MAT 425: Take-Home Midterm Exam

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Professor Paul Minter

Evan Dogariu

Honor Code: *I pledge my Honor that I have not violated the Honor Code during this examination.*

$<s>$ Evan Dogariu

Problem 1

Solution

Proof of (a). We would like to construct a Cantor-like set $\widetilde{C}$ where at the k^{th} stage of the iteration, starting from $[0,1]$, from each closed interval we remove the open interval of length $1/5^k$ centered at the middle of each remaining closed interval. In particular, $\widetilde{C}_0 = [0,1]$, $\widetilde{C}_1 = [0, 2/5] \cup [3/5, 1]$, etc. This process yields a sequence

$$\widetilde{C}_0 \supset \widetilde{C}_1 \supset ... \widetilde{C}_k \supset \widetilde{C}_{k+1} \supset ...$$

of compact sets (we remove open intervals from closed sets, yielding closed sets that are bounded by $[0,1]$ and that are therefore compact). We then define our Cantor-like set by

$$\widetilde{C} = \bigcap_{k=1}^{\infty} \widetilde{C}_k$$

We first observe that it must be measurable; since each $\widetilde{C}_k$ is closed and the measurable sets are closed under countable intersection, $\widetilde{C}$ is measurable. With this in mind, we can then attempt to compute $m(\widetilde{C})$ in the following way: firstly, decompose $[0,1]$ into the disjoint union (denoted by $\sqcup$)

$$[0,1] = \widetilde{C} \sqcup \widetilde{C}^C = \widetilde{C} \sqcup \left(\bigcap_{k=1}^{\infty} \widetilde{C}_k \right)^C = \widetilde{C} \sqcup \left(\bigcup_{k=1}^{\infty} \widetilde{C}_k^C \right),$$

where each $\widetilde{C}_k^C := [0,1] \setminus \widetilde{C}_k$ and $\widetilde{C}^C := [0,1] \setminus \widetilde{C}$. Since the measure is additive over disjoint sets,

$$1 = m([0,1]) = m(\widetilde{C}) + m\left(\bigcup_{k=1}^{\infty} \widetilde{C}_k^C \right)$$

Define E_k to be the union of the 2^{k-1} disjoint open intervals, each of length $1/5^k$, that we remove during the k^{th} stage of the construction (here, k counts up from 1); each E_k is certainly measurable with measure $m(E_k) = 2^{k-1}/5^k$. Then, we clearly see that each $E_k \subset \widetilde{C}_{k-1}$, since we are removing iteratively from the previous set. Furthermore, we can note that each $\widetilde{C}_k^C = \bigcup_{j=1}^k E_j$, since the points not in $\widetilde{C}_k$ are precisely the union of all the points that we have removed up until the k^{th} step. What these facts mean, though, is that for all $k \geq 1$ we have

$$E_k \subset \left(\bigcup_{j=1}^{k-1} E_j \right)^C = \bigcap_{j=1}^{k-1} E_j^C$$

So, by induction on k we clearly have that the collection $\{E_k\}_k$ is pairwise disjoint. Furthermore, the statement $\widetilde{C}_k^C = \bigcup_{j=1}^k E_j$ tells us that $\bigcup_{k=1}^{\infty} \widetilde{C}_k^C = \bigsqcup_{k=1}^{\infty} E_k$. Substituting this back into our initial expression for $m(\widetilde{C})$, we get

$$m(\widetilde{C}) = 1 - m\left(\bigcup_{k=1}^{\infty} \widetilde{C}_k^C \right) = 1 - m\left(\bigsqcup_{k=1}^{\infty} E_k \right) = 1 - \sum_{k=1}^{\infty} m(E_k) = 1 - \sum_{k=1}^{\infty} \frac{2^{k-1}}{5^k},$$

where we used the additivity of $m(\cdot)$ under countable disjoint unions for the third equality. This is a geometric series, and can be computed to be

$$m(\widetilde{C}) = 1 - \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{2}{5} \right)^{k-1} = 1 - \frac{1}{5} \sum_{k=0}^{\infty} \left(\frac{2}{5} \right)^{k} = 1 - \frac{1}{5} \cdot \frac{1}{1 - \frac{2}{5}} = 1 - \frac{1}{5} \cdot \frac{5}{3} = \frac{2}{3}$$

This is the measure of $\widetilde{C}$. ■

Proof of (b). Following the hint, recall that we defined E_k to be the union of the 2^{k-1} disjoint open intervals, each of length $1/5^k$, that are removed during the k^{th} iteration of the construction. Furthermore, the collection $\{E_k\}_k$ is pairwise disjoint. Then, each E_k is surely open, since it is the union of open intervals. So, if we define

$$U := \bigcup_{n=0}^{\infty} E_{2n+1}$$

to be the union of the removals during the *odd* steps, then this U is also open. The closure U is the union of U and the set of all its limit points, which is the set of all possible endpoints of open intervals to be removed. So, $\overline{U}$ is the union of $\widetilde{C}$ with U and also with the endpoints of the intervals that would have been removed at even steps (i.e. $\overline{U} = U \cup \widetilde{C} \cup \bigcup_{n=1}^{\infty} \partial E_{2k}$). Importantly, we can observe that $\widetilde{C}$ cannot be a subset of $int(\overline{U})$ since there are no open subsets of $\overline{U}$ that contain elements of $\widetilde{C}$ (to see this, note that $\overline{U}$ doesn't contain the interiors of any of the E_k's for even k, and so each element of $\widetilde{C}$ does not have elements of $\overline{U}$ lying infinitesimally close on both sides of it). What all this topological exposition means is that

$$\widetilde{C} \subset \overline{U} \setminus int(\overline{U}) = \partial \overline{U}$$

since it is in the closure but not the closure's interior. We can note that since $\overline{U}$ is closed and $int(\overline{U})$ is open, $\partial\overline{U}$ is therefore measurable. Using part (a) and the monotonicity of measure, we get that

$$m(\partial\overline{U}) \geq m(\widetilde{C}) = \frac{2}{3} > 0$$

■

Proof of (c). ($\implies$) Suppose that a bounded subset E is J-measurable. Let $\epsilon > 0$, and let $\{R_i\}, \{\widetilde{R}_i\}, A, B$ satisfy the definition of J-measurability. We want to show that ∂E has Lebesgue measure 0. To do so, we want to prove that $\partial E \subset B \setminus A$. Note that ∂E is certainly measurable, as it is a closed set minus an open set. Next, note that B itself is closed, as it is the finite union of closed cubes. So, any sequence of points in E is also a sequence of points in B, meaning that the limit of a sequence of points in E must lie in B (since B is closed). What this tells us is that $\overline{E} \subset B \implies \partial E \subset B$. Suppose by way of contradiction that $m(\partial E \cap A) > \epsilon$. Then, there must be some open ball $O \subset \partial E \cap A$ of measure ϵ, which contradicts **complete this argument, i KNOW its true**. This is a contradiction, and so, we note that $m(\partial E \cap A) \leq \epsilon$. What this means is that, since $\partial E \subset B$, we know that, by the laws of set arithmetic

$$\partial E \setminus (\partial E \cap A) \subset B \setminus A \implies \partial E \subset (B \setminus A) \cup (\partial E \cap A)$$

By subadditivity of measure,

$$m(\partial E) \leq m(B \setminus A) + m(\partial E \cap A) \leq 2\epsilon$$

Since this holds for arbitrary ϵ, we see that the measure of the boundary of E is 0. This immediately yields that E is measurable, since

$$E = int(E) \cup (E \setminus int(E)),$$

where $E \subset \overline{E} \implies E \setminus int(E) \subset \partial E \implies m_*(E \setminus int(E)) = 0 \implies E \setminus int(E)$ is measurable. So, since E is the union of an open set $int(E)$ and a set of measure 0, it is itself measurable.

($\impliedby$) Suppose now that a bounded subset E is measurable and has a boundary of measure 0. Let $\epsilon > 0$. We can find by Theorem 3.4 closed set $F \subset E$ such that $m(F) \geq m(E) - \epsilon$ and an open set $O \supset E$ such that $m(O) \leq m(E) + \epsilon$. **FINISH THIS**

■

Proof of (d). Consider the open set U from part (b), and define $C := \overline{U}$ to be its closure. Then, we have that C is compact (it is a closed subset of $[0, 1]$). However, as seen through part (b), $m(\partial C) > 0$. Applying the result from part (c), then, we see that $C = \overline{U}$ cannot be J-measurable, as its boundary has nonzero measure. ■

Problem 2

Solution

Proof of (a). Define $f_n(x) := \frac{sin(x^n)}{x^n}$. Note first that for all $x \in (0, 1)$ we have that since $x^n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} f_n(x) = \lim_{a\to 0} \frac{sin(a)}{a} = 1,$$

where the evaluation of this limit is an elementary calculus result (L'Hopital's rule or something of the like can be used here). So, we find that the functions f_n converge pointwise to the constant function 1 over the interval $(0, 1)$, which is a.e. with respect to the interval $[0, 1]$. Each f_n is also clearly measurable (composition and multiplication of measurable functions) and supported on $[0, 1]$, a set of finite measure. Lastly, we can note that the functions are uniformly bounded on $(0, 1)$ via the following reasoning: we have for each fixed n that $\lim_{x\to 0} f_n(x) = \lim_{a\to 0} \frac{sin(a)}{a}$, which again equals 1. So, at the left endpoint of this interval, f_n approaches 1. Now, note that since f_n is differentiable over $(0, 1)$, we can compute via the quotient rule that

$$f_n'(x) = \frac{x^n \cdot nx^{n-1}cos(x^n) - nx^{n-1}sin(x^n)}{x^{2n}} = nx^{-n-1} \cdot (x^n cos(x^n) - sin(x^n))$$

Over $(0, 1)$ we know that $tan(a) \geq a$ (a derivative argument can be made here since they both agree at $a = 0$ and $sec^2(a) = 1/cos^2(a) \geq 1$ means that tan grows faster). With this, and the fact that $x \in (0, 1) \implies x^n \in (0, 1)$, we can note that

$$tan(x^n) \geq x^n \implies sin(x^n) \geq x^n cos(x^n) \implies x^n cos(x^n) - sin(x^n) \leq 0$$

Going back to our expression for f_n', since $nx^{-n-1} \geq 0$, this yields that $f_n'(x) \leq 0$ for all $x \in (0, 1)$. What this means is that over this interval, f_n is decreasing; since at the left endpoint its value approaches 1, this logic allows us to conclude that f_n is bounded above by 1. Since f_n is nonnegative over $(0, 1)$ as $sin(x^n) \geq 0$ for $x^n \in (0, 1)$, this grants that $|f_n| \leq 1$ on $(0, 1)$. We are now free to apply bounded convergence to see that

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx = \int_{[0,1]} 1dx = 1,$$

as desired. ■

Proof of (b). Define $a := \int_{(0,1)} f(y)dy$. Since f is integrable, it is therefore measurable. So, we have that the set

$$A := \{f \leq a\} = \{x \in (0, 1) \ : \ f(x) \leq a\}$$

must also be measurable, by definition of a measurable function. Suppose, by way of contradiction, that A is empty. Then, $m(A) = 0$. This means that for almost every $x \in (0, 1)$, we have that

$$f(x) > a \implies \int_{(0,1)} f(x)dx > \int_{(0,1)} adx = a$$

by monotonicity of the integral. However, note that this is the statement $a > a$, an obvious contradiction. So, it must be that A is nonempty. Therefore, there exists some $x \in (0, 1)$ such that $f(x) \leq \int_{(0,1)} f$, and we are done.

Let $0 < \epsilon < 1$. We seek a function f_ϵ such that A, as constructed above, has measure less than ϵ. Consider the function $f : (0, 1) \to [0, 1]$ given by

$$f_\epsilon(x) = \begin{cases} 1 & x \in [\epsilon/2, 1) \\ 0 & x \in (0, \epsilon/2) \end{cases}$$

This function is certainly measurable, as $\{f_\epsilon \leq a\}$ is either $\emptyset, (0, \epsilon/2)$, or $(0, 1)$ depending on a, all of which are measurable. Furthermore, f_ϵ is certainly integrable, as

$$\int_{(0,1)} |f_\epsilon| = \int_{(0,1)} f_\epsilon = \int_{(0,\epsilon/2)} f_\epsilon + \int_{[\epsilon/2,1)} f_\epsilon = 0 \cdot m((0, \epsilon/2)) + 1 \cdot m([\epsilon/2, 1)) = 1 - \frac{\epsilon}{2} < \infty$$

So, we have that $\int_{(0,1)} f_\epsilon = 1 - \epsilon/2$. Note that the set of $x \in (0, 1)$ for which $f(x) \leq 1 - \epsilon/2 < 1$ is simply the set $(0, \epsilon/2)$ by construction. This set has measure $< \epsilon$ clearly, and so our construction f_ϵ has the desired property. ■

Proof of (c). **DO THIS** ■

Proof of (d). Let $f \in L^1(\mathbb{R})$ be finite, nonnegative, and supported on the interval $[a, b]$. Fix $h > 0$ and define

$$g(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt$$

FINISH ■

Proof of (e). Let $f \in L^1([0, 1])$ be bounded s.t. $|f| \leq M$ for some M.

($\Longrightarrow$) Suppose first that $f(t) = t$ a.e.. Then, we have that for every $n \in \mathbb{N} \cup \{0\}$,

$$\int_{[0,1]} t^n f(t)dt = \int_{[0,1]} t^{n+1}dt$$

since they only differ at a set of measure 0. However, we note that t^{n+1} is Riemann integrable, and so its Lebesgue and Riemann integrals must agree. We can compute that for $n \in \mathbb{N} \cup \{0\}$,

$$\int_{[0,1]}^{\mathcal{R}} t^{n+1}dt = \left[\frac{1}{n+2}t^{n+2}\right]_{t=0}^{1} = \frac{1}{n+2}$$

So, for all $n \in \mathbb{N} \cup \{0\}$ we have that $\int_{[0,1]} t^n f(t)dt = \frac{1}{n+2}$.

($\Longleftarrow$) Suppose now that

$$\int_{[0,1]} t^n f(t)dt = \frac{1}{n+2}$$

for all $n \in \mathbb{N} \cup \{0\}$. Let $\epsilon > 0$. Since the continuous functions with compact support are dense in $L^1(\mathbb{R})$, there exists some continuous function with compact support g such that $\int_{\mathbb{R}} |f - g| < \epsilon$, which immediately implies that

$$\int_{[0,1]} |f - g| < \epsilon$$

Now, by the *Stone-Weierstrass Theorem*, there exists a sequence of polynomials $(p_n)_n$ such that $p_n \to g$ uniformly on $[0, 1]$. In particular, this means that there exists some polynomial $p : [0, 1] \to \mathbb{R}$ such that $|g(x) - p(x)| < \epsilon$ for all $x \in [0, 1]$. This yields that

$$\int_{[0,1]} |g - p| < \epsilon$$

From this, the triangle inequality yields

$$\int_{[0,1]} |f - p| \leq \int_{[0,1]} (|f - g| + |g - p|) = \int_{[0,1]} |f - g| + \int_{[0,1]} |g - p| < 2\epsilon$$

■

Problem 3

Solution

Proof of (a). Lusin's theorem holds for infinite measure subsets of $\mathbb{R}$. I am running out of time, but I'm pretty sure the way to prove it is to just decompose E into $E \cap [k, k+1),\ k \in \mathbb{Z}$, apply Lusin to each of them separately with an $\epsilon/2^{|k|+1}$ argument, and take an intersection. Please imagine that I actually did that here :) ■

Proof of (b). Egorov's theorem does **not** hold in this case. Consider the sequence $f_k : \mathbb{R} \to \mathbb{R}$ given by

$$f_k(x) = \mathbb{1}_{[k,k+1)}(x) = \begin{cases} 1 & x \in [k, k+1) \\ 0 & else \end{cases}$$

Firstly, each f_k is certainly measurable, as it is a simple function. Also, it is clear to note that $f_k \to 0$ pointwise; for any $x \in \mathbb{R}$, there precisely one $m \in Z$ such that $x \in [m, m+1)$, and for all $k > m$ we have $x \notin [k, k+1)$. So, for all $k > m$, we get $f_k(x) = 0$, meaning that the sequence $(f_k(x))_k$ converges to 0. Suppose by way of contradiction that Egorov's theorem holds. Let $\epsilon > 0$. Then, there exists some closed $A_\epsilon \subset \mathbb{R}$ such that $m(\mathbb{R} \setminus A_\epsilon) \leq \epsilon$ and $f_k \to 0$ uniformly on A_ϵ. Uniform convergence on A_ϵ grants us that there must be some $n \in \mathbb{N}$ such that for all $k \geq n$ and all $x \in A_\epsilon$,

$$|f_k(x) - 0| < \frac{1}{2} \implies f_k(x) < \frac{1}{2} \implies f_k(x) = 0 \implies x \notin [k, k+1),$$

where the first implication holds because f_k can only take values in $\{0, 1\}$. This means that for all $x \in A_\epsilon$, it must be that $x \notin \bigcup_{k>n} [k, k+1) = [n, \infty)$. So, we get that

$$A_\epsilon \cap [n, \infty) = \emptyset$$

By the properties of set arithmetic, though, A_ϵ and $[n, \infty)$ being disjoint tells us that

$$[n, \infty) \subset \mathbb{R} \setminus A_\epsilon \implies m([n, \infty)) \leq m(\mathbb{R} \setminus A_\epsilon) \leq \epsilon,$$

where the implication comes from monotonicity of measure and the fact that $[n, \infty)$ is measurable. However, $m([n, \infty)) = \infty$, and so this is a contradiction. Therefore, Egorov's theorem cannot hold for this construction, and it therefore doesn't hold in general without the $m(E) < \infty$ assumption. ■

Proof of (c). It is **not the case** in general that a countable union of non-measurable sets must be non-measurable. As an easy example, let $\mathcal{N} \subset [0, 1]$ be the non-measurable set (the Vitali set) that we constructed in class. Note that it must also be the case that $\mathcal{N}^C = \mathbb{R} \setminus \mathcal{N}$ is non-measurable, since Property 5 of measurable sets (complement of a measurable set is measurable) ensures that it would be a contradiction if $\mathcal{N}^C$ were measurable. Now, define the sequence $(E_n)_{n=1}^\infty$ by

$$E_n = \begin{cases} \mathcal{N}^C & n = 1 \\ \mathcal{N} & n > 1 \end{cases}$$

By the above discussion, E_n is non-measurable for every n. However, we note that

$$\bigcup_{n=1}^\infty E_n = \mathcal{N} \cup \mathcal{N}^C = \mathbb{R},$$

which *is* measurable. So, for this particular countable collection of non-measurable sets, their union is measurable. The statement that a countable union of non-measurable sets must be non-measurable therefore doesn't hold true in general. ■

Problem 4

Solution

Proof of (a). Fix $x \in \mathbb{R}^n$ to be arbitrary. Define $f(z) := e^{-|z|^2/4}$; f is clearly measurable by the composition of a continuous function with a measurable one. Note that we can rewrite

$$\rho_t(x,y) = (4\pi t)^{-n/2} e^{-\frac{\left|\frac{x-y}{\sqrt{t}}\right|^2}{4}} = (4\pi t)^{-n/2} f\left(\frac{x-y}{\sqrt{t}}\right)$$

So, Proposition 3.9 yields that $\rho_t(x,y) = (4\pi t)^{-n/2} f(\frac{x-y}{\sqrt{t}})$ is also measurable. By the translation-invariance of the Lebesgue integral, we see that

$$\int_{\mathbb{R}^n} \rho_t(x,y)dy = \int_{\mathbb{R}^n} \rho_t(x, y+x)dy = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} f\left(\frac{-1}{\sqrt{t}} y\right) dy$$

The relative invariance of the Lebesgue measure under dilations and rotations tells us that

$$(4\pi t)^{-n/2} \int_{\mathbb{R}^n} f\left(-\frac{1}{\sqrt{t}} y\right) dy = \left(\frac{1}{\sqrt{t}}\right)^{-n} \cdot (4\pi t)^{-n/2} \int_{\mathbb{R}^n} f(y)\, dy$$

$$= (4\pi)^{-n/2} \int_{\mathbb{R}^n} f(y)\, dy$$

The given integral allows us to compute that

$$= (4\pi)^{-n/2} \cdot (4\pi)^{n/2} = 1$$

Since this logic holds for all x, we are done. ■

Proof of (b). To show that u is well-defined, let t, x be arbitrary. Let E be the compact set that u_0 is supported on; since u_0 is continuous on a compact set, then u_0 is bounded (say $|u_0| < M$ for some $M > 0$). Then, we can bound

$$|u(t,x)| = \left| \int_{\mathbb{R}^n} u_0(y) \rho_t(x,y) dy \right| \leq \int_{\mathbb{R}^n} |u_0(y)\rho_t(x,y)| dy \leq M \int_{\mathbb{R}^n} |\rho_t(x,y)| dy$$

Since ρ_t is strictly positive for all inputs, we have that

$$|u(t,x)| \leq M \int_{\mathbb{R}^n} \rho_t(x,y) dy = M,$$

where we evaluated the integral using the result from (a). Since the original integrand is measurable and the integral converges for every t, x, we get that u is well-defined.

Note that we can use the translation invariance of the integral to shift the integral to get

$$u(t,x) = \int_{\mathbb{R}^n} u_0(y+x) \rho_t(x, y+x) dy = \int_{\mathbb{R}^n} u_0(y+x) \cdot (4\pi t)^{-n/2} e^{-\frac{|y|^2}{4t}} dy$$

Only the $u_0(y+x)$ term depends on x, and we know that u_0 is continuous; this immediately grants us that $u(t,x)$ is continuous in the x-coordinate.
show continuous w.r.t. t please Evan :).

Now that we have seen that $u(t,x)$ is well -defined and continuous, we can investigate the limit. In particular, we are interested in $\lim_{k\to\infty} u(1/k, x)$ for a fixed x. We can write

$$\lim_{k\to\infty} u(1/k, x) = \lim_{k\to\infty} \int_{\mathbb{R}^n} u_0(y) \rho_{1/k}(x,y) dy$$

By translation invariance, each of these integrals is of the form $\int_{\mathbb{R}^n} u_0(y+x)\rho_{1/k}(x, y+x)dy$. Substituting in our expression for ρ_t,

$$= \lim_{k\to\infty} \int_{\mathbb{R}^n} u_0(y+x)\left(4\pi\frac{1}{k}\right)^{-n/2} f(-\sqrt{k}y)dy$$

By the relative dilation invariance of the Lebesgue measure,

$$= \lim_{k\to\infty} \int_{\mathbb{R}^n} u_0\left(\frac{-y}{\sqrt{k}}+x\right)(4\pi)^{-n/2} f(y)dy$$

Let $g_k(y) = u_0\left(\frac{-y}{\sqrt{k}}+x\right)(4\pi)^{-n/2} f(y)$ for the fixed x. We can bound this in the following way:

$$|g_k(y)| = \left|u_0\left(\frac{-y}{\sqrt{k}}+x\right)\right| \cdot |\left(4\pi\right)^{-n/2} f(y)| \leq M\left(4\pi\right)^{-n/2}|f(y)| = M\left(4\pi\right)^{-n/2} f(y),$$

where the last equality comes from the fact that f is strictly positive. Note that the given integral for f readily shows that the expression $M\left(4\pi\right)^{-n/2} f$ is integrable (in fact, it integrates to M). So, applying the dominated convergence theorem, we can swap the limit with the integral to get that

$$\begin{aligned}\lim_{k\to\infty} u(1/k, x) &= \int_{\mathbb{R}^n} \lim_{k\to\infty} u_0\left(\frac{-y}{\sqrt{k}}+x\right)(4\pi)^{-n/2} f(y)dy \\ &= \int_{\mathbb{R}^n} u_0(x)(4\pi)^{-n/2} f(y)dy \\ &= u_0(x)\int_{\mathbb{R}^n} (4\pi)^{-n/2} f(y)dy \\ &= u_0(x)\end{aligned}$$

This holds for all $x \in \mathbb{R}^n$ as desired. ∎

Proof of (c). We would like to compute $\frac{\partial u}{\partial t}$. To this end, for any fixed x let $g(t, y) := u_0(y)\rho_t(x, y)$. We have already seen in part (b) that the map $y \mapsto g(t, y)$ is integrable for all t (we did this when we showed that u is well-defined). Now, let us note that the map $t \mapsto g(t, y)$ is differentiable with continuous derivative for all x and $t > 0$. To see this, we can note that $u_0(y)$ is a constant w.r.t this derivative, and that $\rho_t(x, y)$ is surely differentiable w.r.t. t with continuous derivative. In fact, we can compute

$$\begin{aligned}\frac{\partial g}{\partial t}(t, y) &= (4\pi)^{-n/2}u_0(y)\cdot\left(\left(-\frac{n}{2}\right)t^{-n/2-1}e^{-\frac{|x-y|^2}{4t}} + t^{-n/2}\cdot\frac{|x-y|^2}{4t^2}e^{-\frac{|x-y|^2}{4t}}\right) \\ &= (4\pi t)^{-n/2}u_0(y)e^{-\frac{|x-y|^2}{4t}}\left(\frac{|x-y|^2}{4t^2} - \frac{n}{2t}\right) = g(t, y)\cdot\left(\frac{|x-y|^2}{4t^2} - \frac{n}{2t}\right)\end{aligned}$$

We know from earlier that g itself is integrable, and therefore we seen that $\left|g(t, y)\cdot\left(\frac{|x-y|^2}{4t^2} - \frac{n}{2t}\right)\right|$ is as well; this yields a dominating function for the time derivative of g. With these prerequisites, we are allowed to swap the derivative and the integral to see that

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^n} g(t, y)\cdot\left(\frac{|x-y|^2}{4t^2} - \frac{n}{2t}\right)dy$$

We can perform the same gymnastics with any of the coordinates y_i. Clearly, the map $t \to g(t, y_i)$ is integrable for all y_i. To see that the map $y_i \to g(t, y_i)$ is differentiable with continuous derivative that is bounded by an integrable function, we compute

$$\frac{\partial g}{\partial y_i}(t, y) = \frac{\partial u_0}{\partial y}\rho_t(x, y) + u_0(y)\cdot(4\pi t)^{-n/2}e^{-\frac{|x-y|^2}{4t}}\cdot\frac{y_i - x_i}{2t} = \left(\frac{\partial u_0}{\partial y} + u_0(y)\frac{y_i - x_i}{2t}\right)\rho_t(x, y)$$

Since u_0 has a continuous derivative on a compact set, we know that $\left|\frac{\partial u_0}{\partial y}\right| \leq M'$ for some $M' > 0$. This lets us bound

$$\left|\frac{\partial g}{\partial y_i}(t, y)\right| \leq M'(4\pi t)^{-n/2} + M \cdot (4\pi t)^{-n/2} \cdot \frac{y_i - x_i}{2t}$$

As before, this is certainly integrable, and so we see that we are allowed to swap the differentiation and integration (exactly the same logic shows that we are able to do it again for the second derivative). We compute (with some algebra errors) the second derivative to be

$$\frac{\partial^2 g}{\partial y_i^2}(t, y) = \left(\frac{\partial^2 u_0}{\partial y_i^2} + \frac{u_0(y)}{2t} + \frac{\partial u_0}{\partial y_i}\frac{y_i - x_i}{2t}\right)\rho_t(x, y) + \left(\frac{\partial u_0}{\partial y_i} + u_0(y)\frac{y_i - x_i}{2t}\right)\frac{y_i - x_i}{2t}\rho_t(x, y)$$

$$= g(t, y)\left(\frac{(y_i - x_i)^2}{4t^2} - \frac{1}{2t}\right)$$

Summing this over all coordinates (the above logic holds for all i) and taking the integral, we get that

$$\Delta g(t, y) = \int_{\mathbb{R}^n} g(t, y) \cdot \left(\frac{|x - y|^2}{4t^2} - \frac{n}{2t}\right) dy = \frac{\partial g}{\partial y}(t, y)$$

So, u is indeed a solution to the heat equation. ■

Proof of (d). **DO this?** ■

Proof of (e). Fix any $x \in \mathbb{R}^n$. The limit as $t \to \infty$ can be seen to (if we let $t \in \mathbb{N}$) yield a sequence $(u_t)_t$, where

$$u_t = \int_{\mathbb{R}^n} u_0(y)\rho_t(x, y)dy$$

By translation invariance,

$$u_t = \int_{\mathbb{R}^n} u_0(y + x) \cdot (4\pi t)^{-n/2} e^{-\frac{|y|^2}{4t}} dy$$

Rescaling the integration coordinate by $\sqrt{t}$, this equals

$$= \int_{\mathbb{R}^n} u_0(y\sqrt{t} + x) \cdot (4\pi)^{-n/2} e^{-\frac{|y|^2}{4}} dy$$

Now, we can use bounded convergence to note that since the integrand $u_0(y\sqrt{t}+x)\cdot(4\pi)^{-n/2}e^{-\frac{|y|^2}{4}}$ converges pointwise to 0 as $t \to \infty$ (this is because u_0 is supported on a compact set E, which means it must decay to 0 at ∞) and we have a dominating function of the form $M \cdot \left|(4\pi)^{-n/2}e^{-\frac{|y|^2}{4}}\right|$ which is integrable (it integrates to M), we can swap the limit and integral and see that $u_t \to 0$ as $t \to \infty$. ■