MAT 520: Problem Set 9

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Problem 1

Prove Weyl's criterion for the spectrum of an operator. Let $A = A^* \in \mathcal{B}(\mathcal{H})$ be given. We have $\lambda \in \sigma(A)$ iff there exists some $\{\varphi_n\}_{n\in\mathbb{N}}$ with $\|\varphi_n\| = 1$ such that

$$\lim_{n\to\infty} \|(A - \lambda\mathbb{1})\varphi_n\| = 0$$

Solution

Proof. ($\Longrightarrow$) Suppose that $\lambda \in \sigma(A)$. We know that A has no residual spectrum since it is self-adjoint (Theorem 9.21 in the lecture notes), and so λ is either in the point or continuous spectrum. Suppose first that $\lambda \in \sigma_p(A)$, in which case there is some $\psi \in \mathcal{H}$ such that $(A - \lambda\mathbb{1})\psi = 0$. Letting $\{\varphi_n\}_n$ be the constant sequence $\varphi_n := \frac{\psi}{\|\psi\|}$, we trivially have the result

$$\lim_{n\to\infty} \|(A - \lambda\mathbb{1})\varphi_n\| = \|\psi\|\|(A - \lambda\mathbb{1})\psi\| = 0$$

The only other option is that $\lambda \in \sigma_c(A)$, in which case $A - \lambda\mathbb{1}$ is injective with dense image. In particular, the image of $A - \lambda\mathbb{1}$ cannot be closed, since then it would be surjective and invertible and λ wouldn't be in the spectrum at all. Therefore, Lemma 7.20 in the lecture notes tells us that $A - \lambda\mathbb{1}$ is not bounded below, and so there does not exist an $\epsilon > 0$ for which

$$\left\|(A - \lambda\mathbb{1})\left(\frac{\psi}{\|\psi\|}\right)\right\| \geq \epsilon \quad (\psi \in \ker(A - \lambda\mathbb{1})^\perp = \mathcal{H})$$

Thus, for all $n \in \mathbb{N}$ there is a $\varphi_n \in \mathcal{H}$ such that $\|\varphi_n\| = 1$ and $\|(A - \lambda\mathbb{1})\varphi_n\| < \frac{1}{n}$. Choosing such a φ_n for each $n \in \mathbb{N}$ yields a sequence of unit vectors $\{\varphi_n\}_n$ for which

$$\|(A - \lambda\mathbb{1})\varphi_n\| < \frac{1}{n} \ \forall n \implies \lim_{n\to\infty} \|(A - \lambda\mathbb{1})\varphi_n\| = 0$$

as desired.

($\Longleftarrow$) We will show this direction by contrapositive. So, suppose now that $\lambda \notin \sigma(A)$, which implies $0 \notin \sigma(A - \lambda\mathbb{1}) \implies 0 \notin \sigma\left((A - \lambda\mathbb{1})^2\right)$ by the spectral mapping theorem. Since $(A - \lambda\mathbb{1})^2 = |A - \lambda\mathbb{1}|^2$ by self-adjointness, we find that $0 \notin \sigma\left(|A - \lambda\mathbb{1}|^2\right)$. By Lemma 7.20 in the lecture notes, there exists some $\epsilon > 0$ such that

$$\|(A - \lambda\mathbb{1})\psi\| \geq \epsilon\|\psi\| \quad (\psi \in \ker(A - \lambda\mathbb{1})^\perp)$$

However, since $\lambda \notin \sigma(A)$ we know that $A - \lambda\mathbb{1}$ is invertible and therefore has trivial kernel, meaning that

$$\|(A - \lambda\mathbb{1})\psi\| \geq \epsilon\|\psi\| \quad (\psi \in \mathcal{H})$$

In particular, for every sequence $\{\varphi_n\}_n$ with $\|\varphi_n\| = 1$ we know that

$$\|(A - \lambda\mathbb{1})\varphi_n\| \geq \epsilon\|\varphi_n\| = \epsilon,$$

which cannot go to 0 in norm. ■

Problem 2

Let $A \in \mathcal{B}(\mathcal{H})$ be compact, and $\{\varphi_n\}_n \subseteq \mathcal{H}$ converge weakly (in the sense of the Banach space weak topology on $\mathcal{H}$) to some $\varphi \in \mathcal{H}$. Show that $A\varphi_n \to A\varphi$ in norm.

Solution

Proof. Suppose $\varphi_n \to \varphi$ weakly. Then, for all continuous linear $\lambda : \mathcal{H} \to \mathbb{C}$, we know that $\lambda(\varphi_n) \to \lambda(\varphi)$ in $\mathbb{C}$. For all $\psi \in \mathcal{H}$, we know

$$\langle \psi, A\varphi_n \rangle = \langle A^*\psi, \varphi_n \rangle$$

Since $\langle A^*\psi, \cdot \rangle$ is a continuous linear functional on $\mathcal{H}$ by boundedness of $\|A^*\psi\|$, weak convergence tells us that

$$\langle A^*\psi, \varphi_n \rangle \to \langle A^*\psi, \varphi \rangle \implies \langle \psi, A\varphi_n \rangle \to \langle \psi, A\varphi \rangle$$

As this holds for all $\psi \in \mathcal{H}$ and by Riesz representation all continuous linear functionals are of the form $\langle \psi, \cdot \rangle$, we find that $A\varphi_n \to A\varphi$ weakly. Suppose by way of contradiciton that $A\varphi_n \not\to A\varphi$ in norm. Then, there is an $\epsilon > 0$ and a subsequence $\{A\varphi_{n_k}\}_k$ such that $\|A\varphi_{n_k} - A\varphi\| \geq \epsilon$ for all k. By compactness of A, the subsequence $\{A\varphi_{n_k}\}_k$ itself contains a convergent subsubsequence $\{A\varphi_{n_{k_j}}\}_j$, which we know by selection converges to some $\phi \neq A\varphi$. So, as $j \to \infty$ we see that $A\varphi_{n_{k_j}} \to \phi$ in norm, implying that $A\varphi_{n_{k_j}} \to \phi$ weakly. However, $A\varphi_n \to A\varphi$ weakly implies that $A\varphi_{n_{k_j}} \to A\varphi$ weakly as well. Since $\phi \neq A\varphi$, this is a contradiciton. ■

Problem 3

Determine whether the following operators are compact or not (and prove what you think):

(a) $\mathbb{1}$.

(b) $u \otimes v^*$ for some $u, v \in \mathcal{H}$.

(c) On the Banach space $X = C([0,1] \to \mathbb{C})$ with $\|\cdot\|_\infty$, let $A : X \to X$ be given by

$$(A\varphi)(x) = \int_{[0,1]} K(x,y)\varphi(y)\,dy,$$

where $K : [0,1]^2 \to \mathbb{C}$ is some *continuous* function.

(d) $A := \frac{1}{1+X^2}$ on $\ell^2(\mathbb{Z})$, where X is the position operator given by

$$(X\psi)(n) \equiv n\psi(n) \quad (n \in \mathbb{Z}, \psi \in \ell^2(\mathbb{Z})),$$

and we employ the holomorphic functional calculus to define A.

Solution

Proof. (a) $\mathbb{1}$ is not compact in an infinite-dimensional Banach space. This can be seen immediately since the image of the unit ball is the unit ball, which we know to be compact if and only if the space is finite-dimensional. So, by Lemma 9.33 in the lecture notes, $\mathbb{1}$ is not compact when the space is infinite-dimensional since it sends a bounded set to a set with noncompact closure.

(b) Fix two $u, v \in \mathcal{H}$ and define $A \in \mathcal{B}(\mathcal{H})$ via $A := u \otimes v^*$. Equivalently, $A(\psi) \equiv \langle v, \psi \rangle\, u$. Note that the image of this operator is contained in $\text{span}\{u\}$, which is a space of dimension 1. So, A is a rank-1 operator; in particular, it is finite-rank and therefore compact.

(c) A is clearly linear, and it is bounded since

$$|(A\varphi)(x)| \leq \sup_{x,y\in[0,1]} |K(x,y)| \int_{[0,1]} |\varphi(y)|dy \leq \sup_{x,y\in[0,1]} |K(x,y)|\|\varphi\| \implies \|A\| \leq \sup_{x,y\in[0,1]} |K(x,y)| < \infty,$$

which follows because K is continuous on a compact domain. To see that A is compact, let $\mathcal{F} \subseteq X$ be a bounded set of elements of X (i.e. $\|f\|_\infty < M \ \forall f \in \mathcal{F}$); we will show that $\overline{A(\mathcal{F})} \subseteq X$ is compact. To do so, we will show that every sequence in $\overline{A(\mathcal{F})}$ contains a convergent subsequence. Note that $\overline{A(\mathcal{F})}$ is clearly uniformly bounded since $\mathcal{F}$ is. So, if we can show that $\overline{A(\mathcal{F})}$ is uniformly equicontinuous, then every sequence of functions in $\overline{A(\mathcal{F})}$ will be both uniformly bounded and uniformly equicontinuous, which by the Arzela-Ascoli theorem means that each one will have a uniformly converging subsequence. Therefore, to prove that A is compact it suffices to show that $\overline{A(\mathcal{F})}$ is uniformly equicontinuous.

To this end, let $\epsilon > 0$ be arbitrary. Let $\delta > 0$ be such that $|(x_1, y_1) - (x_2, y_2)| < \delta \implies |K(x_1, y_1) - K(x_2, y_2)| < \frac{\epsilon}{3M}$, which exists by continuity of K. Then, for all $f \in \mathcal{F}$ and all $a, b \in [0,1]$ with $|a - b| < \delta$, we have that

$$\begin{aligned}
|(Af)(a) - (Af)(b)| &= \left| \int_{[0,1]} (K(a,y) - K(b,y))f(y)dy \right| \\
&\leq \int_{[0,1]} |K(a,y) - K(b,y)||f(y)|dy \\
&\leq \frac{\epsilon}{3M} \int_{[0,1]} |f(y)|dy \leq \frac{\epsilon}{3M} M = \frac{\epsilon}{3}
\end{aligned}$$

Problem 3 continued on next page...

For any $g \in \overline{A\mathcal{F}}$, we know g is the uniform limit of functions in $A\mathcal{F}$ and so there is a $f \in \mathcal{F}$ such that $\|g - Af\|_\infty < \frac{\epsilon}{3}$. Thus, for all $a, b \in [0,1]$ with $|a - b| < \delta$,

$$\begin{aligned}|g(a) - g(b)| &\leq |g(a) - (Af)(a)| + |(Af)(a) - (Af)(b)| + |(Af)(b) - g(b)| \\ &\leq \|g - Af\|_\infty + \frac{\epsilon}{3} + \|Af - g\|_\infty < \epsilon\end{aligned}$$

So, since our choice of δ does not depend on g, a, or b, we see that $\overline{A\mathcal{F}}$ is a uniformly equicontinuous family of functions. Since it is also uniformly bounded, Arzela-Ascoli tells us that it is sequentially compact in the $\|\cdot\|_\infty$ topology, and so it is compact in X. Since this holds for all bounded $\mathcal{F}$, we see that A is compact.

(d) It is clear from the properties of the functional calculus that A is a multiplication operator given by

$$(A\psi)(n) = \frac{1}{1+n^2}\psi(n) \quad (n \in \mathbb{Z},\ \psi \in \ell^2(\mathbb{Z}))$$

By Claim 9.38 in the lecture notes, since $\langle \delta_n, A\delta_n \rangle \equiv \frac{1}{1+n^2} \to 0$ as $|n| \to \infty$ (where $\{\delta_n\}_{n\in\mathbb{Z}}$ is the standard position basis), we may realize A as a norm limit of finite-rank operators. Thus, A is compact. ■

Problem 4

On $\mathcal{H} \oplus \mathcal{H}$, let

$$H := \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix}$$

for some $S \in \mathcal{B}(\mathcal{H})$. Find the polar decomposition of H.

Solution

Proof. Note that

$$|H|^2 = H^*H = \begin{bmatrix} S^*S & 0 \\ 0 & SS^* \end{bmatrix} = \begin{bmatrix} |S|^2 & 0 \\ 0 & |S^*|^2 \end{bmatrix},$$

and so

$$|H| = \begin{bmatrix} |S| & 0 \\ 0 & |S^*| \end{bmatrix}$$

Write the polar decomposition of S to be $S = U|S|$ for some partial isometry $U \in \mathcal{B}(\mathcal{H})$ with $\ker(U) = \ker(S)$. Note that

$$(U|S|U^*)^2 = U|S|U^*U|S|U^* = U|S||U|^2|S|U^* = S|U|^2S^*$$

We want to show that $S|U|^2 = S$ on $\mathcal{H}$. Let $M := \ker(|U|^2)$. By Lemma 7.18 in the notes, we know that $M = \ker(|U|^2) = \ker(U) = \ker(S)$. Furthermore, since U is a partial isometry, $|U|^2$ is a self-adjoint projection onto $M^\perp$. We may decompose $\mathcal{H} = M \oplus M^\perp$. Let $\psi \in \mathcal{H}$ be arbitrary, and so we may write $\psi = \psi_M + \psi_{M^\perp}$ with $\psi_M \in M$ and $\psi_{M^\perp} \in M^\perp$. So,

$$S|U|^2\psi = S|U|^2\psi_M + S|U|^2\psi_{M^\perp} = S\psi_{M^\perp} = S\psi$$

Since $|U|^2\psi_{M^\perp} = \psi_{M^\perp}$ and $S\psi = S\psi_{M^\perp}$. Since this holds for all ψ, we see that $S|U|^2 = S$. Therefore,

$$(U|S|U^*)^2 = SS^* = |S^*|^2$$

By the uniqueness of square roots, we find that

$$|S^*| = U|S|U^* = US^*$$

So,

$$|H| = \begin{bmatrix} |S| & 0 \\ 0 & US^* \end{bmatrix}$$

Now, define an operator V on $\mathcal{H} \oplus \mathcal{H}$ via

$$V := \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}$$

To see that V is a partial isometry, we compute

$$|V|^2 = V^*V = \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}\begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix} = \begin{bmatrix} U^*U & 0 \\ 0 & UU^* \end{bmatrix} = \begin{bmatrix} |U|^2 & 0 \\ 0 & |U^*|^2 \end{bmatrix}$$

Since both $|U|^2$ and $|U^*|^2$ are self-adjoint projections, so is $|V|^2$, and thus V is a partial isometry. We note that

$$V|H| = \begin{bmatrix} 0 & U^*US^* \\ U|S| & 0 \end{bmatrix} = \begin{bmatrix} 0 & |U|^2S^* \\ S & 0 \end{bmatrix}$$

We have already shown that $S|U|^2 = S$; taking the adjoint shows that $|U|^2S^* = S^*$. Therefore,

$$V|H| = \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix} = H$$

Thus, $H = V|H|$ is the polar decomposition of H. ■

Problem 5

Show that an idempotent is compact if and only if it is of finite rank.

Solution

Proof. ($\implies$) Suppose that A is idempotent and compact. We claim that $\text{im}(A) = \ker(A - \mathbb{1})$. To show the first inclusion, suppose that $\varphi \in \text{im}(A)$, and so $\varphi = A\psi$ for some $\psi \in \mathcal{H}$. Then, $A\varphi = A^2\psi = A^\psi = \varphi$ because $A^2 = A$, and so $\varphi \in \ker(A - \mathbb{1})$. Therefore, $\text{im}(A) \subseteq \ker(A - \mathbb{1})$. Conversely, if $\varphi \in \ker(A - \mathbb{1})$, then $A\varphi = \varphi \implies \varphi \in \text{im}(A)$, and so $\text{im}(A) = \ker(A - \mathbb{1})$.

Now, suppose by way of contradiction that A is not finite rank. Then, $\dim(\ker(A - \mathbb{1})) = \dim(\text{im}(A)) = \infty$. So, $\ker(A - \mathbb{1})$ is a closed and infinite-dimensional subspace of $\mathcal{H}$, which means it is itself an infinite-dimensional Hilbert space. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $\ker(A - \mathbb{1})$. As $\{\varphi_n\}_n$ is a bounded sequence, we know that $\{A\varphi_n\}_n$ must have a convergent subsequence by compactness of A. However, since each $\varphi_n \in \ker(A - \mathbb{1})$, we know that $A\varphi_n = \varphi_n$ for all n. Together, these facts tell us that $\{\varphi_n\}_n$ must itself have a convergent subsequence. This is impossible, since for all $n \neq m$ we know

$$\|\varphi_n - \varphi_m\|^2 = \|\varphi_n\|^2 + \|\varphi_m\|^2 - 2\,\mathbb{R}\text{e}\{\langle \varphi_n, \varphi_m \rangle\} = 2 \not\to 0$$

by orthonormality. So, no subsequence of $\{\varphi_n\}_n$ is Cauchy, which yields a contradiction. Therefore, A must be finite rank.

($\impliedby$) Suppose that A is finite rank. Then, it is trivially compact. ■

Problem 6

Show that no nonzero multiplication operator on $\mathcal{H} := L^2([0,1])$ is compact.

Solution

Proof. Let A be a compact multiplication operator on $L^2([0,1])$. Then, there is some $F \in \mathcal{H}$ such that for any $f \in \mathcal{H}$ we have

$$(Af)(x) = F(x)f(x)$$

Suppose by way of contradiction that F is not the zero element of $\mathcal{H}$. Then, letting λ denote the Lebesgue measure on $\mathbb{R}$, there is an $\epsilon > 0$ for which $\lambda(\{x \in [0,1] : \ |F(x)| \geq \epsilon\}) > 0$. For this ϵ, define $E := \{x \in [0,1] : \ |F(x)| \geq \epsilon\}$. Consider the set $S \subseteq \mathcal{H}$ given by

$$S := \{f \in \mathcal{H} : \ f(x) = 0 \text{ for a.e. } x \in [0,1] \setminus E\}$$

S is certainly a vector space. To see that S is closed, let $\{f_n\}_n \subseteq S$ be a sequence that converges in the $\mathcal{H}$-norm to some $f \in \mathcal{H}$. Then, for any $\delta > 0$ we may find an $n \in N$ large enough that

$$\delta > \|f_n - f\|^2 = \int_0^1 |f_n(x) - f(x)|^2 \mathrm{d}x \geq \int_{[0,1]\setminus E} |f_n(x) - f(x)|^2 \mathrm{d}x = \int_{[0,1]\setminus E} |f(x)|^2 \mathrm{d}x$$

Since this holds for any $\delta > 0$, we see that $\int_{[0,1]\setminus E} |f|^2 = 0$, and so f is 0 a.e. on $[0,1] \setminus E$. Therefore, $f \in S$, which shows that S is closed, and so S is a Banach space in its own right. We claim that S is infinite dimensional. Since $\lambda(E) > 0$ we know that E is uncountable, and so there is a bijection $\Psi : [0,1] \to E$. Then, we have that $S \cong L^2(E, \lambda) \cong L^2([0,1], \lambda_\# \Psi)$ where $\lambda_\# \Psi$ is the pullback measure $\lambda_\# \Psi(V) = \lambda(\Psi^{-1}(V))$. In particular, S has the same dimensionality as $L^2([0,1], \lambda_\# \Psi)$ and is therefore an infinite-dimensional Banach space. Furthermore, we certainly have that $A(S) \subseteq S$, and so we may consider the bounded linear operator $A|_S : S \to S$. We claim that $A|_S$ is surjective. To see this, note that for any $f \in S$, we also have that the function g sending $x \mapsto \frac{1}{F(x)} f(x)$ is in S (it certainly is supported on E, and it is in L^2 since $|\frac{1}{F(x)} f(x)| \leq \frac{1}{\epsilon} |f(x)|$ on E, and so $\|g\| \leq \frac{\|f\|}{\epsilon} < \infty$). Since $(Ag)(x) = F(x) \frac{1}{F(x)} f(x) = f(x)$ over E, we see that $A|_S g = f$. Since this holds for all $f \in S$, the map $A|_S$ is surjective. By the open mapping theorem, it is therefore open. It is also compact by compactness of A. Let $B := \{f \in S : \ \|f\| < 1\}$ denote the open unit ball in S. Then, $A|_S(B) \subseteq S$ is an open set with compact closure by the openness and compactness of $A|_S$ respectively. This is a contradiction, since no open set in an infinite-dimensional Banach space may have compact closure. ∎

Problem 7

Show that if $A \in \mathcal{B}(\mathcal{H})$ is compact and $\{e_n\}_n$ is an ONB then $\|Ae_n\| \to 0$. Find a counter-example of the converse.

Solution

We first show that $e_n \to 0$ weakly. Let $\psi \in \mathcal{H}$ be arbitrary. By Theorem 7.27 in the lecture notes,

$$\|\psi\|^2 = \sum_{n \in \mathbb{N}} |\langle e_n, \psi \rangle|^2 < \infty$$

So, since the sum converges and each element is nonnegative, $\langle e_n, \psi \rangle \to 0$. Since this holds for all ψ, we know that $e_n \to 0$ weakly. Therefore, by Problem 2, we find that $Ae_n \to A0 = 0$ in norm. Equivalently, $\|Ae_n\| \to 0$.

Now, we construct a counterexample to the converse by exhibiting a Hilbert space $\mathcal{H}$, an orthonormal basis $\{e_n\}_n$ of $\mathcal{H}$, and an operator $A \in \mathcal{B}(\mathcal{H})$ for which $\|Ae_n\| \to 0$ yet A is not compact. To do so, for each $m \in \mathbb{N}$ let $\mathcal{H}_m := \mathbb{R}^m$ be the m-dimensional Euclidean space with the Euclidean inner product. Define

$$\mathcal{H} := \bigoplus_{m \in \mathbb{N}} \mathcal{H}_m$$

to be our Hilbert space with the direct sum inner product. For each m, let $\{\tilde{e}_j^{(m)}\}_{j=1}^m \subseteq \mathcal{H}_m$ be an ONB of $\mathcal{H}_m$. Let

$$e_j^{(m)} := (\ldots, 0, \tilde{e}_j^{(m)}, 0, \ldots) \in \mathcal{H}$$

be the element of $\mathcal{H}$ with $\tilde{e}_j^{(m)}$ in the m^{th} coordinate and 0's everywhere else. Then, we find that

$$\bigcup_{m \in \mathbb{N}} \{e_j^{(m)} : \ j \in \{1, \ldots, m\}\}$$

is an ONB for $\mathcal{H}$. Lastly, for each $m \in \mathbb{N}$ define the operator $A_m \in \mathcal{B}(\mathcal{H}_m)$ via the $m \times m$ matrix

$$A_m := \frac{1}{m} \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix}$$

with respect to the basis $\{\tilde{e}_j^{(m)}\}_j$ of $\mathcal{H}_m$. We note that A_m is idempotent for all m, as

$$A_m^2 = \frac{1}{m^2} \begin{bmatrix} m & \ldots & m \\ \vdots & \ddots & \vdots \\ m & \ldots & m \end{bmatrix} = A_m$$

Define the operator $A \in \mathcal{B}(\mathcal{H})$ via $A := \bigoplus_{m \in \mathbb{N}} A_m$, i.e. A acts on $\mathcal{H}$ by applying A_m coordinatewise. Then, A is clearly also idempotent. Furthermore, observe that for each $e_j^{(m)}$, we have that

$$Ae_j^{(m)} = (\ldots, 0, A_m \tilde{e}_j^{(m)}, 0, \ldots) = \frac{1}{m} \sum_{j=1}^m e_j^{(m)}$$

Thus,

$$\|Ae_j^{(m)}\|^2 = \frac{1}{m^2} \left\| \sum_{j=1}^m e_j^{(m)} \right\|^2 = \frac{1}{m^2} \sum_{j=1}^m \|e_j^{(m)}\|^2 = \frac{1}{m^2} \cdot m = \frac{1}{m},$$

where the second equality holds by orthogonality. Thus, as $m \to \infty$, we have that $\|Ae_j^{(m)}\| \to 0$. If we let $\{e_n\}_n$ be the ONB of $\mathcal{H}$ enumerated as $\{e_1^{(1)}, e_1^{(2)}, e_2^{(2)}, e_1^{(3)}, \ldots, e_1^{(m)}, e_2^{(m)}, \ldots\}$, then it holds that $\|Ae_n\| \to 0$ as $n \to \infty$. However, A cannot be compact since it is idempotent and not finite-rank (see Problem 5). To see that A is not finite-rank, note that for any $k \in \mathbb{N}$ we have that $\bigoplus_{m=1}^{k+1} A_m$ has $k+1$-dimensional range, and so A cannot have finite-dimensional range. So, $A \in \mathcal{B}(\mathcal{H})$ is not compact yet $\|Ae_n\| \to 0$ for an ONB $\{e_n\}_n$ of $\mathcal{H}$.