MAT 520: Problem Set 8

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Problem 1

Let P, Q be two orthogonal projections onto closed subspaces M, N in a Hilbert space $\mathcal{H}$ such that $[P, Q] = 0$. Write $P^\perp := \mathbb{1} - P$ and $Q^\perp := \mathbb{1} - Q$.

(a) Show $P^\perp, Q^\perp, PQ, P+Q-PQ$ and $P+Q-2PQ$ are orthogonal projections.

(b) What is the relation between the projections in the previous item and M, N?

Solution

Proof. (a) Note that $(PQ)^* = Q^*P^* = QP = PQ$ since P and Q are self-adjoint and $QP - PQ = 0$ by assumption. Therefore, all of the operators of interest are self-adjoint, and so we just need to show that they are all idempotent.

($P^\perp$ and $Q^\perp$) We show it for $P^\perp$, since the result for $Q^\perp$ will follow. Now, we may decompose $\mathcal{H} = M \oplus M^\perp$. For all $\psi \in \mathcal{H}$, we may write $\psi = \psi_M + \psi_{M^\perp}$ with $\psi_M \in M$ and $\psi_{M^\perp} \in M^\perp$. So,

$$P^\perp(\psi) = \psi - P\psi = \psi_{M^\perp}$$

Therefore, $P^\perp$ is the projection onto the closed subspace $M^\perp$.

(PQ) Note that

$$(PQ)^2 = PQPQ = PPQQ = PQ$$

since $QP = PQ$ and P, Q are idempotent. So, PQ is also idempotent.

($P + Q - PQ$) We compute

$$\begin{aligned}(P+Q-PQ)^2 &= P^2 + PQ - P^2Q + QP + Q^2 - QPQ - PQP - PQ^2 + (PQ)^2 \\ &= P + PQ - PQ + PQ + Q - PQ - PQ - PQ + (PQ)^2 \\ &= P + Q - PQ\end{aligned}$$

($P + Q - 2PQ$) We compute

$$\begin{aligned}(P+Q-2PQ)^2 &= P^2 + PQ - 2P^2Q + QP + Q^2 - 2QPQ - 2PQP - 2PQ^2 + (2PQ)^2 \\ &= P + PQ - 2PQ + PQ + Q - 2PQ - 2PQ - 2PQ + 4(PQ)^2 \\ &= P + Q - 2PQ\end{aligned}$$

(b) $P^\perp$ and $Q^\perp$ are the orthogonal projections onto the closed subspaces $M^\perp$ and $N^\perp$, respectively. We claim PQ is the orthogonal projection onto $M \cap N$, or equivalently that $PQ\psi = \psi \iff \psi \in M \cap N$. To see this, note that $\psi = PQ\psi \in \operatorname{im}(P) = M$ and similarly $\psi = QP\psi \in \operatorname{im}(Q) = N$, and so $\psi = PQ\psi \implies \psi \in M \cap N$. Conversely, if $\psi \in M \cap N$, then $P\psi = \psi$ and $Q\psi = \psi$, and so $PQ\psi = \psi$.

Next, we claim that $\operatorname{im}(P + Q - PQ) = M \oplus N$. Clearly, $M \oplus N$ is closed. So, for any $\psi \in \mathcal{H}$ we may uniquely write $\psi = \psi_M + \psi_N + \psi_r$ for $\psi_M \in M$, $\psi_N \in N$, and $\psi_r \in (M \oplus N)^\perp = M^\perp \cap N^\perp$. We have that $P\psi = \psi_M + P\psi_N$ and $Q\psi = Q\psi_M + \psi_N$. Thus, $PQ\psi = PQ\psi_M + P\psi_N = QP\psi_M + P\psi_N = Q\psi_M + P\psi_N$. So,

$$(P+Q-PQ)\psi = \psi_M + P\psi_N + Q\psi_M + \psi_N - Q\psi_M - P\psi_N = \psi_M + \psi_N$$

We see that $P + Q - PQ$ removes the $(M \oplus N)^\perp$ part of ψ. Therefore, $P + Q - PQ$ is the orthogonal projection onto the closed subspace $M \oplus N$.

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Lastly, note that

$$P + Q - 2PQ = P - PQ + Q - QP = PQ^{\perp} + QP^{\perp}$$

We know that $[P, Q^{\perp}] = [Q, P^{\perp}] = 0$ since $[P, Q] = 0$, and so $PQ^{\perp}$ is the orthogonal projection onto $M \cap N^{\perp}$ and $QP^{\perp}$ is the orthogonal projection onto $M^{\perp} \cap N$. By Problem 6 on Problem Set 7, we know that since the sum of these self-adjoint projections is another self-adjoint projection, the spaces must be orthogonal. Therefore, we see that $P + Q - 2PQ$ is an orthogonal projection onto the closed subspace

$$(M \cap N^{\perp}) \oplus (M^{\perp} \cap N)$$

■

Problem 2

Let P, Q be two orthogonal projections onto closed subspaces M, N in a Hilbert space $\mathcal{H}$. Show that

$$\operatorname*{s-lim}_{n\to\infty} (PQ)^n$$

exists and is the orthogonal projection onto $M \cap N$.

Solution

Proof. Observe that for any n,

$$(PQ)^n = PQPQ \dots PQ = P(QPQ)(QPQ)\dots(QPQ) = P(QPQ)^{n-1}$$

So, we know that

$$\operatorname*{s-lim}_{n\to\infty}(PQ)^n = P \operatorname*{s-lim}_{n\to\infty}(QPQ)^n$$

and that these strong limits exist or don't exist together. We claim that

$$\operatorname*{s-lim}_{n\to\infty} (QPQ)^n$$

exists and is the orthogonal projection onto $M \cap N$, from which the problem's result will follow immediately. Let $A := QPQ$. Then, $A^* = Q^*P^*Q^* = QPQ = A$, and so A is self-adjoint. Also, $\|A\| \le \|Q\|\|P\|\|Q\| \le 1$.

Lemma 1. *Let $A \in \mathcal{B}(\mathcal{H})$ be positive with $\|A\| \le 1$. Then, A^n strongly converges to the orthogonal projection onto* $\ker(A - \mathbb{1})$.

Proof of Lemma. As A is self-adjoint, we may apply the measurable functional calculus. Since A is positive with $\|A\| \le 1$, we know that $\sigma(A) \subseteq [0, 1]$. Let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(\lambda) = \lambda^n$ and $\chi : [0, 1] \to \mathbb{R}$ be given by $\chi_{\{1\}}(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{else} \end{cases}$. Then, over $\sigma(A)$ we see that $f_n \to \chi_{\{1\}}$ pointwise and $\|f_n\|_\infty \le 1 < \infty$. So, the measurable functional calculus (Theorem VII.2(d) in Reed & Simon) gives that $f_n(A) \to \chi_{\{1\}}(A)$ strongly. Therefore, $A^n \to \chi_{\{1\}}(A)$ strongly. Lastly, we know that $\chi_{\{1\}}(A)$ is the orthogonal projection onto $\ker(A - \mathbb{1})$ by Theorem 12.29 in Rudin (which states that $\ker(A - \lambda\mathbb{1}) = \operatorname{im}(\chi_{\{\lambda\}}(A))$). ■

In light of the above lemma, all we must show is that $\ker(QPQ - \mathbb{1}) = M \cap N$. Clearly, for any $\psi \in M \cap N$ we have that $QPQ\psi = \psi$, and so $M \cap N \subseteq \ker(QPQ - \mathbb{1})$. To see the other direction, suppose that $\psi \in \ker(QPQ - \mathbb{1})$ in which case $QPQ\psi = \psi$. Then,

$$\|\psi\| = \|QPQ\psi\| \le \|Q\psi\| \le \|\psi\|$$

and so $\|Q\psi\| = \|\psi\|$. Similarly, $\|P\psi\| = \|\psi\|$. However, the Pythagorean theorem tells us that

$$\|\psi\|^2 = \|P\psi\|^2 + \|(\mathbb{1} - P)\psi\|^2 \implies (\mathbb{1} - P)\psi = 0 \implies P\psi = \psi \implies \psi \in M$$

and similarly $Q\psi = \psi \implies \psi \in N$. Therefore, $\psi \in M \cap N$ and we are done. ■

Problem 3

Let $A \in \mathcal{B}(\mathcal{H})$. Show that the set of $\lambda \in \sigma(A)$ such that λ is not an eigenvalue of A and $\text{im}(A - \lambda\mathbb{1})$ is closed but not the whole of $\mathcal{H}$ is an open subset of $\mathbb{C}$.

Solution

Proof. Write

$$S := \{\lambda \in \sigma(A) : \ \ker(A - \lambda\mathbb{1}) = \{0\} \text{ and } \text{im}(A - \lambda\mathbb{1}) = \overline{\text{im}(A - \lambda\mathbb{1})} \neq \mathcal{H}\}$$

Let $\lambda \in S$ be arbitrary. By Lemma 7.20 in the lecture notes, since the image of $A - \lambda\mathbb{1}$ is closed, there is some $\epsilon > 0$ such that for all $\varphi \in \ker(A - \lambda\mathbb{1})^\perp$, we have

$$\|(A - \lambda\mathbb{1})\varphi\| \geq \epsilon\|\varphi\|$$

However, since $A - \lambda\mathbb{1}$ is injective by definition of S, $\ker(A - \lambda\mathbb{1})^\perp = \mathcal{H}$. So, we get that

$$\|(A - \lambda\mathbb{1})\varphi\| \geq \epsilon\|\varphi\| \quad (\forall\varphi \in \mathcal{H})$$

Define $\delta := \frac{\epsilon}{2} > 0$ and suppose by way of contradiction that $B_\delta(\lambda) \not\subseteq S$. Then, there is some $\gamma \in \mathbb{C}$ with $|\gamma - \lambda| < \delta$ and $\gamma \notin S$. We have that for all $\varphi \in \mathcal{H}$,

$$\begin{aligned}
\epsilon\|\varphi\| &\leq \|(A - \lambda\mathbb{1})\varphi\| = \|((A - \gamma\mathbb{1}) + (\gamma\mathbb{1} - \lambda\mathbb{1}))\varphi\| \\
&\leq \|(A - \gamma\mathbb{1})\varphi\| + |\gamma - \lambda|\|\varphi\| \\
&\leq \|(A - \gamma\mathbb{1})\varphi\| + \frac{\epsilon}{2}\|\varphi\|,
\end{aligned}$$

where the second line comes from the triangle inequality and the third uses our selection of δ. So,

$$\|(A - \gamma\mathbb{1})\varphi\| \geq \frac{\epsilon}{2}\|\varphi\| \quad (\forall\varphi \in \mathcal{H})$$

From this, we see that $(A - \gamma\mathbb{1})\varphi = 0 \implies \|\varphi\| = 0$ and so $A - \gamma\mathbb{1}$ has trivial kernel. Also, Lemma 7.20 grants that $\text{im}(A - \gamma\mathbb{1})$ is closed. So, since $\gamma \notin S$ it must be that $\text{im}(A - \gamma\mathbb{1}) = \mathcal{H}$. So, $A - \gamma\mathbb{1}$ is both injective and surjective, which means it's invertible. We will show that this ends in a contradiction.

To this end, let $\varphi \in \mathcal{H}$ be arbitrary, and let $\psi := (A - \gamma\mathbb{1})^{-1}\varphi$. By the bound from earlier,

$$|(A - \gamma\mathbb{1})\psi\| \geq \frac{\epsilon}{2}\|\psi\| \implies \|(A - \gamma\mathbb{1})^{-1}\varphi\| \leq \frac{2}{\epsilon}\|\varphi\|$$

Since this holds for all $\varphi \in \mathcal{H}$, we find that

$$\|(A - \gamma\mathbb{1})^{-1}\|_{\text{op}} \leq \frac{2}{\epsilon} \implies \frac{\epsilon}{2} \leq \|(A - \gamma\mathbb{1})^{-1}\|_{\text{op}}^{-1}$$

Let $T_\gamma := A - \gamma\mathbb{1}$ and $T_\lambda := A - \lambda\mathbb{1}$ for notation. Then, $\delta \leq \|T_\gamma^{-1}\|_{\text{op}}^{-1}$ and so

$$\|T_\gamma - T_\lambda\|_{\text{op}} = \|(\lambda - \gamma)\mathbb{1}\|_{\text{op}} = |\lambda - \gamma| < \delta \leq \|T_\gamma^{-1}\|_{\text{op}}^{-1}$$

So, $T_\lambda \in B_{\|T_\gamma^{-1}\|_{\text{op}}^{-1}}(T_\gamma) \subseteq \mathcal{B}(\mathcal{H})$. Thus, by the logic in the proof of Claim 6.6 in the notes (openness of $\mathcal{G}_\mathcal{A}$ for Banach algebra $\mathcal{A}$), this means that T_λ is also invertible, and so $\lambda \notin \sigma(A)$. This is obviously a contradiction, and so we find that $B_\delta(\lambda) \subseteq S$. Since such a neighborhood exists for all $\lambda \in S$, we have shown that S is open in $\mathbb{C}$. ■

Problem 4

Define the numerical range $N(A)$ of $A \in \mathcal{B}(\mathcal{H})$ via

$$N(A) := \{\langle \psi, A\psi \rangle : \psi \in \mathcal{H} \text{ and } \|\psi\| = 1\}.$$

(a) Show that

$$\sigma(A) \subseteq \overline{N(A)}$$

(b) Find an example where $N(A)$ is not closed and

$$\sigma(A) \not\subseteq N(A)$$

(c) Find an example where

$$\sigma(A) \neq N(A) = \overline{N(A)}$$

Solution

Proof. (a) Consider any $\lambda \in \mathbb{C}$. Note that $\sigma(A) - \lambda = \sigma(A - \lambda\mathbb{1})$ by the spectral mapping theorem. Similarly, $\lambda \in N(A) \iff$ there is a unit vector ψ s.t. $\langle \psi, A\psi \rangle = \lambda \iff \langle \psi, (A - \lambda\mathbb{1})\psi \rangle = 0$. So, $N(A) - \lambda = N(A - \lambda\mathbb{1})$. Since $\mathcal{H}$ is a TVS and so translation is homeomorphic, we see that $\overline{N(A)} - \lambda = \overline{N(A - \lambda\mathbb{1})}$. All this goes to show that it suffices to prove that $0 \in \sigma(A) \implies 0 \in \overline{N(A)}$, since we could apply the same logic to $A - \lambda\mathbb{1}$ to get the result for any λ. We proceed.

Suppose that $0 \in \sigma(A)$. If we have $0 \in \sigma_p(A)$, then there is a nonzero $\psi \in \mathcal{H}$ such that $A\psi = 0$, and so

$$\left\langle \frac{\psi}{\|\psi\|}, A\frac{\psi}{\|\psi\|} \right\rangle = 0 \implies 0 \in N(A)$$

and we are done. If instead we have that $0 \in \sigma_r(A)$, then Claim 9.20 in the notes shows that $0 \in \sigma_p(A^*)$. So, there is some nonzero $\psi \in \mathcal{H}$ such that $A^*\psi = 0$, and so

$$\left\langle \frac{\psi}{\|\psi\|}, A\frac{\psi}{\|\psi\|} \right\rangle = \left\langle A^*\frac{\psi}{\|\psi\|}, \frac{\psi}{\|\psi\|} \right\rangle = 0 \implies 0 \in N(A)$$

So, suppose that $0 \in \sigma_c(A)$, the only other option. In this case, we know that $\text{im}(A)$ is not closed in $\mathcal{H}$. So, by Lemma 7.20 in the notes, for all $\epsilon > 0$ there is a $\varphi \in \mathcal{H}$ such that $\|A\varphi\| < \epsilon\|\varphi\|$. Thus, for all $\epsilon > 0$ there exists a unit vector $\psi = \frac{\varphi}{\|\varphi\|} \in \mathcal{H}$ such that $\|A\psi\| < \epsilon$. Therefore, for all $\epsilon > 0$ we may apply Cauchy-Schwartz to see that

$$|\langle \psi, A\psi \rangle| \leq \|\psi\|\|A\psi\| = \|A\psi\| < \epsilon$$

So, for all $\epsilon > 0$ there is a $\lambda \in N(A)$ with $|\lambda| < \epsilon$, which certainly means that $0 \in \overline{N(A)}$. We see that in any case, $0 \in \overline{N(A)}$, proving the result.

(b) Consider the example where $\mathcal{H}$ is separable and A is the multiplication operator defined on an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$ via

$$A\varphi_j := \frac{1}{j}\varphi_j$$

and extended linearly. This is clearly a bounded linear operator, and so $A \in \mathcal{B}(\mathcal{H})$. Since each φ_j is an eigenvector with eigenvalue $\frac{1}{j}$, we see that $\frac{1}{j} \in \sigma(A)$ for all $j \in \mathbb{N}$. Since $\sigma(A)$ is closed, this also means that $0 \in \sigma(A)$. Furthermore, for each j we have that

$$\langle \varphi_j, A\varphi_j \rangle = \left\langle \varphi_j, \frac{1}{j}\varphi_j \right\rangle = \frac{1}{j}\|\varphi_j\|^2 = \frac{1}{j} \implies \frac{1}{j} \in N(A)$$

since $\|\varphi_j\| = 1$. However, we claim that $0 \notin N(A)$. Suppose by way of contradiction that there were some $\psi \in \mathcal{H}$ with $\|\psi\| = 1$ and $\langle \psi, A\psi \rangle = 0$. By Theorem 7.27 in the notes, we can express

$$\psi = \sum_{j \in \mathbb{N}} \langle \varphi_j, \psi \rangle \varphi_j$$

By continuity of A, we may distribute it into the infinite sum to see

$$A\psi = \sum_{j \in \mathbb{N}} \langle \varphi_j, \psi \rangle A\varphi_j = \sum_{j \in \mathbb{N}} \frac{\langle \varphi_j, \psi \rangle}{j} \varphi_j$$

So, we see that

$$0 = \langle \psi, A\psi \rangle = \sum_{j \in \mathbb{N}} \overline{\langle \varphi_j, \psi \rangle} \frac{\langle \varphi_j, \psi \rangle}{j} = \sum_{j \in \mathbb{N}} \frac{1}{j} | \langle \varphi_j, \psi \rangle |^2$$

Since this sum of nonnegative terms equals 0, each term must also be 0, and so

$$\langle \varphi_j, \psi \rangle = 0 \; \forall j \in \mathbb{N} \implies \|\psi\|^2 = \sum_{j \in \mathbb{N}} | \langle \varphi_j, \psi \rangle |^2 = 0,$$

where the implication follows from the other part of Theorem 7.27. This contradicts that $\|\psi\| = 1$, and so $0 \notin N(A)$. Thus, $N(A)$ is not closed and $\sigma(A) \not\subseteq N(A)$.

(c) Let $\mathcal{H}$ be an arbitrary Hilbert space, $M \subsetneq \mathcal{H}$ a nonempty proper closed subspace, and P be the orthogonal projection operator onto M. We know that $\sigma(P) \subseteq \{0, 1\}$ by Claim 8.20 in the notes. However, we claim that $N(P) = [0, 1]$. Firstly, note that $P^2 = P^* = P \implies P = |P|^2$, and so P is positive. By Lemma 9.5 in the notes, $N(P) \subseteq [0, \infty)$. Next, note that by Problem 27 on Problem Set 7, self-adjointness of P tells us that $1 = \|P\| = \sup\{|z| : \; z \in N(P)\}$, and so $N(P) \subseteq [0, 1]$. For the reverse inclusion, suppose that $\lambda \in [0, 1]$. Let $v \in M$ be a unit vector and $w \in M^\perp$ be a unit vector orthogonal to v. Define $\psi := \sqrt{\lambda} v + \sqrt{1 - \lambda} w$. Then, we have that

$$\|\psi\|^2 = \left\langle \sqrt{\lambda} v + \sqrt{1-\lambda} w, \sqrt{\lambda} v + \sqrt{1-\lambda} w \right\rangle = \lambda \|v\|^2 + (1-\lambda)\|w\|^2 + 2 \left\langle \sqrt{\lambda} v, \sqrt{1-\lambda} w \right\rangle = \lambda + (1 - \lambda) = 1,$$

where this follows since $v \perp w$ and $\|v\| = \|w\| = 1$. So, ψ is a unit vector. Furthermore,

$$\langle \psi, P\psi \rangle = \left\langle \sqrt{\lambda} v + \sqrt{1-\lambda} w, \sqrt{\lambda} v \right\rangle = \|\sqrt{\lambda} v\|^2 = \lambda,$$

where the above again follows since $v \perp w$ and $\|v\| = 1$. So, $\lambda \in N(P)$. Since this holds for all $\lambda \in [0, 1]$, we find that $N(P) = [0, 1]$. Therefore, $N(P)$ is closed but not equal to $\sigma(P)$. ■

Problem 5

Show that if $A \in \mathcal{B}(\mathcal{H})$ has $A = A^*$ then

$$\left\|(A - z\mathbb{1})^{-1}\right\| \leq \frac{1}{|\,\mathbb{I}\text{m}\,\{z\}\,|} \quad (\forall z \in \mathbb{C} : \ |\,\mathbb{I}\text{m}\,\{z\}\,| > 0)$$

Solution

Proof. Let $z \in \{w \in \mathbb{C} : \ |\,\mathbb{I}\text{m}\,\{w\}\,| > 0\}$ be arbitrary. Noting that $A = A^* \implies \sigma(A) \subseteq \mathbb{R}$, we see that z is not in the spectrum, which means that $A - z\mathbb{1}$ is indeed invertible. To proceed, let $\varphi \in \mathcal{H}$ be an arbitrary vector with $\|\varphi\| = 1$. In Lemma 10.2 in the notes, we derived that

$$\mathbb{I}\text{m}\,\{z\} \left\|(A - z\mathbb{1})^{-1}\varphi\right\|^2 = \mathbb{I}\text{m}\left\{\left\langle \varphi, (A - z\mathbb{1})^{-1}\varphi \right\rangle\right\}$$

Taking the magnitude of both sides,

$$|\,\mathbb{I}\text{m}\,\{z\}\,| \left\|(A - z\mathbb{1})^{-1}\varphi\right\|^2 = \left|\mathbb{I}\text{m}\left\{\left\langle \varphi, (A - z\mathbb{1})^{-1}\varphi \right\rangle\right\}\right|$$

We note that by Cauchy-Schwartz and the fact that $|\,\mathbb{I}\text{m}\,\{\gamma\}\,| \leq |\gamma|$ for all $\gamma \in \mathbb{C}$, we have

$$\left|\mathbb{I}\text{m}\left\{\left\langle \varphi, (A - z\mathbb{1})^{-1}\varphi \right\rangle\right\}\right| \leq \left|\left\langle \varphi, (A - z\mathbb{1})^{-1}\varphi \right\rangle\right| \leq \|\varphi\| \|(A - z\mathbb{1})^{-1}\varphi\| = \|(A - z\mathbb{1})^{-1}\varphi\|,$$

where the last equality follows since φ is a unit vector. So, we find that

$$|\,\mathbb{I}\text{m}\,\{z\}\,| \left\|(A - z\mathbb{1})^{-1}\varphi\right\|^2 \leq \|(A - z\mathbb{1})^{-1}\varphi\|$$

Since $\varphi \neq 0 \implies (A - z\mathbb{1})^{-1}\varphi \neq 0$ by invertibility, and so we divide and see that

$$|\,\mathbb{I}\text{m}\,\{z\}\,| \left\|(A - z\mathbb{1})^{-1}\varphi\right\| \leq 1$$

Thus,

$$\left\|(A - z\mathbb{1})^{-1}\varphi\right\| \leq \frac{1}{|\,\mathbb{I}\text{m}\,\{z\}\,|}$$

Since this holds for all unit vectors $\varphi \in \mathcal{H}$, we find the operator norm bound

$$\left\|(A - z\mathbb{1})^{-1}\right\| \leq \frac{1}{|\,\mathbb{I}\text{m}\,\{z\}\,|}$$

As this holds for all $z \in \{w \in \mathbb{C} : \ |\,\mathbb{I}\text{m}\,\{w\}\,| > 0\}$, we have proven the desired result. ■

Problem 6

Show that if $A \in \mathcal{B}(\mathcal{H})$ is an isometry then $\operatorname{im}(A)$ is closed in $\mathcal{H}$.

Solution

Proof. Suppose that $A \in \mathcal{B}(\mathcal{H})$ is an isometry. Then, we know that

$$\|A\varphi\| = \|\varphi\| \quad (\forall \varphi \in \mathcal{H})$$

So, we clearly have that

$$\|A\varphi\| \geq \|\varphi\| \quad \big(\forall \varphi \in (\ker A)^{\perp}\big)$$

By Lemma 7.20 in the notes, this gives us that $\operatorname{im}(A) \in \operatorname{Closed}(\mathcal{H})$. ∎

Problem 7

Define $\mathcal{H} := L^2([0,1] \to \mathbb{C})$. Let $V \in \mathcal{B}(\mathcal{H})$ be given by

$$V(\psi) := \int_0^{\cdot} \psi \quad (\forall \psi \in \mathcal{H})$$

(a) Show that V is well-defined (it is a bounded linear map) with

$$V^*(\psi) = \int_{\cdot}^{1} \psi \quad (\forall \psi \in \mathcal{H})$$

(b) Show that the spectral radius $r(V)$ equals 0 and that $\sigma(V) = \{0\}$.

(c) Show that $\|V\| = \frac{2}{\pi}$.

Solution

Proof. For this problem, note that $\mathcal{H} \subseteq L^1([0,1] \to \mathbb{C})$ since the domain $[0,1]$ is finite measure. By Holder's inequality, $\|f\|_{L^1} \leq \|f\|_{L^2}$ for all $f \in \mathcal{H}$. We apply this estimate in this problem often to see that $\int_0^1 |f| \leq \|f\|_{\mathcal{H}}$.

(a) We note that for $\alpha \in \mathbb{C}$ and $\psi, \varphi \in \mathcal{H}$, we have that for all $x \in [0,1]$,

$$V(\alpha\psi + \varphi)(x) = \int_0^x (\alpha\psi + \varphi) = \alpha \int_0^x \psi + \int_0^x \varphi = \alpha V(\psi)(x) + V(\varphi)(x)$$

by linearity of the integral, and so $V(\alpha\psi + \varphi) = \alpha V(\psi) + V(\varphi)$ and V is linear. To show boundedness, note that for all $\psi \in \mathcal{H}$,

$$\|V\psi\|^2 = \int_0^1 \mathrm{d}x \left| \int_0^x \psi \right|^2 \leq \int_0^1 \mathrm{d}x \left(\int_0^x |\psi| \right)^2 \leq \int_0^1 \mathrm{d}x \left(\int_0^1 |\psi| \right)^2 = \left(\int_0^1 |\psi| \right)^2$$

By an application of Holder's inequality ($\|\cdot\|_{L^1} \leq \|\cdot\|_{L^p}$), we know that for all $\psi \in \mathcal{H}$,

$$\int_0^1 |\psi| \leq \|\psi\|,$$

where the above norm is the $\mathcal{H}$ norm. Thus,

$$\|V\psi\| \leq \|\psi\|,$$

and so V is a bounded operator. To see the adjoint result, define $W \in \mathcal{B}(\mathcal{H})$ to be

$$W(\psi) := \int_{\cdot}^{1} \psi \quad (\psi \in \mathcal{H}),$$

where we know that W is bounded and linear by identical logic to the above. Now, observe that for all $\psi, \varphi \in \mathcal{H}$,

$$\langle \psi, V\varphi \rangle = \int_0^1 \overline{\psi(x)} \int_0^x \varphi(t) dt \mathrm{d}x = \int_0^1 \int_0^x \overline{\psi(x)} \varphi(t) dt \mathrm{d}x$$

For each $x \in [0,1]$, define $\chi_x := \chi_{[0,x]}$ to be the indicator function of the interval $[0,x]$. Then,

$$\langle \psi, V\phi \rangle = \int_0^1 \int_0^1 \chi_x(t) \overline{\psi(x)} \varphi(t) dt \mathrm{d}x$$

Problem 7 continued on next page...

By Tonelli's theorem, we compute

$$\begin{aligned}\int_0^1 \int_0^1 |\chi_x(t)\overline{\psi(x)}\varphi(t)| dt \mathrm{d}x &= \int_0^1 \int_0^1 |\chi_x(t)\overline{\psi(x)}\varphi(t)| \mathrm{d}x dt \\ &\leq \int_0^1 |\varphi(t)| \int_0^1 |\psi(x)| \mathrm{d}x dt \\ &\leq \|\psi\| \int_0^1 |\varphi(t)| dt \\ &\leq \|\psi\| \|\varphi\| < \infty\end{aligned}$$

where we again used the Holder estimate $\int_0^1 |\psi(x)| \mathrm{d}x \leq \|\psi\|$. Since this is finite, we may apply Fubini's theorem and switch the order of integration, getting

$$\langle \psi, V\varphi \rangle = \int_0^1 \int_0^1 \chi_x(t)\overline{\psi(x)}\varphi(t) \mathrm{d}x dt = \int_0^1 \varphi(t) \left(\int_0^1 \chi_x(t)\overline{\psi(x)} \mathrm{d}x \right) dt$$

We note that $\chi_x(t) = 1$ if and only if $t \leq x$, and otherwise it is 0. So, $\chi_x(t) = \chi_{[t,1]}(x)$, and so

$$\int_0^1 \chi_x(t)\overline{\psi(x)} \mathrm{d}x = \int_t^1 \overline{\psi(x)} \mathrm{d}x = \overline{\int_t^1 \psi(x) \mathrm{d}x} = \overline{(W\psi)(t)}$$

Plugging this in,

$$\langle \psi, V\varphi \rangle = \int_0^1 \phi(t)\overline{(W\psi)(t)} dt = \langle W\psi, \varphi \rangle$$

Since this holds for all $\psi, \varphi \in \mathcal{H}$, we have that $V^* = W$ as desired.

(b) For all $n \in \mathbb{N}$ and all $\varphi \in \mathcal{H}$,

$$\begin{aligned}\|V^n\varphi\|^2 &= \int_0^1 \left| \int_0^x dt_n \int_0^{t_n} dt_{n-1} \dots dt_2 \int_0^{t_2} \varphi(t_1) dt_1 \right|^2 \mathrm{d}x \\ &\leq \int_0^1 \left(\int_0^x dt_n \int_0^{t_n} dt_{n-1} \dots dt_2 \int_0^{t_2} |\varphi(t_1)| dt_1 \right)^2 \mathrm{d}x \\ &\leq \int_0^1 \left(\int_0^x dt_n \int_0^{t_n} dt_{n-1} \dots dt_2 \int_0^{1} |\varphi(t_1)| dt_1 \right)^2 \mathrm{d}x \\ &\leq \int_0^1 \left(\int_0^x dt_n \int_0^{t_n} dt_{n-1} \dots dt_2 \|\varphi\| \right)^2 \mathrm{d}x \\ &= \|\varphi\|^2 \int_0^1 \left(\int_0^x dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_3} dt_2 \right)^2 \mathrm{d}x \\ &= \|\varphi\|^2 \int_0^1 \left(\frac{x^{n-1}}{(n-1)!} \right)^2 \mathrm{d}x = \frac{\|\varphi\|^2}{(n-1)!^2} \int_0^1 x^{2n-2} \mathrm{d}x \\ &= \frac{\|\varphi\|^2}{(n-1)!^2} \frac{1}{2n-1},\end{aligned}$$

where we used a Holder estimate for the third inequality. Since this holds for all $\varphi \in \mathcal{H}$, we find that

$$\|V^n\| \leq \frac{1}{(n-1)!\sqrt{2n-1}} \leq \frac{1}{(n-1)!} = \frac{n}{n!}$$

Note that $(n!)^2 = (1(n))(2(n-1))(3(n-2)) \dots (n(1)) = \prod_{k=0}^{n-1} (k+1)(n-k)$. We know $(k+1)(n-k) = nk + n - k(k+1)$. Since $n \geq k+1$, we see that $(k+1)(n-k) \geq nk + n - kn = n$. Thus,

$$(n!)^2 \geq \prod_k = 0^{n-1} n = n^n \implies (n!)^{1/n} \geq \sqrt{n}$$

So,

$$\|V^n\|^{1/n} \leq \frac{n^{1/n}}{\sqrt{n}}$$

We know that $n^{1/n} \to 1$ as $n \to \infty$, and so

$$\lim_{n\to\infty} \|V^n\|^{1/n} = 0$$

Thus, by Gelfand's formula, $r(V) = 0$ and therefore $\sigma(V) = \{0\}$ (the spectrum is nonempty and can only contain 0).

(c) Let $K(x,t) := \chi_{[0,x]}(t)$. Then, we have that

$$(V\psi)(x) = \int_0^x \psi(t)dt = \int_0^1 K(x,t)\psi(t)dt$$

Certainly, $K \in L^2([0,1]\times[0,1])$, and so V is a Hilbert-Schmidt integral operator and is therefore compact. So, $|V|^2 = V^*V$ is a compact, positive operator. The Riesz-Schauder theorem tells us that $r(|V|^2)$ is the magnitude of the largest eigenvalue, since the spectrum of $|V|^2$ is discrete and can only accumulate at 0 (in particular, a magnitude of $r(|V|^2)$ is attained by some eigenvalue). Since $|V|^2$ is self-adjoint, we know that $\|V\|^2 = \||V|^2\| = r(|V|^2)$ by the C^* identity, and so we seek the magnitude of the largest eigenvalue. To this end, suppose that ψ is an eigenvector of $|V|^2$ with eigenvalue $\lambda \neq 0$ (since $|V|^2$ is positive, then $\lambda > 0$). Then, for a.e. $x \in [0,1]$ we have that

$$\lambda\psi(x) = (V^*V\psi)(x) = \int_x^1 \left(\int_0^t \psi(s)ds\right) dt$$

Note that

$$|\psi(y) - \psi(x)| \leq \frac{1}{|\lambda|}\int_x^y \left|\int_0^t \psi(s)ds\right| dt \leq \frac{1}{|\lambda|}\int_x^y \int_0^t |\psi(s)|dsdt$$

Using our favorite Holder estimate $\int_0^t |\psi(s)|ds \leq \int_0^1 |\psi(s)|ds \leq \|\psi\|$, we see that

$$|\psi(y) - \psi(x)| \leq \frac{\|\psi\|}{|\lambda|}|y-x|$$

In particular, ψ is Lipschitz and so differentiable a.e.. Taking a derivative of our initial expression, we see that

$$\lambda\psi'(x) = -\int_0^x \psi(s)ds$$

From this we see that $\psi'(0) = 0$. Applying very similar logic, as above, we have that

$$|\psi'(y) - \psi'(x)| \leq \frac{1}{|\lambda|}\int_x^y |\psi(s)|ds$$

ψ is Lipschitz, and so continuous, which means it is bounded on $[0,1]$, i.e. $|\psi(s)| \leq M < \infty$ for $s \in [0,1]$. Therefore ψ' is $\frac{M}{|\lambda|}$-Lipschitz, which means that ψ' is a.e. differentiable. So, we may take another derivative and see that for a.e. $x \in [0,1]$,

$$\lambda\psi''(x) = -\psi(x) \implies \psi(x) = C_1\cos(x/\sqrt{\lambda}) + C_2\sin(x/\sqrt{\lambda})$$

for some constants C_1, C_2. We know that $\psi'(0) = 0$, and so $C_2 = 0$. Also, since $(V^*V\psi)(1) = 0$ we have $\psi(1) = 0$. Therefore,

$$\cos(1/\sqrt{\lambda}) = 0 \implies \frac{1}{\sqrt{\lambda}} = \left(k + \frac{1}{2}\right)\pi \text{ for some } k \in \mathbb{N}$$

The above holds for any $k \in \mathbb{N}$ (note that we cannot have $k < \frac{1}{2}$ since the LHS is positive), and so we seek the k that maximizes λ_k. We write

$$\lambda_k = \left(\frac{2}{(2k+1)\pi}\right)^2 \implies \max_{k \in \mathbb{N}}\{\lambda_k\} = \frac{4}{\pi^2}$$

Therefore, $r(|V|^2) = \frac{4}{\pi^2}$ and we get our answer that $\|V\| = \frac{2}{\pi}$. ■

Problem 8

Let $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{S}^1)$ be the Fourier series given by

$$\ell^2(\mathbb{Z}) \ni \psi \mapsto \left([0, 2\pi] \ni k \mapsto \sum_{n\in\mathbb{Z}} e^{-ikn}\psi_n =: \hat{\psi}(k) \right)$$

Let $A \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be the discrete Laplacian:

$$A = R + R^*,$$

where R is the bilateral right shift operator.

$$R\delta_n := \delta_{n+1} \quad (\forall n \in \mathbb{Z})$$

and $\{\delta_n\}_{n\in\mathbb{Z}}$ is the standard basis of $\ell^2(\mathbb{Z})$. Calculate

$$\mathcal{F}A\mathcal{F}^* \in \mathcal{B}(L^2(\mathbb{S}^1))$$

Solution

Proof. Write $\mathcal{H}_1 := \ell^2(\mathbb{Z})$ and $\mathcal{H}_2 := L^2(\mathbb{S}^1)$ for notation. For each $n \in \mathbb{Z}$, define $\varphi_n \in \mathcal{H}_2$ via $\varphi_n(x) = e^{-inx}$ for $x \in [0, 2\pi]$. We know by elementary Fourier analysis that $\{\varphi_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{H}_2$ (assuming the inner product is normalized by $\frac{1}{2\pi}$). Then, letting $\{\delta_n\}_{n\in\mathbb{Z}}$ denote the standard orthonormal basis of $\mathcal{H}_1$, we see that

$$\mathcal{F}(\delta_n) = \varphi_n \quad (n \in \mathbb{Z})$$

So, $\mathcal{F}$ is unitary. For any $f \in \mathcal{H}_1$, we may express

$$f = \sum_{n\in\mathbb{Z}} \langle \varphi_n, f \rangle \varphi_n,$$

where this convergence is in the $\mathcal{H}_2$ norm. Applying $\mathcal{F}^*$, we get that

$$\mathcal{F}^* f = \sum_{n\in\mathbb{Z}} \langle \varphi_n, f \rangle \delta_n,$$

where this convergence is in the $\mathcal{H}_1$ norm since $\mathcal{F}^*$ preserves the norm. Applying A,

$$(A\mathcal{F}^*)(f) = \sum_{n\in\mathbb{Z}} \langle \varphi_n, f \rangle (\delta_{n+1} + \delta_{n-1})$$

Applying $\mathcal{F}$ again,

$$(\mathcal{F}A\mathcal{F}^*)(f) = \sum_{n\in\mathbb{Z}} \langle \varphi_n, f \rangle (\varphi_{n+1} + \varphi_{n-1}) = \sum_{n\in\mathbb{N}} \langle \varphi_{n+1} + \varphi_{n-1}, f \rangle \varphi_n,$$

where we shifted indices to get the second sum. This convergence is again in the $\mathcal{H}_2$ norm. Consider the function $g \in \mathcal{H}_2$ given by $g(x) = 2\cos(x)f(x)$ (which is certainly in $\mathcal{H}_2$ since $|\cos(x)| \leq 1$). We claim that $g = (\mathcal{F}A\mathcal{F}*)(f)$, or equivalently that as $N \to \infty$,

$$\left\| g - \sum_{n=-N}^{N} \langle \varphi_{n+1} + \varphi_{n-1}, f \rangle \varphi_n \right\|_{\mathcal{H}_2} \to 0$$

To see this, note that $g = \sum_{n\in\mathbb{Z}} \langle \varphi_n, g \rangle \varphi_n$ in norm, and so by the triangle inequality,

$$\left\| g - \sum_{n=-N}^{N} \langle \varphi_{n+1} + \varphi_{n-1}, f \rangle \varphi_n \right\|_{\mathcal{H}_2} \leq \left\| \sum_{|n|>N} \langle \varphi_n, g \rangle \varphi_n \right\|_{\mathcal{H}_2} + \left\| \sum_{n=-N}^{N} (\langle \varphi_n, g \rangle - \langle \varphi_{n+1} + \varphi_{n-1}, f \rangle)\varphi_n \right\|_{\mathcal{H}_2}$$

To see that the second term is actually always 0, we apply the Pythagorean identity to see

$$\left\| \sum_{n=-N}^{N} (\langle \varphi_n, g\rangle - \langle \varphi_{n+1} + \varphi_{n-1}, f\rangle)\varphi_n \right\|_{\mathcal{H}_2}^2 = \sum_{n=-N}^{N} |\langle \varphi_n, g\rangle - \langle \varphi_{n+1} + \varphi_{n-1}, f\rangle|^2$$

For each n, we may compute that since $g(x) = 2\cos(x)f(x)$,

$$\begin{aligned}\langle \varphi_n, g\rangle - \langle \varphi_{n+1} + \varphi_{n-1}, f\rangle &= \frac{1}{2\pi}\int_{[0,2\pi]} e^{inx} 2\cos(x)f(x) - (e^{i(n+1)x} + e^{i(n-1)x})f(x)\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{[0,2\pi]} e^{inx} f(x)(2\cos(x) - (e^{ix} + e^{-ix}))\mathrm{d}x\end{aligned}$$

Since $e^{ix} + e^{-ix} = 2\cos(x)$ for all x, this integral is identically 0. Therefore, we find that

$$\left\| g - \sum_{n=-N}^{N} \langle \varphi_{n+1} + \varphi_{n-1}, f\rangle \varphi_n \right\|_{\mathcal{H}_2} \leq \left\| \sum_{|n|>N} \langle \varphi_n, g\rangle \varphi_n \right\|_{\mathcal{H}_2} = \left\| g - \sum_{n=-N}^{N} \langle \varphi_n, g\rangle \varphi_n \right\|_{\mathcal{H}_2}$$

We know that the sum $g = \sum_{n\in\mathbb{Z}} \langle \varphi_n, g\rangle \varphi_n$ converges in norm, and so the right hand side of the above inequality must go to 0. Thus, $(\mathcal{F}A\mathcal{F}^*)(f) = g$. Since this holds for each $f \in \mathcal{H}_2$, we find that $\mathcal{F}A\mathcal{F}^* \in \mathcal{B}(\mathcal{H}_2)$ is the multiplication operator by the map $[0, 2\pi] \ni \theta \to 2\cos(\theta)$. ■