MAT 425: Final Exam

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Professor Paul Minter

Evan Dogariu

Honor Code: *I pledge my Honor that I have not violated the Honor Code during this examination.*

$<s>$ Evan Dogariu

Problem 1

Solution

Proof of (a).
Let $\mathcal{C} \subset [0,1]$ denote the usual Cantor set. Let $S := \{x \in \mathbb{R} : |\sin(x)| \in \mathcal{C}\}$. Now, let

$$S_0 := S \cap \left[0, \frac{\pi}{2}\right) = \left\{x \in \left[0, \frac{\pi}{2}\right) : |\sin(x)| \in \mathcal{C}\right\}$$

Note that for $x \in \left[0, \frac{\pi}{2}\right)$ we have that $|\sin(x)| = \sin(x)$, and that $\sin(\cdot)$ on this domain is a strictly increasing and bijective function from $\left[0, \frac{\pi}{2}\right) \to [0,1)$. As such, it has an inverse function $\arcsin : [0,1) \to \left[0, \frac{\pi}{2}\right)$. We know from elementary calculus that arcsin (restricted to the domain $[0,1)$) is continuous and differentiable everywhere with

$$\arcsin'(t) = \frac{1}{\sqrt{1-t^2}}$$

So, since $\frac{1}{\sqrt{1-t^2}} \in L^1([0,1))$ (we know that $\int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{\pi}{2} < \infty$), we can apply the second part of Theorem 3.11 from Chapter 3 to see that

$$\arcsin(x) = \int_{[0,x]} \frac{1}{\sqrt{1-t^2}} dt$$

As such, this grants that arcsin is absolutely continuous. Now, let $\epsilon > 0$. Let δ be such that for all finite collections of disjoint intervals $\{(a_k, b_k)\}_{k=1}^N$ we have

$$\sum_{k=1}^N (b_k - a_k) < \delta \implies \sum_{k=1}^N |\arcsin(b_k) - \arcsin(a_k)| < \epsilon$$

We know that the Cantor set $\mathcal{C}$ is compact and has measure 0. So, there must exist an open set $U \supset \mathcal{C}$ such that $m(U) < \delta$ by the definition of Lebesgue measurability. Since $U \subset \mathbb{R}$, we know that we can express U as a countable union of disjoint open intervals $\{(a_k, b_k)\}_{k \in \mathbb{N}}$. The union of these intervals is an open cover of $\mathcal{C}$; since $\mathcal{C}$ is compact, there must therefore be a finite subcover $\{(a_k, b_k)\}_{k=1}^N$ such that

$$\mathcal{C} \subset \bigcup_{k=1}^N (a_k, b_k)$$

Firstly, note that as $\bigcup_{k=1}^N (a_k, b_k) \subset U$, monotonicity of measure grants

$$\sum_{k=1}^N (b_k - a_k) = m\left(\bigcup_{k=1}^N (a_k, b_k)\right) \leq m(U) < \delta$$

So, by the absolute continuity of arcsin, we have that

$$\sum_{k=1}^N |\arcsin(b_k) - \arcsin(a_k)| < \epsilon$$

Now, note that the image of the Cantor set under arcsin is precisely equal to S_0; in other words, $S_0 = \arcsin(\mathcal{C})$. As such, we see that

$$S_0 \subset \arcsin\left(\bigcup_{k=1}^N (a_k, b_k)\right)$$

The right hand side is the image of an open set under a continuous map, and is thus open and therefore measurable. On the interval $[0,1)$, we know that arcsin is strictly increasing (the derivative exists everywhere is always > 0). This means that each interval (a_k, b_k) gets mapped to the interval $(\arcsin(a_k), \arcsin(b_k))$,

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and that the images of the intervals are all disjoint ($a_k > b_j$ for $k > j$ implies that $\arcsin(a_k) > \arcsin(b_j)$, and so the images of nonoverlapping intervals can't overlap). This tells us that

$$S_0 \subset \arcsin\left(\bigcup_{k=1}^{N}(a_k, b_k)\right) = \bigcup_{k=1}^{N}(\arcsin(a_k), \arcsin(b_k))$$

Taking the exterior measure of both sides and applying monotonicity of the exterior measure,

$$m_*(S_0) \le m\left(\bigcup_{k=1}^{N}(\arcsin(a_k), \arcsin(b_k))\right) = \sum_{k=1}^{N}|\arcsin(b_k) - \arcsin(a_k)| < \epsilon,$$

where the ϵ bound was derived earlier using the absolute continuity of arcsin. So,

$$m_*(S_0) \le \epsilon$$

for all $\epsilon > 0$, which means that S_0 is measurable with $m(S_0) = 0$. Now, note that by the oddness and periodicity of $\sin(\cdot)$, the function $|\sin(x)|$ is periodic in the sense that:

$$|\sin(x)| = |\sin(k\pi + x)| = |\sin(k\pi - x)| \qquad \forall k \in \mathbb{Z}$$

So, if we define for all $n \in \mathbb{Z}$ the set

$$S_n := S \cap \left[\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right) = \left\{x \in \left[\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right) : |\sin(x)| \in \mathcal{C}\right\},$$

then we see that S_n is a translate of S_0 for all $n \in \mathbb{Z}$ (i.e. $S_n = S_0 + \frac{n\pi}{2}$), which means that $m(S_n) = 0$ by the translation invariance of the Lebesgue measure. So, as $S = \bigsqcup_{n\in\mathbb{Z}} S_n$ (here, $\bigsqcup$ denotes a disjoint union), we get that

$$m(S) \le \sum_{n\in\mathbb{Z}} m(S_n) = \sum_{n\in\mathbb{Z}} 0 = 0$$

Therefore, $m(S) = 0$. ■

Proof of (b). Let $E \subset \mathbb{R}^n$ be measurable with finite measure. Suppose $f : E \to \mathbb{R}$ is measurable and finite a.e.. For each $n \in \mathbb{N}$, define

$$E_n := \{x \in E : f(x) \in [-n, n]\} = f^{-1}([-n, n])$$

Each E_n is certainly measurable since is is the preimage of an interval under a measurable function. Furthermore, $E_n \subset E_{n+1}$ as $f(x) \in [-n, n] \implies f(x) \in [-(n+1), n+1]$, and so $(E_n)_n$ is a monotonically increasing sequence of sets. Let $G := \{x \in E : |f(x)| = +\infty\}$ be the set of points over which f is infinite; then, $m(G) = 0$. For all $x \in E \setminus G$ we have that $|f(x)| < \infty$, which means that $f(x) \in [-n, n]$ for some large enough n. This then gives that, since $E_n \subset E$ for all n,

$$E \setminus G \subset \bigcup_{n=1}^{\infty} E_n \subset E$$

We know that $m(E \setminus G) = m(E)$, as $m(G) = 0$ and both E and G are measurable. By monotonicity of measure and Corollary 3.3(i) of Chapter 1 (measure of limit of monotonic sets), we then have

$$m(E) = m(E \setminus G) \le \lim_{n\to\infty} m(E_n) \le m(E) \implies \lim_{n\to\infty} m(E_n) = m(E)$$

In particular, this means that

$$m(E \setminus E_n) \to 0 \quad \text{as} \quad n \to \infty,$$

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as $m(E \setminus E_n) = m(E) - m(E_n) \to 0$. Let $\epsilon > 0$. Then, there must be an N such that

$$m(E \setminus E_N) < \epsilon$$

Also, E_N is measurable by our earlier reasoning and

$$\sup_{x \in E_N} |f(x)| \leq N < \infty$$

by construction of $(E_n)_n$. So, we have found a measurable set E_N such that $m(E \setminus E_N) < \epsilon$ and f is bounded on E_N. This proves the desired result. ■

Proof of (c). Suppose that $E \subset \mathbb{R}$ is measurable with $m(E) = 0$. Let $f \equiv +\infty$ on E, and be 0 elsewhere (the value of f on E^C doesn't change the integral over E). Note that f is measurable since for all reals $a \in \mathbb{R}$ we find that $\{f < a\}$ must be either $\emptyset$ or E^C, both of which are measurable sets; so f is a non-negative, measurable function. Now, recall that we define $\int_E f$ by

$$\int_E f := \sup_g \int_{\mathbb{R}} g,$$

where the supremum runs over all measurable functions g such that $0 \leq g \leq f \cdot \mathbb{1}_E$ over $\mathbb{R}$ and g is bounded and supported on a set of finite measure. Clearly, since $0 \leq g \leq f \cdot \mathbb{1}_E$, we see that $g \equiv 0$ on E^C, and so $\text{supp}(g) \subset E$. This means that for all such g, each of which are bounded above (say by M_g), we have

$$\int_{\mathbb{R}} g = \int_E g \leq M_g \int_E 1 = M_g \cdot m(E) = 0,$$

where the first equality used that $\text{supp}(g) \subset E$ and the inequality is because of the boundedness of g. Since g is non-negative, $\int_{\mathbb{R}} g \geq 0$, which means that $\int_{\mathbb{R}} g = 0$ for all qualifying g. Since every such g integrates to 0, the supremum of all their integrals must also be 0, and so

$$\int_E f = 0$$

■

Proof of (d). Suppose that $g : [a, b] \to \mathbb{R}$ is monotone increasing. We want to show that g is measurable. To this end, let $c \in \mathbb{R}$ be arbitrary, and let

$$E_c := \{x \in [a, b] : g(x) < c\} = \{g < c\}$$

Now, note that if a point $x \in E_c$, then for all $y < x$ we have

$$g(y) \leq g(x) \implies c > g(x) \geq g(y) \implies y \in E_c,$$

where we used the monotonicity of g and the definition of E_c. Since having an element of E_c means that all smaller elements of $[a, b]$ are also in E_c, then E_c must be either the empty set or an interval of the form $[a, d)$ or $[a, d]$ for some $d \in [a, b]$. All of these possibilities are measurable, and so E_c certainly must be measurable. Since $E_c := \{g < c\}$ is measurable for all $c \in \mathbb{R}$, then g is measurable as desired. ■

Proof of (e). Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence of measurable, non-negative functions that decreases to a function f (i.e. $f_n \searrow f$, and so f is measurable and non-negative). We want to show that

$$\lim_{n \to \infty} \int f_n = \int f$$

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To this end, define a new sequence of functions given by

$$g_n := f_1 - f_n \qquad \text{for all } n \in \mathbb{N}$$

Clearly, each g_n is measurable. We also know that $g_1 \equiv 0$ and $g_n \to f_1 - f$ pointwise. Furthermore, since $f_n \geq f_{n+1}$, then $f_1 - f_n \leq f_1 - f_{n+1}$, and so $g_n \leq g_{n+1}$. So, $(g_n)_n$ is an increasing sequence of measurable functions that increases to $g_n \nearrow f_1 - f$. Since $g_1 \equiv 0$, then all of the g_n's are non-negative. This means that we can apply the monotone convergence theorem (Corollary 1.9 from Chapter 2) directly to $(g_n)_n$ and find that

$$\lim_{n\to\infty} \int g_n = \int (f_1 - f)$$

Plugging in the form of g_n and using the linearity of the Lebesgue integral, we get that

$$\int f_1 - \lim_{n\to\infty} \int f_n = \int f_1 - \int f$$

Rearranging,

$$\lim_{n\to\infty} \int f_n = \int f$$

So, the result of the monotone convergence theorem does indeed hold for decreasing sequences of non-negative functions. ■

Proof of (f). Suppose that $g : [0,1] \to [0,1]$ is measurable and that $f : [0,1] \to \mathbb{R}$ is continuous. Let $E := \{x \in [0,1] : g(x) = 1\}$ be the preimage of 1 under E; we know E is measurable because g is a measurable function and $E = [0,1] \setminus \{g < 1\}$, both of which are measurable sets. Now, for all $x \in E$, we know that $g(x)^n = 1$ for all n by construction of E, and so $\lim_{n\to\infty} g(x)^n = 1$. Next, for all $x \notin E$ we have

$$g(x) \in [0,1) \implies \lim_{n\to\infty} g(x)^n = 0$$

So, for each $x \in [0,1]$ the limit $\lim_{n\to\infty} g(x)^n$ exists, and in fact $g^n \to \mathbb{1}_E$ pointwise. Since f is continuous and therefore inherits limits, we know that

$$\lim_{n\to\infty} f(g(x)^n) = f\left(\lim_{n\to\infty} g(x)^n\right) = f(\mathbb{1}_E(x)) \qquad \text{for all } x \in [0,1]$$

By properties 2 and 5 of measurable functions, we know that $f \circ g^n$ is measurable for all n. Now, note that the image $f([0,1])$ is the image of a compact set under a continuous map, and is therefore compact and thus bounded. So, f is a bounded function, say by $|f(y)| \leq M$ for all $y \in [0,1]$. Then, for all $x \in [0,1]$ and for all $n \in \mathbb{N}$,

$$g(x) \in [0,1] \implies g(x)^n \in [0,1] \implies |f(g(x)^n)| \leq M$$

So, we have a sequence of functions $(f \circ g^n)_{n\in\mathbb{N}}$ such that $|f \circ g^n| \leq M$ on $[0,1]$ for all n. Since the constant function M is integrable on $[0,1]$ ($\int_{[0,1]} |M| = M < \infty$), M is therefore a dominating function for this sequence. So, we can apply dominated convergence to find that

$$\lim_{n\to\infty} \int_{[0,1]} f(g(x)^n)dx = \int_{[0,1]} \lim_{n\to\infty} f(g(x)^n) = \int_{[0,1]} f(\mathbb{1}_E(x))dx,$$

where the first equality comes from Theorem 1.13 and the second uses our earlier calculation of the limit. Now, for $x \in E$ we know that $f(\mathbb{1}_E(x)) = f(1)$, while for $x \notin E$ we know that $f(\mathbb{1}_E(x)) = f(0)$. This tells us that, since $m([0,1] \setminus E) = 1 - m(E)$,

$$\lim_{n\to\infty} \int_{[0,1]} f(g(x)^n)dx = f(1) \cdot m(E) + f(0) \cdot (1 - m(E)) = f(0) + m(E) \cdot [f(1) - f(0)],$$

where $E := g^{-1}(\{1\}) = \{x \in [0,1] : g(x) = 1\}$. ■

Problem 2

Let $E \subset \mathbb{R}^n$ be measurable. A family $\mathcal{F}$ of measurable functions on E is said to be *uniformly integrable over* E if for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $f \in \mathcal{F}$, whenever $A \subset E$ is measurable with $m(A) < \delta$, we have $\int_A |f| < \epsilon$.

Solution

Proof of (a). Suppose that $\{f_k\}_{k=1}^N$ is a finite collection of measurable functions, each of which is integrable over E. Let $\epsilon > 0$ be arbitrary. By Proposition 1.12(ii) of Chapter 2, for each k there exists a $\delta_k > 0$ such that whenever $A \subset E$ is measurable, we have

$$m(A) < \delta_k \implies \int_A |f_k| < \epsilon$$

Let $\delta = \min_{k\in\{1,\ldots,N\}} \delta_k$; since there are finitely many δ_k's this minimum is attained, and so $\delta > 0$. Furthermore, for all $f_k \in \mathcal{F}$, if $A \subset E$ is measurable with $m(A) < \delta \leq \delta_k$ then $\int_A |f_k| < \epsilon$ by construction of δ_k. Since such a δ exists that works for all $f_k \in \mathcal{F}$ and the above logic applies for all ϵ, we see that $\mathcal{F}$ is uniformly integrable over E. ■

Proof of (b). This statement is not true. As a counterexample, let $\mathcal{F}$ denote the family of functions $\{f_n\}_{n\in\mathbb{N}}$ where each $f_n : [0,1] \to \mathbb{R}$ is given by

$$f_n(x) := n \cdot \mathbb{1}_{[0,\frac{1}{n}]}$$

In other words, $f_n(x) = \begin{cases} n & x \leq \frac{1}{n} \\ 0 & else \end{cases}$. Then, each f_n is certainly measurable (it is a simple function) and we find that

$$\int_{[0,1]} |f_n| = n\int_{[0,\frac{1}{n}]} 1 = n \cdot \frac{1}{n} = 1 \qquad \text{for all } n \in \mathbb{N}$$

So, $\mathcal{F}$ is an example of a family of measurable functions from $[0,1] \to \mathbb{R}$ such that for all $f \in \mathcal{F}$,

$$\int_{[0,1]} |f| \leq 1$$

We wish to show that $\mathcal{F}$ is **not** uniformly integrable over $[0,1]$. To this end, suppose by way of contradiction that $\mathcal{F}$ is uniformly integrable over $[0,1]$. Then, let $\delta > 0$ be the value such that whenever $A \subset E$ is measurable with $m(A) \leq \delta$ we know that for all $f \in \mathcal{F}$,

$$\int_A |f| < \frac{1}{2}$$

Let $N > \frac{1}{\delta}$, and let $A := [0, \frac{1}{N}]$. Then, $m(A) = \frac{1}{N} < \delta$, and so it should be that $\int_A |f| < \frac{1}{2}$ for all $f \in \mathcal{F}$ by our selection of δ. However, note that

$$\int_A |f_N| = \int_{[0,\frac{1}{N}]} N \cdot \mathbb{1}_{[0,\frac{1}{N}]} = \int_{[0,\frac{1}{N}]} N = 1 \not< \frac{1}{2},$$

where $f_N \in \mathcal{F}$. This is an obvious contradiction, and so this $\mathcal{F}$ is not uniformly integrable over $[0,1]$. ■

Proof of (c). Let $E \subset \mathbb{R}^n$ be a measurable set of finite measure. Suppose $\mathcal{F} = \{f_k\}_{k\in\mathbb{N}}$ is uniformly integrable over E and that $f_k \to f$ pointwise a.e. on E for some f. First, we wish to show that f is

Problem 2 continued on next page...

integrable. Note that, since $\{|f_k|\}_{k\in\mathbb{N}}$ is a sequence of non-negative functions and $|f_k| \to |f|$ pointwise a.e. ($|\cdot|$ is continuous and $f_k \to f$), Fatou's lemma gives that

$$\int_E |f| \leq \liminf_{k\to\infty} \int_E |f_k|$$

Let $\epsilon > 0$, and let $\delta > 0$ be such that for all measurable $A \subset E$ with $m(A) < \delta$ we know

$$\int_A |f_k| < \epsilon \qquad \forall f_k \in \mathcal{F}$$

(such a δ is guaranteed by the uniform integrability criterion). Write $E := \bigsqcup_{n=1}^N E_n$ where each E_n is measurable with $m(E_n) < \delta$ (we can do this because E has finite measure, and so it can be covered by a finite disjoint union of sets each of measure $< \delta$). Then, for all k we have

$$\int_E |f_k| = \sum_{n=1}^N \int_{E_n} |f_k| \leq \sum_{n=1}^N \epsilon = N\epsilon$$

by the uniform integrability property and our selection of the E_n's. Since $\int_E |f_k| \leq N\epsilon$ holds for all k, we find that

$$\liminf_{k\to\infty} \int_E |f_k| < \infty \implies \int_E |f| < \infty,$$

and so f is integrable.

Let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ be selected as before. Now, since f is integrable, Proposition 1.12(ii) yields a $\delta' > 0$ such that for all measurable $A \subset E$ with $m(A) < \delta'$ we know $\int_A |f| < \epsilon$. Let $\widetilde{\delta} := \min\{\delta, \delta'\}$. By Egorov's Theorem (since $m(E) < \infty$ and f_k is measurable $\forall k$), there exists a closed set $A_{\widetilde{\delta}} \subset E$ such that $m(E \setminus A_{\widetilde{\delta}}) \leq \widetilde{\delta}$ and $f_k \to f$ uniformly on $A_{\widetilde{\delta}}$. Let K be such that for all $k > K$ we know that

$$|f_k(x) - f(x)| \leq \epsilon \qquad \forall x \in A_{\widetilde{\delta}}$$

(we know such a K exists by uniform convergence). So, we can say that for all $k > K$,

$$\begin{aligned}
\int_E |f_k - f| &= \int_{A_{\widetilde{\delta}}} |f_k - f| + \int_{E\setminus A_{\widetilde{\delta}}} |f_k - f| \\
&\leq \int_{A_{\widetilde{\delta}}} |f_k - f| + \int_{E\setminus A_{\widetilde{\delta}}} |f_k| + \int_{E\setminus A_{\widetilde{\delta}}} |f| \\
&\leq \int_{A_{\widetilde{\delta}}} \epsilon + \int_{E\setminus A_{\widetilde{\delta}}} |f_k| + \int_{E\setminus A_{\widetilde{\delta}}} |f| \\
&\leq \left(\int_{A_{\widetilde{\delta}}} \epsilon\right) + \epsilon + \epsilon \\
&= (m(A_{\widetilde{\delta}}) + 2) \cdot \epsilon \\
&\leq (m(E) + 2) \cdot \epsilon,
\end{aligned}$$

where in the second line we used the triangle equality, in the third line we used the fact that $f_k \to f$ uniformly on $A_{\widetilde{\delta}}$, in the fourth line we used the fact that $m(E \setminus A_{\widetilde{\delta}}) \leq \widetilde{\delta} = \min\{\delta, \delta'\}$ as well as the definitions of δ and δ' to bound the integrals of $|f_k|$ and $|f|$, respectively, over $E \setminus A_{\widetilde{\delta}}$, and in the last line we used that $A_{\widetilde{\delta}} \subset E$ and the monotonicity of measure. So, for each ϵ there exists an K such that for all $k > K$,

$$\int_E |f_k - f| \leq (m(E) + 2) \cdot \epsilon$$

Problem 2 continued on next page...

Since $m(E) < \infty$ and does not depend on ϵ, we can take ϵ to 0 and find that

$$\lim_{k\to\infty} \int_E |f_k - f| = 0 \implies \left|\int_E f_k - \int_E f\right| \to 0 \implies \int_E f_k \to \int_E f$$

as desired (the first implication is because of the triangle inequality $\left|\int_E (f_k - f)\right| \leq \int_E |f_k - f| \to 0$). ■

Proof of (d). Both results from part (c) can fail when $m(E) \not< \infty$. We will produce counterexamples for both results in the setting $E = \mathbb{R} \implies m(E) = \infty$.

Firstly, let $\mathcal{F} = \{f_k\}_{k\in\mathbb{N}}$ be the family of functions $f_k : E \to \mathbb{R}$ given by

$$f_k := \mathbb{1}_{[-k,k]} \qquad \forall k \in \mathbb{N}$$

To see that $\mathcal{F}$ is indeed uniformly integrable over E, let $\epsilon > 0$ be arbitrary and let $\delta := \epsilon$. Then, for any measurable $A \subset E$ with $m(A) < \delta$, for all $k \in \mathbb{N}$ we get

$$\int_A |f_k| = \int_A \mathbb{1}_{[-k,k]} = \int_{A\cap[-k,k]} 1 = m(A \cap [-k,k]) \leq m(A) < \delta = \epsilon,$$

where we used that $A \cap [-k,k] \subset A$ and the monotonicity of measure. Since this holds for all k, we find that $\mathcal{F}$ is indeed uniformly integrable over E. However, note that $f_k \to \mathbb{1}_E$ pointwise a.e. over E (this is because for each $x \in E$ and all $k > |x|$ we have $f_k(x) = \mathbb{1}_E(x) = 1$). The function $\mathbb{1}_E$ is not integrable over E since $\int_E |\mathbb{1}_E| = m(E) = \infty$, and so the first conclusion from part (c) doesn't hold for this example.

Next, let $\mathcal{F} = \{f_k\}_{k\in\mathbb{N}}$ be the family of functions $f_k : E \to \mathbb{R}$ given by

$$f_k := \frac{1}{2k} \cdot \mathbb{1}_{[-k,k]} \qquad \forall k \in \mathbb{N}$$

To see that $\mathcal{F}$ is indeed uniformly integrable over E, let $\epsilon > 0$ be arbitrary and let $\delta := \epsilon$. Then, for any measurable $A \subset E$ with $m(A) < \delta$, for all $k \in \mathbb{N}$ we get

$$\int_A |f_k| = \int_A \frac{1}{2k} \cdot \mathbb{1}_{[-k,k]} = \int_{A\cap[-k,k]} \frac{1}{2k} \leq \int_{A\cap[-k,k]} 1 = m(A \cap [-k,k]) \leq m(A) < \delta = \epsilon,$$

where we used that $\frac{1}{2k} \leq 1$ and $A \cap [-k,k] \subset A$. Since this holds for all k, we find that $\mathcal{F}$ is indeed uniformly integrable over E. However, note that for all $x \in E$,

$$|f_k(x)| \leq \frac{1}{2k} \implies \lim_{k\to\infty} f_k(x) = 0$$

So, $f_k \to f$ pointwise, where $f \equiv 0$ is the zero function. We can compute that for all $k \in \mathbb{N}$,

$$\int_E f_k = \int_E \frac{1}{2k} \cdot \mathbb{1}_{[-k,k]} = \frac{1}{2k} \cdot m([-k,k]) = \frac{1}{2k} \cdot 2k = 1$$

So, $\int_E f_k = 1$ for all k but $\int_E f \equiv 0$, which means that $\int_E f_k \not\to \int_E f$ (a sequence of 1's cannot approach 0). Therefore, the second conclusion from part (c) doesn't hold for this example. ■

Problem 3

Solution

Throughout this solution, functions in $L^2(E)$ are real-valued.

Proof of (a). Let $E \subset \mathbb{R}^n$ be measurable with $m(E) < \infty$. Suppose that $(f_n)_n$ is a bounded sequence in $L^2(E)$ and $f \in L^2(E)$. We will present the answers in the order that they are asked on the exam sheet, but for clarity and to avoid circular logic we make clear here that **the order that these are proved in is (iii), (i), (ii), and then (iv)**. This allows us to make use of results from (iii) in the proof of (i).

(i) Suppose first that (P1) holds, in which case $f_n \to f$ in $L^2(E)$. We will show that (P2) and (P3) hold for certain subsequences of $(f_n)_n$ (though perhaps not for the same subsequence). As $n \to \infty$,

$$||f_n - f||_{L^2(E)}^2 \to 0 \implies \int_E |f_n - f|^2 \to 0$$

Let $\epsilon > 0$. For each k, we can find an element of the sequence n_k such that

$$\int_E |f_{n_k} - f|^2 \leq \frac{\epsilon^2}{k}$$

Recall Chebyshev's Identity, which was proven in Problem 1 on PSET 3 and states that for non-negative, integrable g,

$$m(\{g > \alpha\}) \leq \frac{1}{\alpha} \int g,$$

where $\alpha > 0$. We apply this for each k with $\alpha = \epsilon^2$ and $g := |f_{n_k} - f|^2$, which is certainly non-negative and integrable. Then,

$$m\left(\{|f_{n_k} - f|^2 > \epsilon^2\}\right) \leq \frac{1}{\epsilon^2} \cdot \int_E |f_{n_k} - f|^2 \leq \frac{1}{k} \qquad \forall k \in \mathbb{N}$$

Equivalently,

$$m\left(\{|f_{n_k} - f| > \epsilon\}\right) \leq \frac{1}{k} \qquad \forall k \in \mathbb{N}$$

The above tells us that

$$m(\{x \in E : |f_{n_k}(x) - f(x)| > \epsilon\}) \to 0 \qquad \text{as } k \to \infty$$

Since this holds for all $\epsilon > 0$, we have proven exactly the criterion for convergence in measure to f as $k \to \infty$. So, there exists a subsequence $(f_{n_k})_k \subset (f_n)_n$ such that (P3) holds. Now, applying the result from part (iii), which states that a sequence converging in measure to f has a subsequence that converges pointwise a.e. to f, we find that there is a subsubsequence $(f_{n_{k_j}})_j \subset (f_{n_k})_k \subset (f_n)_n$ such that (P2) holds; i.e. $f_{n_{k_j}} \to f$ pointwise a.e. on E as $j \to \infty$. To sum up, we see that if (P1) holds, then there exists a subsequence of $(f_n)_n$ for which (P2) holds and also a subsequence of $(f_n)_n$ for which (P3) holds.

(ii) Suppose that $f_n \to f$ pointwise a.e. on E. We will first prove that there exists a subsequence for which (P3) holds, and then we will construct a counterexample in this setting such that no subsequence can have property (P1). To prove the first part, we will apply Egorov's Theorem. Let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ also be arbitrary. Then, as $m(E) < \infty$ and all the f_n's are measurable with $f_n \to f$ pointwise a.e. on E, we can find a closed set $A_\delta \subset E$ such that $m(E \setminus A_\delta) < \delta$ and $f_n \to f$ uniformly on A_δ. Now, by uniform convergence we can find a $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|f_n(x) - f(x)| \leq \epsilon \qquad \forall x \in A_\delta$$

Problem 3 continued on next page...

Then, for all $n > N$ we find that

$$\{|f_n - f| > \epsilon\} \subset E \setminus A_\delta \implies m(\{|f_n - f| > \epsilon\} < \delta$$

Since such an N exists for all δ, taking $\delta \to 0$ implies that

$$\lim_{n\to\infty} m(\{|f_n - f| > \epsilon\} = 0$$

Since this statement holds for all $\epsilon > 0$, we find that $f_n \to f$ in measure as well (importantly, the whole sequence converges in measure and not just a subsequence; we will use this later).

Now, we will construct a counterexample for which (P1) doesn't hold for any subsequence. Let $E = [0, 1]$ and consider the sequence of functions $(f_n)_n$ given by

$$f_n(x) := \sqrt{n} \cdot \mathbb{1}_{[0,\frac{1}{n}]}(x)$$

(this is a similar counterexample to the one we used in Problem 2(b)). Firstly, note that each $f_n \in L^2(E)$ and that the sequence is bounded in $L^2(E)$, as

$$||f_n||^2_{L^2(E)} = \int_E |f_n|^2 = n \int_E \mathbb{1}_{[0,\frac{1}{n}]} = n \cdot m\left(\left[0, \frac{1}{n}\right]\right) = 1,$$

where we used that $[0, \frac{1}{n}] \subset E$ for all $n \in \mathbb{N}$. So, the sequence $(f_n)_n$ is indeed a bounded sequence in $L^2(E)$. We claim that $f_n \to 0$ pointwise a.e. on E, yet that $f_n \nrightarrow 0$ in $L^2(E)$. To see the first part, let $x \in E$ be nonzero and arbitrary. Then, for all $n > \frac{1}{x}$ we have that $x > \frac{1}{n}$, which means that $x \notin [0, \frac{1}{n}]$ and therefore that $f_n(x) = 0$. So, we find that

$$\lim_{n\to\infty} f_n(x) = 0$$

Since this holds for all nonzero $x \in E$, we find that $f_n \to 0$ pointwise a.e. on E. However, for every n we have already seen that

$$||f_n - 0||_{L^2(E)} = ||f_n||_{L^2(E)} = 1,$$

which means that no subsequence of $(f_n)_n$ can converge to 0 (since any subsequence will not decay in norm to 0). Therefore, we have constructed a counterexample where (P1) cannot hold for any subsequence of $(f_n)_n$, yet $f_n \to f \equiv 0$ pointwise a.e..

(iii) Suppose that $f_n \to f$ in measure. We will first show that (P2) holds for a certain subsequence of $(f_n)_n$, and then we will construct a counterexample to show that (P1) need not hold for any subsequence. For every $\epsilon > 0$ we have

$$m(\{x \in E : |f_n(x) - f(x)| > \epsilon\}) \to 0 \qquad \text{as } n \to \infty$$

For each k, define the measurable set

$$G_n^{(k)} := \left\{|f_n - f| > \frac{1}{k}\right\}$$

Then, $\lim_{n\to\infty} m(G_n^{(k)}) = 0$ for all k by the criterion for convergence in measure with $\epsilon = \frac{1}{k}$. This means that for each k there exists an element n_k such that

$$m\left(G^k_{n_k}\right) \le \frac{1}{2^k}$$

So, $\left\{G^k_{n_k}\right\}_{k\in\mathbb{N}}$ is a countable family of measurable subsets of E for which

$$\sum_{k=1}^\infty m\left(G^k_{n_k}\right) \le \sum_{k=1}^\infty \frac{1}{2^k} = 1 < \infty$$

Let

$$G := \limsup_{k\to\infty} G^k_{n_k} = \{x \in E : x \in G^k_{n_k} \text{ for infinitely many } k\}$$

The Borel-Cantelli Lemma, which was proven as Problem 5 on PSET 1, states that in precisely this setting we have that G is measurable and that $m(G) = 0$. Now, note that this means that for almost every x, we know that $x \notin G$ and so $x \in G^k_{n_k}$ for only finitely many k. For each $x \notin G$ for which aforementioned property holds, let k_x be the largest k such that $x \in G^k_{n_k}$. Then, for all $k > k_x$ we know that $x \notin G^k_{n_k}$. Equivalently, for $x \notin G$ and all $k > k_x$,

$$|f_{n_k}(x) - f(x)| \leq \frac{1}{k}$$

Taking $k \to \infty$ for all $x \notin G$, we find that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$. Since $x \notin G$ for a.e. $x \in E$, we find that this subsequence $\{f_{n_k}\}_k$ converges pointwise a.e. on E, and so (P2) holds for a subsequence of $(f_n)_n$.

Now, we will construct a counterexample for which (P1) doesn't hold for any subsequence. Let $E = [0, 1]$ and consider the sequence of functions $(f_n)_n$ given by

$$f_n(x) := \sqrt{n} \cdot \mathbb{1}_{[0,\frac{1}{n}]}(x)$$

(this is the same counterexample we just used for (ii)). Firstly, note that each $f_n \in L^2(E)$ and that the sequence is bounded in $L^2(E)$, as

$$||f_n||^2_{L^2(E)} = \int_E |f_n|^2 = n \int_E \mathbb{1}_{[0,\frac{1}{n}]} = n \cdot m\left(\left[0, \frac{1}{n}\right]\right) = 1,$$

where we used that $[0, \frac{1}{n}] \subset E$ for all $n \in \mathbb{N}$. So, the sequence $(f_n)_n$ is indeed a bounded sequence in $L^2(E)$. We claim that $f_n \to 0$ in measure, yet that $f_n \not\to 0$ in $L^2(E)$. To see the first part, note that $f_n(x)$ can only take values in $\{0, n\}$. So, for all ϵ and all $n > \epsilon$ we have that for all $x \in E$,

$$|f_n(x) - 0| > \epsilon \iff f_n(x) = n \iff x \in \left[0, \frac{1}{n}\right]$$

So, for all $\epsilon > 0$ we know that for large enough n,

$$m(\{x \in E : |f_n(x) - 0| > \epsilon\}) = m\left(\left[0, \frac{1}{n}\right]\right) = \frac{1}{n}$$

Then, for all $\epsilon > 0$, we can take the limit as $n \to \infty$ to see that

$$m(\{x \in E : |f_n(x) - 0| > \epsilon\}) \to 0 \qquad \text{as } n \to \infty$$

So, $f_n \to 0$ in measure. However, for every n we have already seen that

$$||f_n - 0||_{L^2(E)} = ||f_n||_{L^2(E)} = 1,$$

which means that no subsequence of $(f_n)_n$ can converge to 0 (since any subsequence will not decay in norm to 0). Therefore, we have constructed a counterexample where (P1) cannot hold for any subsequence of $(f_n)_n$, yet $f_n \to f \equiv 0$ in measure.

(iv) Note that the property $\int_E f_n g \to \int_E fg$ for all $g \in L^2(E)$ is equivalent to the weak convergence property that $\langle f_n, g\rangle_{L^2(E)} \to \langle f, g\rangle_{L^2(E)}$ for all $g \in L^2(E)$. We first show that there exists a sequence $(f_n)_n$ of functions for which the given property holds, but there is no subsequence of $(f_n)_n$ for which (P1) holds. To construct this, let $\{\varphi_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for $L^2(E)$, which we know to be a separable, infinite-dimensional Hilbert space. Consider the sequence $(\varphi_k)_k$; this sequence clearly has unit norm for all

k. Furthermore, we know that this has no convergent subsequence since no subsequence can be Cauchy; indeed, for any $m \neq n$ we have that

$$||\varphi_n - \varphi_m||^2_{L^2(E)} = ||\varphi_n||^2_{L^2(E)} + ||\varphi_m||^2_{L^2(E)} - 2Re\,\langle \varphi_n, \varphi_m \rangle_{L^2(E)} = 2,$$

and so no subsequence can meet the Cauchy criterion. However, we do have that $\langle \varphi_{k_n}, g \rangle_{L^2(E)} \to \langle f, g \rangle_{L^2(E)}$ for some $f \in L^2(E)$ and some subsequence $(\varphi_{k_n})_n$ by the result of Problem 9 from PSET 7, which stated that all sequences of unit vectors in an infinite-dimensional separable Hilbert space have a weakly convergent subsequence. Define the sequence $(f_n)_n$ by $f_n := \varphi_{k_n}$ for each $n \in \mathbb{N}$. We then have that $\langle f_n, g \rangle_{L^2(E)} \to \langle f, g \rangle_{L^2(E)}$ for all $g \in L^2(E)$, yet that no subsequence of $(f_n)_n$ can converge in $L^2(E)$ to anything, let alone to f. So, in this setting there is no subsequence for which (P1) holds, yet $(f_n)_n$ has the weak convergence property.

Suppose now that $E = [0, 2\pi]$. We will now show that there exists a sequence $(f_n)_n$ of functions for which the given weak convergence property holds, but there is no subsequence of $(f_n)_n$ for which (P2) holds. To see this, let $f_n := \cos(-nx)$ for all $n \in \mathbb{N}$; then, $f_n \in L^2(E)$ clearly since $|\cos|^2 \leq 1$ and 1 is integrable over $[0, 2\pi]$. Let $g \in L^2(E)$ be arbitrary. Note that the identity function $1 \in L^2(E)$ as $\int_E |1|^2 = m(E) < \infty$. So, we can use Cauchy-Schwartz to note that since $||g||_{L^2(E)} < \infty$,

$$\int_E |g| = \langle |g|, 1 \rangle_{L^2(E)} \leq ||\,|g|\,||_{L^2(E)} \cdot ||1||_{L^2(E)} = m(E) \cdot ||g||_{L^2(E)} < \infty$$

So, $g \in L^1(E)$. Then, we have that

$$\int_E f_n g = \int_E g(x)\cos(-nx)dx = \int_E g(x) Re(e^{-inx})dx = Re\left(\int_E g(x)e^{-inx}dx\right),$$

where we used that the real part of an integral is the integral of the real part (this property is inherited from finite summations via the definition of an integral in terms of simple functions). Now, by the first result from Problem 6 of PSET 5, which states that if $g \in L^1(E)$ then $\int_E g(x)e^{-inx}dx \to 0$ as $|n| \to \infty$, we know that $\int_E f_n g \to 0$ as $n \to \infty$. Since this holds for all $g \in L^2(E)$, we find that $\int_E f_n g = \int_E 0g$ for all $g \in L^2(E)$, which means that $(f_n)_n$ converges weakly to the function 0. However, there is no subsequence of $(f_n)_n$ that converges pointwise a.e. to 0. To see this, suppose by way of contradiction there was some subsequence $(f_{n_k})_k$ that converged pointwise a.e. to 0. Then, for almost every $x \in E$, we would have that $\cos(-nx) \to 0$ as $n \to \infty$; by the Cauchy criterion, this would imply that for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ large enough that $|\cos(-nx) - \cos(-mx)| < \epsilon$ for all $n, m > N$. Then, since this property holds for a.e. $x \in E$, we should have that for all $n, m > N$,

$$\int_E (\cos(-nx) - \cos(-mx))^2 dx \leq \int_E \epsilon^2 = 2\pi\epsilon^2$$

However, for any $n \neq m$, we can compute that

$$\int_E (\cos(-nx) - \cos(-mx))^2 dx = \int_E \cos^2(nx)dx + \int_E \cos^2(mx)dx - 2\int_E \cos(nx)\cos(mx)dx$$

The first two integrals on the right hand side both evaluate to π for $n, m \in \mathbb{N}$, and the third integral evaluates to 0 for $n \neq m$. So, since $2\pi \nleq 2\pi\epsilon^2$ for some ϵ, we arrive at a contradiction. Thus, no subsequence of $(f_n)_n$ can converge pointwise a.e. to 0. So, even though $(f_n)_n$ has the given weak convergence property, no subsequence can satisfy (P2).

We will reuse the previous counterexample to also show that (P3) cannot hold for any subsequence of $(f_n)_n$ for $(f_n)_n$ defined as in the previous paragraph (i.e. $f_n(x) = \cos(-nx)$). Suppose by way of contradiction that some subsequence $(f_{n_k})_k$ converges in measure to 0. Then, using the result from part (iii), we

Problem 3 continued on next page...

see that convergence in measure of $(f_{n_k})_k$ implies that there is some subsequence $(f_{n_{k_j}})_j$ that converges pointwise a.e. to 0. However, from the previous paragraph we know that no subsequence of $(f_n)_n$ converges pointwise to 0. This is a contradiction, and so we find that no subsequence of $(f_n)_n$ can satisfy (P3). ■

Proof of (b). Suppose now that $m(E) = \infty$. We can be sure that only the affirmative answers from (a)(i)-(iv) will change, as none of the counterexamples that were made when $m(E) < \infty$ relied on finiteness of E, and so their proofs carry over directly to the case when $m(E) = \infty$. If you are unconvinced of this, note that we used the same counterexample for (ii) and (iii)'s (P1), which is certainly unchanged if $m(E) = \infty$ as all we require is that $[0, \frac{1}{n}] \subset E$ for all $n \in \mathbb{N}$. Next, the (P1) counterexample from (iv) uses results that hold for all infinite-dimensional Hilbert spaces, and so finiteness of E is irrelevant. For the (P2) and (P3) counterexamples from (iv) (which are the same counterexample), it actually does change when $m(E) = \infty$, as we no longer can apply Problem 6 of PSET 5 since we no longer know that $g \in L^2(E) \implies g \in L^1(E)$. However, we can simply multiply f_n by $\mathbb{1}_{[0,2\pi]}$ for each n, which ensures that our computation of $\int f_n g = 0$ can still be done for all $g \in L^2(E)$ (just with $g \cdot \mathbb{1}_{[0,2\pi]} \in L^1([0, 2\pi]$ being used in the application of Problem 6 of PSET 5 instead). The rest of the reasoning for those counterexamples (namely, that no subsequence of $(f_n)_n$ converges to 0) still holds when $m(E) = \infty$ if we multiply f_n by $\mathbb{1}_{[0,2\pi]}$ as discussed earlier. So, since all our counterexamples are either unchanged or fixable when $m(E) = \infty$, we only need to investigate how the affirmative answers (which are (P2) and (P3) in (a)(i), (P3) in (a)(ii), and (P2) in (a)(iii)) change.

The logic in (a)(iii)'s (P2) proof that convergence in measure implies a subsequence converges pointwise a.e. makes no use of finiteness of E, and so that result still stands. Furthermore, nowhere in the proof of (a)(i)'s two affirmative results do we use that $m(E) < \infty$, and so those results hold too (we do make use of (a)(iii)'s result about (P2), but we just saw that that holds as well); so, (P2) and (P3) in (a)(i) hold.

We claim that the (P3) result from (a)(ii) does not hold. To see this, let $E = \mathbb{R}$ and let $f_n := \mathbb{1}_{[n,n+1)}$ for all $n \in \mathbb{N}$. Then, for each $x \in E$ we find that $f_n(x) = 0$ for all $n > x$, and so $f_n \to 0$ pointwise everywhere in E. However, no subsequence of $(f_n)_n$ can converge in measure to 0, since for all $\epsilon < 1$ and every $n \in \mathbb{N}$ we have that

$$m(\{x \in E : |f_n(x) - 0| > \epsilon\}) = m(\{x \in E : x \in [n, n + 1)\}) = m([n, n + 1)) = 1,$$

which means that this quantity cannot decay to 0 along any subsequence. So, the (P3) result from (a)(ii) does not hold. ■

Proof of (c). Suppose that $E \subset \mathbb{R}^n$ is measurable and $m(E) < \infty$. For any two measurable functions g, h on E, define

$$\rho(g, h) := \int_E \frac{|g - h|}{1 + |g - h|}$$

Let $(f_k)_k$ be a sequence of measurable functions that is bounded in $L^2(E)$, and let f be a measurable function. We want to show that $f_k \to f$ in measure on E if and only if $\rho(f_k, f) \to 0$.

($\implies$) Suppose first that $f_k \to f$ in measure on E. Fix $\epsilon > 0$ to be arbitrary. For each k, define

$$E_k := \{x \in E : |f_k(x) - f(x)| > \epsilon\}$$

Then, we know that $m(E_k) \to 0$ as $k \to \infty$ by the criterion for convergence in measure. So, there exists a $K \in \mathbb{N}$ such that for all $k > K$ we have

$$m(E_k) < \epsilon$$

For all $k > K$ we can write

$$\begin{aligned}
\rho(f_k, f) &= \int_E \frac{|f_k - f|}{1 + |f_k - f|} \\
&= \int_{E_k} \frac{|f_k - f|}{1 + |f_k - f|} + \int_{E \setminus E_k} \frac{|f_k - f|}{1 + |f_k - f|} \\
&\leq \int_{E_k} 1 + \int_{E \setminus E_k} \frac{|f_k - f|}{1 + |f_k - f|} \\
&\leq \epsilon + \int_{E \setminus E_k} \frac{|f_k - f|}{1 + |f_k - f|} \\
&\leq \epsilon + \int_{E \setminus E_k} \frac{\epsilon}{1 + |f_k - f|} \\
&\leq \epsilon + \int_{E \setminus E_k} \epsilon \\
&= (m(E \setminus E_k) + 1) \cdot \epsilon \leq (m(E) + 1) \cdot \epsilon,
\end{aligned}$$

where for the second line we split the integral, in the third line we used that $\frac{|f_k - f|}{1+|f_k - f|} \leq 1$ always (denominator is larger than numerator), in the fourth line we used that $m(E_k) < \epsilon$ for all such k, in the fifth line we used that $|f_k - f| \leq \epsilon$ over $E \setminus E_k$ by construction of E_k, in the sixth line we used that $1 + |f_k - f| \geq 1$, and in the last line we used that $E \setminus E_k \subset E$ and the monotonicity of the Lebesgue measure. So, we find that for every $\epsilon > 0$ there exists a K such that for all $k > K$ we have

$$\rho(f_k, f) \leq (m(E) + 1) \cdot \epsilon$$

Since $m(E)$ is finite and doesn't depend on ϵ, we can take $\epsilon \to 0$ to see that

$$\rho(f_k, f) \to k \qquad \text{as } k \to \infty$$

as desired.

$(\Longleftarrow)$ Suppose now that $\rho(f_k, f) \to 0$ as $k \to \infty$. Note that for all k and all x, we have that

$$\left| \frac{|f_k(x) - f(x)|}{1 + |f_k(x) - f(x)|} \right| \leq 1$$

So, the constant function 1 is a dominating function for the integrand. Since $m(E) < \infty$, we see that

$$||1||^2_{L^2(E)} = \int_E |1|^2 = \int_E 1 = m(E) < \infty,$$

and so $\left| \frac{|f_k(x)-f(x)|}{1+|f_k(x)-f(x)|} \right|$ is dominated by an integrable function. So, by the dominated convergence theorem,

$$0 = \lim_{k\to\infty} \rho(f_k, f) = \lim_{k\to\infty} \int_E \frac{|f_k - f|}{1 + |f_k - f|} = \int_E \lim_{k\to\infty} \left(\frac{|f_k(x) - f(x)|}{1 + |f_k(x) - f(x)|} \right) dx$$

We can be sure that for all $x \in E$ we have that $\lim_{k\to\infty} \left(\frac{|f_k(x)-f(x)|}{1+|f_k(x)-f(x)|} \right) \geq 0$ since it is the limit of non-negative terms. So, we can apply Proposition 1.6(vi) of Chapter 2 to see that

$$\lim_{k\to\infty} \left(\frac{|f_k(x) - f(x)|}{1 + |f_k(x) - f(x)|} \right) = 0 \qquad \text{for a.e. } x \in E$$

For such x, we know that $\lim_{k\to\infty} |f_k(x) - f(x)| \in [0, \infty]$, and we want to show that it equals 0. First, suppose by way of contradiction that the limit equals $+\infty$. Then,

$$\lim_{k\to\infty} \left(\frac{|f_k(x) - f(x)|}{1 + |f_k(x) - f(x)|} \right) = \lim_{k\to\infty} \left(\frac{1}{1 + \frac{1}{|f_k(x)-f(x)|}} \right) = \frac{1}{1 + \lim_{k\to\infty} \frac{1}{|f_k(x)-f(x)|}} = \frac{1}{1+0} = 1 \neq 0,$$

which is a contradiction. So, $\lim_{k\to\infty} |f_k(x) - f(x)| < \infty$. If $\lim_{k\to\infty} |f_k(x) - f(x)| = M$ for some $M \neq 0$, then we would have that

$$\lim_{k\to\infty} \left(\frac{|f_k(x) - f(x)|}{1 + |f_k(x) - f(x)|} \right) = \frac{M}{1+M} \neq 0,$$

which is again a contradiction. So, this means that

$$\lim_{k\to\infty} |f_k(x) - f(x)| = 0 \qquad \text{for a.e. } x \in E$$

as well. This reveals that $f_k \to f$ pointwise a.e. on E. Now, note that in the proof of (a)(ii) above, we actually showed that if $f_n \to f$ pointwise a.e. then $f_n \to f$ in measure (importantly, the whole sequence converges in measure, and not just a subsequence). We can apply this here to see that $f_k \to f$ in measure on E, and so we are done. ■

Problem 4

Solution

Proof of (a). Let the total variation of a function $g : [a, b] \to \mathbb{R}$ be denoted by

$$T_g := \sup \sum_{j=1}^{N} |g(t_j) - g(t_{j-1})|,$$

where the supremum is over all partitions of $[a, b]$. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ from $[a, b] \to \mathbb{R}$ is a sequence with $f_n \to f$ pointwise everywhere on $[a, b]$. Now, let $a = t_0 < t_1 < ... < t_N = b$ be an arbitrary partition of $[a, b]$. By definition of total variation as a supremum, we know that for all $n \in \mathbb{N}$

$$\sum_{j=1}^{N} |f_n(t_j) - f_n(t_{j-1})| \leq T_{f_n}$$

Taking the lim inf as $n \to \infty$ of both sides,

$$\begin{aligned}
\liminf_{n\to\infty} T_{f_n} &\geq \liminf_{n\to\infty} \sum_{j=1}^{N} |f_n(t_j) - f_n(t_{j-1})| \\
&= \sum_{j=1}^{N} \liminf_{n\to\infty} (|f_n(t_j) - f_n(t_{j-1})|) \\
&= \sum_{j=1}^{N} |f(t_j) - f(t_{j-1})|,
\end{aligned}$$

where we are allowed to switch the lim inf with the summation in the second line because the summation is finite, and the third line comes from the fact that $f_n \to f$ everywhere (and that $|\cdot|$ is continuous). So, we know that for any arbitrary partition of $[a, b]$,

$$\sum_{j=1}^{N} |f(t_j) - f(t_{j-1})| \leq \liminf_{n\to\infty} T_{f_n}$$

This bound will hold after taking a supremum over all partitions of $[a, b]$, and so

$$T_f = \sup \sum_{j=1}^{N} |f(t_j) - f(t_{j-1})| \leq \liminf_{n\to\infty} T_{f_n}$$

as desired. ■

Proof of (b). Suppose that $F : [a, b] \to \mathbb{R}$ is increasing. Clearly, F is bounded on $[a, b]$ because F increasing $\implies F(a) \leq F(x) \leq F(b) \quad \forall x \in [a, b]$. By Lemma 3.12 of Chapter 3, we know that F has at most countably many discontinuities; let us label them $\{x_n\}_{n\in\mathbb{N}}$. At each discontinuity x_n, we know that there is a jump

$$\alpha_n := F(x_n^+) - F(x_n^-)$$

such that

$$F(x_n^+) = F(x_n^-) + \alpha_n \qquad \text{and} \qquad F(x_n) = F(x_n^-) + \theta_n \alpha_n$$

for some collection $\{\theta_n\}_n$. Let us construct the jump function given by

$$F_J(x) := \sum_{n=1}^{\infty} \alpha_n j_n(x), \qquad \text{where } j_n(x) := \begin{cases} 0 & x < x_n \\ \theta_n & x = x_n \\ 1 & x > x_n \end{cases}$$

Clearly, F_J is increasing as more $j_n(x)$'s will be nonzero as we increase x and $\alpha_n \geq 0$ for all n. Also, by our construction of the jump function, we know that F_J' exists a.e. and is 0 where it exists (this comes from Theorem 3.14). We know from Lemma 3.13 and the remarks above that result that because F is increasing and bounded on $[a, b]$, then the difference $F - F_J$ is increasing and continuous on $[a, b]$. We will denote this difference by C; i.e. $C : [a, b] \to \mathbb{R}$ is a continuous, increasing function such that

$$C(x) := F(x) - F_J(x) \qquad \forall x \in [a, b]$$

Now, we know from Example 1 in Section 3.1 of Chapter 3 that because F is increasing, bounded, and real-valued on $[a, b]$, then it is of bounded variation. So, by Theorem 3.4, F' exists a.e. on $[a, b]$. By Corollary 3.7, C' therefore exists a.e., is non-negative and measurable, and satisfies

$$\int_{[a,b]} C' \leq C(b) - C(a) \implies \int_{[a,b]} |C'| < \infty,$$

which tells us that C' is integrable. Note that, by linearity of the derivative, we know that for almost every $x \in [a, b]$,

$$F'(x) = C'(x) + F_J'(x) = C'(x),$$

and so F' itself is non-negative almost everywhere, measurable, and integrable (here, we used the fact that $F_j' = 0$ a.e.). As such, let us define a function $F_A : [a, b] \to \mathbb{R}$ given by

$$F_A(x) := \int_{[a,x]} F'$$

Since F' is non-negative a.e., we see that F_A must be increasing (when $y > x$, $F_A(y) - F_A(x) = \int_{(x,y]} F' \geq 0$). Furthermore, by construction we have that F_A is absolutely continuous (see the last remark on page 128 of Stein). Lastly, we know by the statements in Theorem 3.11 that $F_A' = F'$ a.e..

To conclude, let us define a function $F_C : [a, b] \to \mathbb{R}$ by

$$F_C := F - F_J - F_A = C - F_A$$

Since C was continuous and F_A is absolutely continuous (and therefore continuous), F_C is certainly continuous as well. Furthermore, by linearity of the derivative (and the fact that F', F_J', and F_A' exist a.e.), for a.e. $x \in [a, b]$ we have

$$F_C'(x) = F'(x) - F_J'(x) - F_A'(x) = F'(x) - 0 - F'(x) = 0,$$

where we used that $F_A' = F'$ a.e. and that $F_J' = 0$ a.e.. So, $F_C' = 0$ a.e.. The last thing we wish to show is that F_C is increasing. To that end, for every $x, y \in [a, b]$ with $x < y$ we have

$$\begin{aligned}
F_C(y) - F_C(x) &= C(y) - C(x) - F_A(y) + F_A(x) \\
&\geq F_A(x) - F_A(y) + \int_{[x,y]} C' \\
&= F_A(x) - F_A(y) + \int_{[x,y]} F' \\
&= \int_{[a,x]} F' - \int_{[a,y]} F' + \int_{[x,y]} F' \\
&= \int_{[a,y]} F' - \int_{[a,y]} F' = 0,
\end{aligned}$$

where for the second line we used Corollary 3.7 (because $C = F_A + F_C$ is increasing and continuous), for the third line we used that $C' = F'$ a.e., for the fourth line we used our defintion of F_A, and for the last line we used properties of integrals. So, $F_C(y) \geq F_C(x)$ whenever $y > x$, which means that F_C is increasing. To sum up, we have written

$$F = C + F_J = F_A + F_C + F_J,$$

where F_A, F_C, F_J are all increasing, F_A is absolutely continuous, F_C is continuous with $F_C' = 0$ a.e., and F_J is a jump function.

To see the uniqueness (up to additive constants) of this form for F, suppose that there exists two decompositions

$$F = F_A + F_C + F_J = G_A + G_C + G_J,$$

where the component functions have the desired properties. Firstly, note that

$$G_J - F_J = F_A + F_C - G_A - G_C$$

The left hand side is a jump function and is discontinuous at precisely its jump points, while the right hand side is a combination of continuous functions and therefore has no discontinuities. So, $G_J - F_J$ must be a jump function with no jumps, and therefore must be constant.

Next, observe that

$$G_A - F_A = F_C + F_J - G_C - G_J$$

The left hand side is an absolutely continuous function (the difference of two absolutely continuous functions is absolutely continuous by application of the triangle inequality on the $|f(t_j) - f(t_{j-1})|$ part of the criterion for absolute continuity). All of the involved functions are increasing on a closed interval and therefore bounded, which means they are of bounded variation and so differentiable a.e.. Then, we can take a derivative to see that

$$G_A' - F_A' = F_C' + F_J' - G_C' - G_J' = 0 \qquad a.e.,$$

where we used that $F_C', F_J', G_C', G_J' = 0$ a.e.. By Theorem 3.8, since $G_A - F_A$ is absolutely continuous and $(G_A - F_A)' = 0$ a.e., then F_A and G_A differ by a constant as well.

So, we know that F_A and G_A differ by a constant, as do F_J and G_J. Then, as

$$G_C - F_C = F_A - G_A + F_J - G_J,$$

we find that $G_C - F_C$ must be constant as well. So, the constructed F_A, F_C, F_J are unique up to additive constants. ■

Proof of (c). Suppose that F, G are absolutely continouous on $[a, b]$. Then, they are both continuous; since $[a, b]$ is compact, it must be that both F and G are bounded over $[a, b]$, say by $|F(x)| \leq M_F$ and $|G(x)| \leq M_G$ for all $x \in [a, b]$. Then, for any set of disjoint intervals $\{(a_k, b_k)\}_{k=1}^N$,

$$\sum_{k=1}^N |(FG)(b_k) - (FG)(a_k)| = \sum_{k=1}^N |F(b_k) \cdot G(b_k) - F(a_k) \cdot G(a_k)|$$

$$= \sum_{k=1}^N |F(b_k) \cdot G(b_k) - F(a_k) \cdot G(b_k) + F(a_k) \cdot G(b_k) - F(a_k) \cdot G(a_k)|$$

$$\leq \sum_{k=1}^N |F(b_k) \cdot G(b_k) - F(a_k) \cdot G(b_k)| + \sum_{k=1}^N |F(a_k) \cdot G(b_k) - F(a_k) \cdot G(a_k)|$$

$$= \sum_{k=1}^N |G(b_k)| \cdot |F(b_k) - F(a_k)| + \sum_{k=1}^N |F(a_k)| \cdot |G(b_k) - G(a_k)|$$

$$\leq M_G \sum_{k=1}^{N} |F(b_k) - F(a_k)| + M_F \sum_{k=1}^{N} |G(b_k) - G(a_k)|,$$

where in the second line we added and subtracted the same value $(F(a_k) \cdot G(b_k))$, in the third line we used the triangle inequality, and in the last line we applied the boundedness of F and G.

Now, let $\epsilon > 0$. By absolute continuity of F, there exists a $\delta_F > 0$ such that whenever $\sum_{k=1}^{N}(b_k - a_k) < \delta_F$, then $\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \frac{\epsilon}{2M_G}$. Similarly, there exists a $\delta_G > 0$ such that whenever $\sum_{k=1}^{N}(b_k - a_k) < \delta_G$, then $\sum_{k=1}^{N} |G(b_k) - G(a_k)| < \frac{\epsilon}{2M_F}$. Let $\delta := \min\{\delta_F, \delta_G\}$. Then, whenever $\sum_{k=1}^{N}(b_k - a_k) < \delta$, by our earlier derivation we find that

$$\sum_{k=1}^{N} |(FG)(b_k) - (FG)(a_k)| \leq M_G \cdot \frac{\epsilon}{2M_G} + M_F \cdot \frac{\epsilon}{2M_F} = \epsilon$$

Since such a result holds for all $\epsilon > 0$, FG is indeed absolutely continuous.

Now, by Theorem 3.8, $(FG)'$ exists a.e.. By the product rule,

$$(FG)' = FG' + F'G$$

Finally, by Theorem 3.11,

$$F(b)G(b) - F(a)G(a) = \int_{[a,b]} (FG)' = \int_a^b F(x)G'(x)dx + \int_a^b F'(x)G(x)dx,$$

from which the desired result follows. ■

Problem 5

Solution

Throughout this solution, $\mathcal{H}$ will denote an infinite-dimensional, separable, complex Hilbert space.

Proof of (a). Suppose that $\{\varphi_\alpha\}_{\alpha\in A}$ is an orthonormal subset of $\mathcal{H}$. We can then compute that for any $\alpha \neq \beta$, we have

$$||\varphi_\alpha - \varphi_\beta||_{\mathcal{H}}^2 = \langle \varphi_\alpha - \varphi_\beta, \varphi_\alpha - \varphi_\beta \rangle_{\mathcal{H}} = ||\varphi_\alpha||^2 + ||\varphi_\beta||^2 - 2Re\, \langle \varphi_\alpha, \varphi_\beta \rangle_{\mathcal{H}} = 2$$

Since $\mathcal{H}$ is separable, there exists a countable dense subset, say $E \subset \mathcal{H}$. By density of E, we know that for every $\alpha \in A$ there exists an $x_\alpha \in E$ such that

$$||\varphi_\alpha - x_\alpha||_{\mathcal{H}} < \frac{1}{2}$$

It can be shown that the mapping $\alpha \mapsto x_\alpha$ must be injective. To see this, suppose by way of contradiction that it is not injective; that is, suppose by way of contradiction that $x_\alpha = x_\beta$ for two $\alpha \neq \beta$. Then, by the triangle inequality we have

$$||\varphi_\alpha - \varphi_\beta||_{\mathcal{H}} = ||\varphi_\alpha - x_\alpha + x_\alpha - x_\beta + x_\beta - \varphi_\beta||_{\mathcal{H}} \leq ||\varphi_\alpha - x_\alpha||_{\mathcal{H}} + ||x_\alpha - x_\beta||_{\mathcal{H}} + ||\varphi_\beta - x_\beta||_{\mathcal{H}}$$

By the selection of $\{x_\alpha\}_{\alpha\in A} \subset E$ and the fact that $x_\alpha = x_\beta$, we find that

$$||\varphi_\alpha - \varphi_\beta||_{\mathcal{H}} \leq \frac{1}{2} + 0 + \frac{1}{2} = 1,$$

which is a contradiction since $||\varphi_\alpha - \varphi_\beta||_{\mathcal{H}} = \sqrt{2} > 1$ when $\alpha \neq \beta$. So, the mapping from $A \to \{x_\alpha\}_{\alpha\in A}$ that sends $\alpha \mapsto x_\alpha$ must be injective. Suppose by way of contradiction that A is uncountable. Then, since this mapping is injective and it has an uncountable domain, the range $\{x_\alpha\}_{\alpha\in A}$ must also be uncountable. However, $\{x_\alpha\}_{\alpha\in A} \subset E$ and E is countable. This is a contradiction, and so we find that A must be at most countable. ■

Proof of (b). Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. For bounded operators $T : \mathcal{H} \to \mathcal{H}$ define

$$N(T) := \left(\sum_{n=1}^{\infty} ||T(e_n)||_{\mathcal{H}}^2 \right)^{\frac{1}{2}}$$

Suppose first that for some bounded operator T, we have $N(T) < \infty$. Let $\epsilon > 0$. For each $k \in \mathbb{N}$, define by P_k the projection operator onto the span of the first k basis vectors (in other words, P_k is the projection operator onto the closed subspace $span\{e_n\}_{n=1}^k$). Then, for each $k \in \mathbb{N}$ we know that $P_k T$ is a finite-rank bounded operator (its range is finite-dimensional), and is therefore compact (this is stated on page 188 of Stein). Now, since $N(T)^2 < \infty$, the sum $\sum_{n=1}^{\infty} ||T(e_n)||_{\mathcal{H}}^2$ must converge, which means that its tail must get arbitrarily small. So, there exists some $N \in \mathbb{N}$ such that for all $k > N$,

$$\sum_{n=k+1}^{\infty} ||T(e_n)||_{\mathcal{H}}^2 < \epsilon$$

We can compute that for all $k > N$,

$$||P_k T - T||_{op} = \sup\{||(P_k T - T)v||_{\mathcal{H}} : ||v||_{\mathcal{H}} = 1\}$$

(This form of $||S||_{op} = \sup\{||Sv||_{\mathcal{H}} : ||v||_{\mathcal{H}} = 1\}$ is precisely the result of Problem 8(a) on PSET 7). Now, for any unit vector $v \in \mathcal{H}$ with $||v||_{\mathcal{H}} = 1$, write $v = \sum_{n=1}^{\infty} a_n e_n$. Then, Parseval's identity reads

$$1 = ||v||_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |a_n|^2,$$

from which we derive that $|a_n| \leq 1$ for all n. Now,

$$\begin{aligned}(P_kT - T)v &= P_k\left(\sum_{n=1}^{\infty} a_n T(e_n)\right) - \sum_{n=1}^{\infty} a_n T(e_n) \\ &= \sum_{n=1}^{k} a_n T(e_n) - \sum_{n=1}^{\infty} a_n T(e_n) \\ &= -\sum_{n=k+1}^{\infty} a_n T(e_n)\end{aligned}$$

So,

$$||(P_kT - T)v||_{\mathcal{H}} = \left\|\sum_{n=k+1}^{\infty} a_n T(e_n)\right\|_{\mathcal{H}} \leq \sum_{n=k+1}^{\infty} |a_n| \cdot ||T(e_n)||_{\mathcal{H}} \leq \sum_{n=k+1}^{\infty} ||T(e_n)||_{\mathcal{H}},$$

where for the second to last inequality we used the triangle inequality and for the last inequality we used that $|a_n| \leq 1$ for all n (note that to apply the triangle inequality we would really want to apply it finitely and take the limit; continuity of $||\cdot||_{\mathcal{H}}$ ensures that things go well though). So, for all $k > N$, because the tail sum is arbitrarily small, we find that for all unit vectors v,

$$||(P_kT - T)v||_{\mathcal{H}} < \epsilon$$

So, taking the supremum over all unit vectors, we find that for $k > N$, $||P_kT - T||_{op} \leq \epsilon$ Since such an N exists for all $\epsilon > 0$, we find that $||P_kT - T||_{op} \to 0$ as $k \to \infty$. So, $(P_kT)_k$ is a sequence of compact operators that converges in the operator norm to T. By Proposition 6.1(ii) of Chapter 4, we find that T is compact as desired.

The converse, however, is not always true. Let $T : \mathcal{H} \to \mathcal{H}$ be the linear operator defined on the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ by

$$T(e_n) = \frac{1}{\sqrt{n}} e_n$$

T is definitely bounded, as for all vectors $f = \sum_{n=1}^{\infty} a_n e_n \in \mathcal{H}$ we have

$$||Tf||_{\mathcal{H}}^2 = \left\|\sum_{n=1}^{\infty} a_n T(e_n)\right\|_{\mathcal{H}}^2 = \left\|\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a_n e_n\right\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n} \leq \sum_{n=1}^{\infty} |a_n|^2 = ||f||_{\mathcal{H}}^2,$$

which means that $||T||_{op} \leq 1$ (we applied Parseval's identity for the third and fourth equalities). Now, the result of Problem 1 from PSET 8 tells us that for a bounded operator S on a separable Hilbert space that is diagonal with respect to an orthonormal basis (i.e. $S\varphi_k = \lambda_k \varphi_k$ for an orthonormal basis $\{\varphi_k\}_k$), S is compact if and only if $|\lambda_k| \to 0$. Our constructed T satisfies these conditions (it is bounded and each e_n is an eigenvector of T with eigenvalue $\frac{1}{\sqrt{n}}$), and so we find that since $\left|\frac{1}{\sqrt{n}}\right| \to 0$ as $n \to \infty$, then T is compact. However, we can compute that

$$N(T)^2 = \sum_{n=1}^{\infty} ||T(e_n)||_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \left\|\frac{1}{\sqrt{n}} e_n\right\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

In particular, we have constructed a compact operator $T : \mathcal{H} \to \mathcal{H}$ such that $N(T) \not< \infty$, and so the converse does not always hold. ■

Proof of (c). Suppose that $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator.

(i) Let $S := \{f \in \mathcal{H} : U(f) = f\}$ denote the set of U-invariant vectors of $\mathcal{H}$. Certainly S is a subspace, since for all $f, g \in S$ and all $\alpha, \beta \in \mathbb{C}$ we have

$$U(f) = f \quad \text{and} \quad U(g) = g \implies U(\alpha f + \beta g) = \alpha U(f) + \beta U(g) = \alpha f + \beta g \implies \alpha f + \beta g \in S$$

To see that S is closed, let $\{f_k\}_{k\in\mathbb{N}} \subset S$ be a sequence of vectors in S, and suppose that $f_k \to f$ for some $f \in \mathcal{H}$. We want to show that $f \in S$. Note that since U is unitary, then it is bounded (it has operator norm 1), and so U is therefore continuous. So, U inherits limits in the sense that

$$\lim_{k\to\infty} U(f_k) = U\left(\lim_{k\to\infty} f_k\right)$$

Because $f_k \in S \implies U(f_k) = f_k$ for all k, we get

$$\lim_{k\to\infty} f_k = U\left(\lim_{k\to\infty} f_k\right) \implies f = U(f),$$

where we used that $f_k \to f$. So, $f \in S$ and therefore S is a closed subspace of $\mathcal{H}$.

Suppose now that $f \in S$ and $g \in \mathcal{H}$ are arbitrary. We know that $f \in S \implies f = U(f)$. Then,

$$\begin{aligned}\langle U(g) - g, f\rangle_{\mathcal{H}} &= \langle U(g), f\rangle_{\mathcal{H}} - \langle g, f\rangle_{\mathcal{H}} \\ &= \langle U(g), U(f)\rangle_{\mathcal{H}} - \langle g, f\rangle_{h} \\ &= \langle g, f\rangle_{\mathcal{H}} - \langle g, f\rangle_{\mathcal{H}} = 0,\end{aligned}$$

where in the second line we used that $f \in S \implies f = U(f)$, and in the third line we used that U preserves inner products (this can be seen by noting that U is an isometry and applying the result of Problem 10(a) from PSET 7, which states that if T is an isometry then $\langle Tf, Tg\rangle = \langle f, g\rangle \ \ \forall f, g$). So, we achieve the desired result.

(ii) Let P denote the projection operator onto the closed subspace S from the previous part. We want to show that for all $f \in \mathcal{H}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k(f) = P(f)$$

To this end, let $f \in \mathcal{H}$ be arbitrary. Because S is a closed subspace, we can perform the orthogonal decomposition $\mathcal{H} = S \oplus S^\perp$, and so $f = f_1 + f_2$ for some $f_1 \in S$ and some $f_2 \in S^\perp$. Since $f_1 \in S$, then $P(f_1) = f_1 = U^k(f_1)$ for all k, and so

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k(f_1) = \frac{1}{n}\sum_{k=0}^{n-1} f_1 = f_1 = P(f_1) \quad \forall n \in \mathbb{N} \implies \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k(f_1) = P(f_1)$$

Next, note that, by part (i) we have that for all $g \in S$,

$$\langle U(f_2) - f_2, g\rangle_{\mathcal{H}} = 0$$

This means that $U(f_2) - f_2 \in S^\perp$; since $f_2 \in S^\perp$ and $S^\perp$ is a subspace, we find that $U(f_2) \in S^\perp$ as well. Similar logic applied to $U(f_2)$ instead of f_2 shows that $U^2(f_2) \in S^\perp$. Proceeding inductively, we find that $U^k(f_2) \in S^\perp$ for all k. This means that for all n, we have that $\frac{1}{n}\sum_{k=0}^{n-1} U^k(f_2) \in S^\perp$. Let

$$g = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k(f_2)$$

Because $S^\perp$ is a closed subspace and g is the limit of a sequence in $S^\perp$, we know that $g \in S^\perp$. On the other hand, since U is continuous, we can bring the limit outside and apply the linearity of U to see that

$$U(g) = U\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k(f_2)\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^{k+1}(f_2) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} U^k(f_2)$$

So, because the norm $||\cdot||_{\mathcal{H}}$ is continuous (which means we can bring the limit outside),

$$\begin{aligned}
||U(g) - g||_{\mathcal{H}} &= \lim_{n\to\infty} \frac{1}{n}\left\|\sum_{k=1}^{n} U^k(f_2) - \sum_{k=0}^{n-1} U^k(f_2)\right\|_{\mathcal{H}} \\
&= \lim_{n\to\infty} \frac{1}{n}||U^n(f_2) - f_2||_{\mathcal{H}} \\
&\leq \lim_{n\to\infty} \frac{1}{n}\left(||U^n(f_2)||_{\mathcal{H}} + ||f_2||_{\mathcal{H}}\right) \\
&= \lim_{n\to\infty} \frac{1}{n}(||f_2||_{\mathcal{H}} + ||f_2||_{\mathcal{H}}) \\
&= 0,
\end{aligned}$$

where the first line comes from our definition of g and expression for $U(g)$, the second line cancels like terms in the sum, the third line applies the triangle inequality, the fourth line makes use of the fact that U preserves norms, and the last line simply takes the limit. This means that $U(g) = g$, which tells us that $g \in S$ by definition of S. However, we had already found that $g \in S^\perp$, which means that $g \in S \cap S^\perp \implies g = 0$. Therefore, since $f_2 \in S^\perp \implies P(f_2) = 0$, we get that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k(f_2) = g = 0 = P(f_2)$$

So, linearity of U^k and P grant that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k(f) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k(f_1) + \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k(f_2) = P(f_1) + P(f_2) = P(f)$$

Since this holds for all $f \in \mathcal{H}$, we are done. ■

Proof of (d). We will construct an operator T that does not attain its operator norm. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Define $T : \mathcal{H} \to \mathcal{H}$ on the basis by

$$T(\varphi_n) = \left(1 - \frac{1}{n}\right)\varphi_n$$

and extend linearly. Now, let $f \in \mathcal{H}$ be an arbitrary nonzero vector; if we write $f = \sum_{n=1}^\infty a_n\varphi_n$, then

$$||Tf||_{\mathcal{H}}^2 = \left\|\sum_{n=1}^\infty a_n T\varphi_n\right\|_{\mathcal{H}}^2 = \left\|\sum_{n=1}^\infty a_n\left(1 - \frac{1}{n}\right)\varphi_n\right\|_{\mathcal{H}}^2 = \sum_{n=1}^\infty |a_n|^2 \cdot \left(1 - \frac{1}{n}\right)^2,$$

where the last equality is an application of Parseval's identity. Note that $\left(1 - \frac{1}{n}\right)^2 < 1$ for all $n \in \mathbb{N}$, and so

$$||Tf||_{\mathcal{H}}^2 < \sum_{n=1}^\infty |a_n|^2 = ||f||_{\mathcal{H}}^2 \qquad \forall \text{ nonzero } f \in \mathcal{H}$$

This holds for all nonzero vectors f (because for such f one of the a_n's must be nonzero), and so $||T||_{op} \leq 1$ (this automatically grants that T is bounded). However, note that

$$||T\varphi_n||_{\mathcal{H}} = 1 - \frac{1}{n} \qquad \forall n \in \mathbb{N},$$

which means that $\sup_{n\in\mathbb{N}} ||T\varphi_n||_{\mathcal{H}} = 1$. Using the equivalent definition for operator norm provided in Problem 8(a) on PSET 7, we get

$$||T||_{op} = \sup\{||Tf||_{\mathcal{H}} : ||f||_{\mathcal{H}} = 1\} \geq \sup\{||T\varphi_n||_{\mathcal{H}} : n \in \mathbb{N}\} = 1$$

Therefore, $||T||_{op}$ must equal 1. Then, in order for T to attain its operator norm there must be a nonzero vector v such that $||Tf||_{\mathcal{H}} = ||f||_{\mathcal{H}}$. However, we saw already that for all nonzero vectors f,

$$||Tf||_{\mathcal{H}} < ||f||_{\mathcal{H}}$$

So, T does not attain its operator norm, and the construction is complete. ■