $a$ will be of the form $a_n^{(a)} = 0$ and $a_k^{(a)} = 2$ for all $k > n$ (take, for example, the interval $(1/3, 2/3)$ with expansions $1/3 = 0.022222...$ and $2/3 = 0.2$), with all terms $a_k^{(a)} = a_k^{(b)}$ for $k < n$. Then, we find that since there are no differences before the $n^{th}$ term,

$$F(b) - F(a) = \sum_{k=1}^{\infty} \frac{a_k^{(b)} - a_k^{(a)}}{2^{k+1}} = \frac{1}{2^n} - \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}\left(1 - \sum_{k=1}^{\infty} \frac{1}{2^k}\right) = 0$$

So, $F(a) = F(b)$ for all such endpoints of open intervals removed during the construction of the Cantor set. Therefore, if we were to define $F$ to be constant on these intervals, we see that the resulting function $F : [0, 1] \to [0, 1]$ is still surjective and (weakly) monotonically increasing as was the case in part $(b)$, as addition of constant intervals doesn't change monotonicity. Therefore, $F$ is still continuous. ■

# Problem 1.6

## Solution

**Proof.** Let $B$ be the open ball of radius $r > 0$ in $\mathbb{R}^d$. Clearly, $B$ is measurable since it is open. Without loss of generality, suppose that both $B$ and $B_1$ are centered at the origin, since otherwise we can translate to the origin without affecting measure. Let $\nu_d$ be the measure of $B_1$, the open ball of radius 1. Let $B_1 \subset \bigcup_{i=1}^\infty Q_i$ be any countable covering of $B_1$ by closed cubes $\{Q_i\}$. Then, we can simply rescale each cube by a factor of $r$ such that it covers $B$; more precisely, rescale each $Q_i$ to a new cube $\widetilde{Q_i} = \{rx \ : \ x \in Q_i\}$. Then, we will have that $B \subset \bigcup_{i=1}^\infty \widetilde{Q_i}$ (this is true since each element of $B$ can be written as $rx$ for some $x \in B_1$), and have generated a countable covering of $B$ by closed cubes $\{\widetilde{Q_i}\}$. Furthermore, by the definition of volume of a cube we will have that $|\widetilde{Q_i}| = r^d |Q_i|$. Therefore, any countable covering of $B_1$ by closed cubes can generate a countable covering of $B$ of closed cubes with a value scaled by $r^d$. Thus, the infimum satisties

$$m(B) \leq r^d \sum_{i=1}^\infty |Q_i|$$

for all coverings $\{Q_i\}$ of $B_1$. Since this holds for all countable coverings of $B_1$ by closed cubes, we get from another infimum that

$$m(B) \leq r^d m(B_1) = \nu_d r^d$$

Note, however, that we can perform exactly symmetric logic in the other direction. Starting with a countable covering $B \subset \bigcup_{i=1}^\infty \widetilde{Q_i}$ of closed cubes, we can rescale by a factor of $\frac{1}{r} > 0$ in precisely the same way. Identical logic shows that

$$m(B_1) \leq \left(\frac{1}{r}\right)^d m(B) \implies \nu_d r^d \leq m(B)$$

These two results taken together give us that $m(B) = \nu_d r^d$ as desired. ∎

# Problem 1.11

## Solution

**Proof.** We construct the following sequence of subsets $A_k \subset [0,1]$, where each $A_k$ is the subset of $[0,1]$ consisting of numbers without a 4 in the first $k$ digits of their decimal expansions. To make this precise, let

$$A_0 = [0,1]$$

and

$$A_1 = [0, 0.4) \cup [0.5, 1]$$

From this, let

$$A_2 = [0, 0.04) \cup [0.05, 0.14) \cup [0.15, 0.24) \cup [0.25, 0.34) \cup [0.35, 0.4) \cup [0.5, 0.54) \cup ...$$

Continue this, subdividing each interval of $A_k$ into 10 new intervals along the next decimal place, and removing one of them corresponding to having a 4 in the next decimal place. So, each $A_k$ will be a countable union of half-open intervals of $[0,1]$ and will therefore be measurable. The total length of the intervals decreases by a factor of at least $9/10$ at each step. So, we find that $m(A_k) \leq \frac{9}{10} m(A_{k-1})$ for all $k \geq 1$ (this works because the measure of a union of disjoint intervals is the sum of the lengths), and so

$$m(A_k) \leq \left(\frac{9}{10}\right)^k \cdot m(A_0) = \left(\frac{9}{10}\right)^k$$

Let $A = \bigcap_{k=1}^{\infty} A_k$; then, $A$ is the desired set of numbers in $[0,1]$ without a 4 at any point in its decimal expansion, and it is measurable. This setup makes it such that $A_{k+1} \subset A_k$ for all $k$. In other words, $A_k$ decreases to $A$. Therefore, by Corollary 3.3(ii), we have that since $m(A_0) = 1 < \infty$,

$$m(A) = \lim_{k \to \infty} m(A_k) \leq \lim_{k \to \infty} \left(\frac{9}{10}\right)^k = 0$$

Since $m(A)$ is nonnegative, we arrive at the result that $m(A) = 0$. ■

# Problem 1.15

## Solution

**Proof.** Let $E \subset \mathbb{R}^d$. We wish to show that for every countable covering of $E$ by closed rectangles, we can find a countable covering of $E$ by closed cubes that is the same total size. So, let $E \subset \bigcup_{i=1}^\infty R_i$ be a countable covering of $E$ by closed rectangles. Let $D$ be the maximal length of any rectangle along any axis (i.e. if we consider each rectangle to be a product of intervals along dimensions, let $D$ be the max interval length over both rectangles and dimensions). Fix $\epsilon > 0$. For each rectangle $R_i$, consider a grid in $\mathbb{R}^d$ formed by cubes of side length $\frac{\epsilon}{2^i}$. Let $\mathcal{Q}_i$ be the set of cubes that are entirely contained in $R_i$, and let $\mathcal{Q}_i'$ be the set of cubes that intersect both $R_i$ and $R_i^C$. Both $\mathcal{Q}_i$ and $\mathcal{Q}_i'$ must therefore be finite, since $R_i$ is of finite width in each dimension. Then, we clearly have by construction that

$$\bigcup_{Q \in \mathcal{Q}_i} Q \subset R_i \subset \bigcup_{Q \in \mathcal{Q}_i \cup \mathcal{Q}_i'} Q \implies \sum_{Q \in \mathcal{Q}_i} |Q| \leq |R_i| \leq \sum_{Q \in \mathcal{Q}_i \cup \mathcal{Q}_i'} |Q|$$

Now, a simple geometric argument yields that we can always cover each of the $2d$ faces of $R_i$ with at most $\left(\frac{D \cdot 2^i}{\epsilon}\right)^{d-1}$ cubes (each face is $d-1$-dimensional and no axis can fit more than $D/(\epsilon/2^i)$ cubes). With this in mind, and the fact that $|Q| = \frac{\epsilon^d}{2^{id}}$ for all $Q \in \mathcal{Q}_i'$, we can apply Lemma 1.2 and the fact that rectangles have nonnegative area to see that

$$\sum_{Q \in \mathcal{Q}_i'} |Q| = \sum_{Q \in \mathcal{Q}_i'} \frac{\epsilon^d}{2^i d} \leq D^{d-1} \frac{\epsilon}{2^i} \implies \sum_{Q \in \mathcal{Q}_i \cup \mathcal{Q}_i'} |Q| \leq \sum_{Q \in \mathcal{Q}_i} |Q| + \sum_{Q \in \mathcal{Q}_i'} |Q| \leq |R_i| + D^{d-1} \frac{\epsilon}{2^i}$$

This, combined with our earlier inequality, yields that

$$|R_i| \leq \sum_{Q \in \mathcal{Q}_i \cup \mathcal{Q}_i'} |Q| \leq |R_i| + D^{d-1} \frac{\epsilon}{2^i}$$

So, there exists a finite cover of $R_i$ by closed cubes that is not much larger that $R_i$. Applying the same logic for all $i$, we find that there is a countable cover of $E$ by closed cubes $E \subset \bigcup_{i=1}^\infty R_i \subset \bigcup_{i=1}^\infty \bigcup_{Q \in \mathcal{Q}_i \cup \mathcal{Q}_i'} Q$ with

$$\sum_{i=1}^\infty |R_i| \leq \sum_{i=1}^\infty \sum_{Q \in \mathcal{Q}_i \cup \mathcal{Q}_i'} |Q| \leq \sum_{i=1}^\infty |R_i| + \sum_{i=1}^\infty D^{d-1} \frac{\epsilon}{2^i} = \sum_{i=1}^\infty |R_i| + D^{d-1} \epsilon$$

Taking $\epsilon \to 0$, we get that the sum of areas of these closed cubes $Q$ equals the sums of areas of these closed rectangles $R_i$, both of which countably cover $E$. Since this applies for any countable cover of $E$ by closed rectangles, the result also holds after taking an infimum over possible countable covers of $E$ by closed rectangles. The result follows. ■

# Problem 1.16

## Solution

**Proof of (a).** Note first that we can write

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$$

since an element $x$ is in $\bigcup_{k \geq n} E_k$ for all $n \in \mathbb{N}$ if and only if it is in infinitely many $E_k$'s, which happens if and only if $x \in E$ by definition. Since each $E_k$ is measurable, the countable unions $\bigcup_{k \geq n} E_k$ must also be measurable. Next, since each $\bigcup_{k \geq n} E_k$ is measurable and $E$ is a countable intersection of such sets, $E$ is therefore also measurable. ∎

**Proof of (b).** Fix an arbitrary $\epsilon > 0$. The fact that $\sum_{k=1}^{\infty} m(E_k)$, which is a sum of nonnegative values, converges to a finite value in $\mathbb{R}$ implies that there exists an $N \in \mathbb{N}$ large enough that $\sum_{k=N}^{\infty} m(E_k) < \epsilon$. Using the same form for $E$ as in part $(a)$, we can note that

$$E \subset \bigcup_{k \geq N} E_k,$$

and so by subadditivity of measure

$$m(E) \leq \sum_{k=N}^{\infty} m(E_k) < \epsilon$$

Since this holds for all $\epsilon > 0$, we can take the limit $\epsilon \to 0$ to get that $m(E) = 0$ as desired. ∎

# Problem 1.26

## Solution

**Proof.** Suppose that $A \subset E \subset B$, where $A, B$ are measurable sets of finite measure with $m(A) = m(B)$. We therefore know that $m(B - A) = 0$ (since $m(B - A) + m(A) = m(B)$). Now, consider the set $E - A$, which is a subset of $B - A$ since $E \subset B$. Monotonicity yields

$$E - A \subset B - A \implies m_*(E - A) \leq m_*(B - A) = 0$$

This therefore means that $m_*(E - A) = 0$ as well. Since sets of exterior measure 0 are measurable, we know that $E - A$ is measurable, which means that $E = A \cup (E - A)$, the union of two measurable sets, is also measurable. ■