MAT 520: Problem Set 7

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Problem 1

Show that if $a \in \mathcal{A}$ is a partial isometry (i.e. $|a|^2$ is an idempotent) then $a = aa^*a = aa^*aa^*a$.

Solution

Proof. Noting that both $\mathbb{1}$ and $|a^*|^2$ are self-adjoint, we may compute

$$\begin{aligned}\left|(\mathbb{1} - |a^*|^2)a\right|^2 &= a^*(\mathbb{1} - |a^*|^2)(\mathbb{1} - |a^*|^2)a \\ &= a^*(\mathbb{1} - 2|a^*|^2 + (|a^*|^2)^2)a \\ &= a^*(\mathbb{1} - 2aa^* + aa^*aa^*)a \\ &= a^*a - 2a^*aa^*a + a^*aa^*aa^*a \\ &= |a|^2 - 2\left(|a|^2\right)^2 + \left(|a|^2\right)^3\end{aligned}$$

Since $|a|^2$ is an idempotent, $|a|^2 = \left(|a|^2\right)^2 = \left(|a|^2\right)^3$, and so

$$\left|(\mathbb{1} - |a^*|^2)a\right|^2 = |a|^2 - 2|a|^2 + |a|^2 = 0$$

Therefore, by Claim 8.2 we see that

$$(\mathbb{1} - |a^*|^2)a = 0 \implies a - aa^*a = 0 \implies a = aa^*a$$

Then, we see that

$$a = aa^*a = a(a^*a) = (aa^*a)a^*a = aa^*aa^*a$$

and the result is proven. ■

Problem 2

Show that a is a partial isometry iff a^* is a partial isometry.

Solution

Proof. We will only show one direction, as the other follows by symmetry. Suppose that a is a partial isometry. By definition,

$$\left(|a^*|^2\right)^2 = aa^*aa^*$$

By Problem 1, since a is a partial isometry we know $a = aa^*a$. Plugging this in,

$$\left(|a^*|^2\right)^2 - (aa^*a)a^* = aa^* = |a^*|^2$$

So, $|a^*|^2$ is an idempotent, and thus a^* is a partial isometry. ∎

Problem 3

Show that if p, q are self-adjoint projections then $\|p-q\| \leq 1$.

Solution

Proof. We know that $p^2 = p^* = p$ and $q^2 = q^* = q$ by assumption. Using this, we can compute

$$(p-q)^2 + (p+q)^2 = p^2 - pq - qp + q^2 + p^2 + pq + qp + q^2 = 2p^2 + 2q^2 = 2(p+q)$$

So,

$$(p-q)^2 + \mathbb{1} - 2(p+q) + (p+q)^2 = \mathbb{1}$$

Note that as $(\mathbb{1} - (p+q))^2 = \mathbb{1} - 2(p+q) + (p+q)^2$, we may express

$$(p-q)^2 + (\mathbb{1} - p - q)^2 = \mathbb{1} \implies \mathbb{1} - (p-q)^2 = (\mathbb{1} - p - q)^2$$

Since $(\mathbb{1} - p - q)^2 \geq 0$ (it is the square of a self-adjoint element), this means that $\mathbb{1} - (p-q)^2$ is positive as well. In other words,

$$0 \leq (p-q)^2 \leq \mathbb{1} \implies 0 \leq |p-q|^2 \leq \mathbb{1},$$

where the implication follows since $p - q$ is self-adjoint. By Corollary 8.17, $\||p-q|^2\| \leq \|\mathbb{1}\| = 1$. By the C^* identity,

$$1 \geq \||p-q|^2\| = \|p-q\|^2 \implies \|p-q\| \leq 1$$

as desired. ■

Problem 4

Show that if u, v are unitary then $\|u - v\| \leq 2$.

Solution

Proof. We know by Lemma 8.5 in the lecture notes that $\|u\| = \|v\| = 1$, and so $\|-v\| = 1$ as well. The triangle inequality immediately yields

$$\|u - v\| \leq \|u\| + \|-v\| = 2,$$

which is what we wanted to show. ■

Problem 5

Show that if a is self-adjoint with $\|a\| \leq 1$ then

$$a + i\sqrt{\mathbb{1} - a^2}, \qquad a - i\sqrt{\mathbb{1} - a^2}$$

are unitary. Conclude that any $b \in \mathcal{A}$ is the linear combination of four unitaries.

Solution

Proof. Write $a_+ := a + i\sqrt{\mathbb{1} - a^2}$ and $a_- := a - i\sqrt{\mathbb{1} - a^2}$ for notation. Note that

$$a_+a_- = (a + i\sqrt{\mathbb{1} - a^2})(a - i\sqrt{\mathbb{1} - a^2}) = a^2 + ia\sqrt{\mathbb{1} - a^2} - ia\sqrt{\mathbb{1} - a^2} + (\mathbb{1} - a^2) = \mathbb{1}$$

and

$$a_-a_+ = (a - i\sqrt{\mathbb{1} - a^2})(a + i\sqrt{\mathbb{1} - a^2}) = a^2 - ia\sqrt{\mathbb{1} - a^2} + ia\sqrt{\mathbb{1} - a^2} + (\mathbb{1} - a^2) = \mathbb{1}$$

Furthermore,

$$(a_+)^* = a^* + (i\sqrt{\mathbb{1} - a^2})^* = a^* - i\sqrt{\mathbb{1} - a^2} = a - i\sqrt{\mathbb{1} - a^2} = a_-,$$

where we used for the second equality that square roots are positive and therefore self-adjoint (see Theorems 8.12 and 8.14 in the lecture notes) and for the third equality we used that a is self-adjoint. So, a_+ and a_- are adjoints, meaning that the first two relations we derived translate to

$$(a_-)^*a_- = |a_-|^2 = \mathbb{1} \quad \text{and} \quad (a_+)^*a_+ = |a_+|^2 = \mathbb{1}$$

Therefore, both a_+ and a_- are unitary. In particular, since $a = \frac{a_+ + a_-}{2}$, we see that any self-adjoint element in the unit ball can be written as the linear combination of two unitaries.

Now, start with $b \in \mathcal{A}$ arbitrary. Then,

$$b = \frac{b + b^*}{2} + i\frac{b - b^*}{2i} =: b_1 + ib_2,$$

and so $b_1^* = \frac{b^* + b}{2} = b_1$ and $b_2^* = \frac{1}{-2i}(b - b^*)^* = \frac{1}{-2i}(b^* - b) = b_2$. Thus, b_1 and b_2 are self-adjoint, and so $\frac{b_1}{\|b_1\|}$ and $\frac{b_2}{\|b_2\|}$ are self-adjoint and unit norm. We may then apply the previous reasoning with $\frac{b_1}{\|b_1\|}$ and $\frac{b_2}{\|b_2\|}$ in place of a to see that $\frac{b_1}{\|b_1\|}$ and $\frac{b_2}{\|b_2\|}$ may both be expressed as the linear combinations of two unitaries each. So, since b is a linear combination of these two elements, we have expressed b as a linear combination of four unitaries. Explicitly,

$$\begin{aligned} b = \frac{\|b_1\|}{2}\left(\frac{b_1}{\|b_1\|} + i\sqrt{\mathbb{1} - \left(\frac{b_1}{\|b_1\|}\right)^2}\right) + \frac{\|b_1\|}{2}\left(\frac{b_1}{\|b_1\|} - i\sqrt{\mathbb{1} - \left(\frac{b_1}{\|b_1\|}\right)^2}\right) \\ + \frac{i\|b_2\|}{2}\left(\frac{b_2}{\|b_2\|} + i\sqrt{\mathbb{1} + \left(\frac{b_2}{\|b_2\|}\right)^2}\right) + \frac{i\|b_2\|}{2}\left(\frac{b_2}{\|b_2\|} - i\sqrt{\mathbb{1} + \left(\frac{b_2}{\|b_2\|}\right)^2}\right), \end{aligned}$$

where each element in a big parenthesis is unitary. ■

Problem 6

Two self-adjoint projections p, q are said to be orthogonal (written $p \perp q$) if $pq = 0$. Show that the following are equivalent:

(a) $p \perp q$.

(b) $p + q$ is a self-adjoint projection.

(c) $p + q \leq \mathbb{1}$.

Solution

Proof. (a $\implies$ b) Suppose that $pq = 0$. Then, $0 = (pq)^* = q^* p^* = qp$. Now, note that $p + q$ is certainly self-adjoint, and so we must show it is idempotent. To this end, note that

$$(p+q)^2 = p^2 + pq + qp + q^2 = p + 0 + 0 + q = p + q,$$

where we used that both p and q are idempotent. So, $p + q$ is self-adjoint and idempotent, and is therefore a self-adjoint projection.

(b $\implies$ c) Suppose that $p + q$ is a self-adjoint projection, and so $(p + q)^2 = p + q$. We want to show that $\mathbb{1} - (p + q)$ is positive; we will do so by exhibiting its square root. Observe that

$$\begin{aligned}(\mathbb{1} - (p+q))^2 &= \mathbb{1} + (p+q)^2 - 2(p+q) \\ &= \mathbb{1} + (p+q) - 2(p+q) \\ &= \mathbb{1} - (p+q)\end{aligned}$$

Since $\mathbb{1} - (p + q)$ is self-adjoint, we see that $|\mathbb{1} - p - q|^2 = \mathbb{1} - (p + q)$, and so $\mathbb{1} - (p + q)$ is positive by definition of positivity. Then, $p + q \leq \mathbb{1}$.

(c $\implies$ a) Suppose now that $p + q \leq \mathbb{1}$, and so $\mathbb{1} - (p + q)$ is positive. Note that

$$q(\mathbb{1} - (p+q))q = q^2 - qpq - q^3 = -qpq = -qp^2q = -|pq|^2,$$

where we used that $(pq)^* = q^* p^* = qp$. Since $|pq|^2$ is positive by definition, we see that $-|pq|^2 \leq 0$. Therefore,

$$q(\mathbb{1} - (p+q))q \leq 0$$

However, we claim that $q(\mathbb{1} - (p + q))q \geq 0$; to show this, let $\mathbb{1} - (p + q) =: a \geq 0$ for notation. Then, there is a b such that $a = |b|^2$. Thus,

$$qaq = qaq^* = qb^*bq^* = |bq^*|^2 \geq 0$$

So, $0 \leq q(\mathbb{1} - (p + q))q \leq 0$, which means that $q(\mathbb{1} - (p + q))q = 0$. As such, $-|pq|^2 = 0$ as well, directly implying that $pq = 0$. So, $p \perp q$. ■

Problem 7

Let $v_1, \dots, v_n$ be partial isometries and suppose that

$$\sum_{j=1}^{n} |v_j|^2 = \sum_{j=1}^{n} |v_j^*|^2 = \mathbb{1}.$$

Show that $v := \sum_{j=1}^{n} v_j$ is unitary.

Solution

Proof. Write $p_j := |v_j|^2$, which is a self-adjoint projection by partial isometry of v_j. We know that each p_j is positive and $\sum_{j=1}^{n} p_j = \mathbb{1}$. For every $j \neq k$, we then have that

$$\mathbb{1} - (p_j + p_k) = \sum_{\substack{i=1 \\ i \neq j,k}}^{n} p_i \geq 0$$

since the sum of positive operators is positive (Lemma 8.11 in the lecture notes). Thus, $p_j + p_k \leq 1$, which by the previous problem means that they are orthogonal and $p_j p_k = 0$ for all $j \neq k$. So,

$$0 = v_j^* v_j v_k^* v_k$$

Multiplying by v_j on the left and v_k^* on the right and observing that $u = uu^*u$ for partial isometries u (Lemma 8.3 in the lecture notes),

$$0 = (v_j v_j^* v_j)(v_k^* v_k v_k^*) = v_j v_k^*$$

Since this holds for all $j \neq k$, we see that

$$|v^*|^2 = vv^* = \left(\sum_{j=1}^{n} v_j\right)\left(\sum_{k=1}^{n} v_k^*\right) = \sum_{j,k=1}^{n} v_j v_k^* = \sum_{j=1}^{n} v_j v_j^* = \sum_{j=1}^{n} |v_j^*|^2 = \mathbb{1}$$

Applying similar logic starting from $q_j := |v_j^*|^2$ shows that $v_j^* v_k = 0$ for $j \neq k$, and thus that $|v|^2 = \mathbb{1}$. Therefore, v is unitary. ■

Problem 17

Show that

$$\mathcal{H} := \ell^2(\mathbb{R}) \equiv \left\{ f : \mathbb{R} \to \mathbb{C} \mid f^{-1}(\mathbb{C} \setminus \{0\}) \text{ is a countable set and } \sum_{x \in \mathbb{R}} |f(x)|^2 < \infty \right\}$$

is not a separable Hilbert space.

Solution

Proof. Suppose by way of contradiction that $\mathcal{H}$ had a countable dense subset, say $\{f_n\}_{n \in \mathbb{N}}$. For every set $E \subseteq \mathbb{R}$, define the function $\mathbb{1}_E : \mathbb{R} \to \mathbb{C}$ via

$$\mathbb{1}_E(x) := \begin{cases} 0 & x \notin E \\ 1 & x \in E \end{cases}$$

to be the indicator function for the set E. Write

$$\mathcal{S} := \{E \subseteq \mathbb{R} : \ E \text{ is finite}\}$$

Certainly, $\mathcal{S}$ is uncountable since it contains all the singletons of $\mathbb{R}$. For every $E \in \mathcal{S}$, we see that $\mathbb{1}_E \in \mathcal{H}$. However, for any two distinct finite sets $A, B \in \mathcal{S}$, we must have that $\mathbb{1}_A$ and $\mathbb{1}_B$ disagree at at least one point, and so

$$\|\mathbb{1}_A - \mathbb{1}_B\|_{\mathcal{H}} = \sum_{x \in \mathbb{R}} |\mathbb{1}_A(x) - \mathbb{1}_B(x)|^2 \geq 1$$

Therefore, we have that the family of open balls

$$\mathcal{B} := \left\{ B_{1/2}(\mathbb{1}_E) : \ E \in \mathcal{S} \right\}$$

is disjoint, where $B_r(f) := \{g \in \mathcal{H} : \ \|f - g\|_{\mathcal{H}} < r\}$ is the open ball of radius r around $f \in \mathcal{H}$. Furthermore, $\mathcal{B}$ will have the same cardinality as $\mathcal{S}$; in particular, it is uncountable. So, $\mathcal{B}$ is an uncountable collection of disjoint open sets, which will lead us to a contradiction.

Since $\{f_n\}_n$ is dense, we see that for every $B \in \mathcal{B}$ there must be some n for which $f_n \in B$ (this uses the characterization of density as having nonempty intersection with every open set). Since the balls are disjoint, for any distinct $B_1, B_2 \in \mathcal{B}$ we know that if $f_n \in B_1$, then $f_n \notin B_2$. In other words, every ball in $\mathcal{B}$ contains an f_n, and no two balls in $\mathcal{B}$ may contain the same f_n. Thus, the map sending a ball to the index of the element of $\{f_n\}_n$ which it contains is an injective map from $\mathcal{B} \to \mathbb{N}$. This is a contradiction by uncountability of $\mathcal{B}$, and so there can be no countable dense subset. Therefore, $\mathcal{H}$ isn't separable. ■

Problem 18

Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$:

$$Re_j := e_{j+1} \quad (j \in \mathbb{N})$$

where $\{e_j\}_{j\in\mathbb{N}}$ is the standard basis of $\ell^2(\mathbb{N})$ and extend linearly.

(a) Calculate R^*.

(b) Calculate $|R|^2$ and $|R^*|^2$.

(c) Show that R is a partial isometry.

(d) Calculate $\sigma(R)$, $\sigma(R^*)$, $\sigma\left(|R|^2\right)$, and $\sigma\left(|R^*|^2\right)$.

Solution

Proof. (a) We claim that $R^* = L$, the unilateral left shift operator defined on the basis by

$$Le_j := \begin{cases} e_{j-1} & j > 1 \\ 0 & j = 1 \end{cases}$$

and extended linearly. To see that they are adjoints, let $\varphi, \psi \in \ell^2(\mathbb{N})$ be arbitrary. We may therefore express

$$\varphi \equiv \sum_{j\in\mathbb{N}} \varphi_j e_j \quad \text{and} \quad \psi \equiv \sum_{j\in\mathbb{N}} \psi_j e_j$$

for $\varphi_j, \psi_j \in \mathbb{C}$. As such, we see that

$$\langle L\varphi, \psi\rangle = \left\langle \sum_{j>1} \varphi_j e_{j-1}, \psi \right\rangle = \left\langle \sum_{j\in\mathbb{N}} \varphi_{j+1} e_j, \psi \right\rangle = \sum_{j\in\mathbb{N}} \overline{\varphi_{j+1}} \psi_j$$

and

$$\langle \varphi, R\psi\rangle = \left\langle \varphi, \sum_{j\in\mathbb{N}} \psi_j e_{j+1} \right\rangle = \left\langle \varphi, \sum_{j>1} \psi_{j-1} e_j \right\rangle = \sum_{j>1} \overline{\varphi_j} \psi_{j-1} = \sum_{j\in\mathbb{N}} \overline{\varphi_{j+1}} \psi_j,$$

where the last equality simply relabeled indices. So, $\langle L\varphi, \psi\rangle = \langle \varphi, R\psi\rangle$; since this holds for all $\varphi, \psi \in \ell^2(\mathbb{N})$, they are indeed adjoints.

(b) We may now compute $|R|^2 = LR$ and $|R^*|^2 = RL$. For any $j > 1$, we have that

$$LRe_j = Le_{j+1} = e_j \quad \text{and} \quad RLe_j = Re_{j-1} = e_j$$

However, we note that

$$LRe_1 = Le_2 = e_1 \quad \text{yet} \quad RLe_1 = R0 = 0$$

since $Le_1 = 0$. As such, we find that $|R|^2 = LR = \mathbb{1}$, whereas $|R^*|^2$ is defined on the basis as

$$|R^*|^2 e_j = RLe_j = \begin{cases} e_j & j > 1 \\ 0 & j = 1 \end{cases}$$

and extended linearly.

(c) The above shows that $|R|^2 = \mathbb{1}$, which certainly is idempotent. So, R is a partial isometry.

(d) We first claim that the open unit disk is contained in the spectrum of R^*; that is, $B_1(0_\mathbb{C}) \subseteq \sigma(R^*)$. To see this, let $|\lambda| < 1$. Then, we have that $(1, \lambda, \lambda^2, \dots) \in \ell^2(\mathbb{N})$ since it is square summable by the geometric series. Note that

$$R^*(1, \lambda, \lambda^2, \dots) = (\lambda, \lambda^2, \lambda^3, \dots) = \lambda(1, \lambda, \lambda^2, \dots) \implies (\lambda\mathbb{1} - R^*)(1, \lambda, \lambda^2, \dots) = 0$$

So, $\lambda\mathbb{1} - R^*$ has nontrivial kernel and thus $\lambda \in \sigma(R^*)$. Since the spectrum is closed, this tells us that in fact we have $\overline{B_1(0_\mathbb{C})} \subseteq \sigma(R^*)$. However, since R^* is a partial isometry, $|R^*|^2$ is a self-adjoint projection and so $1 = \||R^*|^2\| = \|R^*\|^2$. Thus, the spectral radius $r(R^*) \leq 1$, which means that $\sigma(R^*) \subseteq \overline{B_1(0_\mathbb{C})}$. In total, we find that $\sigma(R^*) = \overline{B_1(0_\mathbb{C})}$. By the continuous functional calculus, since complex conjugation is continuous and maps operators to their adjoints, the spectral mapping theorem and the fact that the set of complex conjugates of the unit disk is still the unit desk together tell us that $\sigma(R) = \overline{B_1(0_\mathbb{C})}$ as well. Since $|R|^2 = \mathbb{1}$, we know immediately that $\sigma(|R|^2) = \{1\}$. Lastly, we know that $\{0, 1\} \in \sigma(|R^*|^2)$ since it is not invertible and there are vectors that get mapped to themselves. Since it is positive and has norm 1 we know that $\sigma(|R^*|^2) \subseteq [0, 1]$. Let $\lambda \in (0, 1)$. Then, for all $\varphi \equiv (\varphi_1, \varphi_2, \dots) \in \ell^2(\mathbb{N})$ we know that

$$(\lambda\mathbb{1} - |R^*|^2)\varphi = (\lambda\varphi_1, (\lambda - 1)\varphi_2, (\lambda - 1)\varphi_3, \dots)$$

Let $\delta := \max\{\lambda, 1 - \lambda\} > 0$. Then, we will have that $\|(\lambda\mathbb{1} - |R^*|^2)\varphi\| \geq \delta\|\varphi\|$. So, $\lambda\mathbb{1} - |R^*|^2 \geq \delta\mathbb{1}$ and $\lambda\mathbb{1} - |R^*|^2$ is self-adjoint; by Corollary 8.18 we find that $\lambda\mathbb{1} - |R^*|^2$ is invertible and $\lambda \notin \sigma(|R^*|^2)$. So, $\sigma(|R^*|^2) = \{0, 1\}$. ∎

Problem 19

Let $\hat{R}$ be the bilateral right shift operator on $\ell^2(\mathbb{Z})$:

$$\hat{R}e_j := e_{j+1} \quad (j \in \mathbb{Z})$$

where $\{e_j\}_{j\in\mathbb{Z}}$ is the standard basis of $\ell^2(\mathbb{Z})$ and extend linearly.

(a) Calculate $\hat{R}^*$.

(b) Calculate $|\hat{R}|^2$ and $|\hat{R}^*|^2$.

(c) Show that $\hat{R}$ is a unitary.

(d) Calculate $\sigma(\hat{R})$, $\sigma(\hat{R}^*)$, $\sigma\left(|\hat{R}|^2\right)$, and $\sigma\left(|\hat{R}^*|^2\right)$.

Solution

Proof. (a) We claim that $\hat{R}^* = \hat{L}$, the bilateral left shift operator defined on the basis by $\hat{L}e_j := e_{j-1}$ and extended linearly. To see that they are adjoints, let $\varphi, \psi \in \ell^2(\mathbb{Z})$ be arbitrary. We may therefore express

$$\varphi \equiv \sum_{j\in\mathbb{Z}} \varphi_j e_j \quad \text{and} \quad \psi \equiv \sum_{j\in\mathbb{Z}} \psi_j e_j$$

for $\varphi_j, \psi_j \in \mathbb{C}$. As such, we see that

$$\left\langle \hat{L}\varphi, \psi \right\rangle = \left\langle \sum_{j\in\mathbb{Z}} \varphi_j e_{j-1}, \psi \right\rangle = \left\langle \sum_{j\in\mathbb{Z}} \varphi_{j+1} e_j, \psi \right\rangle = \sum_{j\in\mathbb{Z}} \overline{\varphi_{j+1}} \psi_j$$

and

$$\left\langle \varphi, \hat{R}\psi \right\rangle = \left\langle \varphi, \sum_{j\in\mathbb{Z}} \psi_j e_{j+1} \right\rangle = \left\langle \varphi, \sum_{j\in\mathbb{Z}} \psi_{j-1} e_j \right\rangle = \sum_{j\in\mathbb{Z}} \overline{\varphi_j} \psi_{j-1} = \sum_{j\in\mathbb{Z}} \overline{\varphi_{j+1}} \psi_j,$$

where the last equality simply relabeled indices. So, $\left\langle \hat{L}\varphi, \psi \right\rangle = \left\langle \varphi, \hat{R}\psi \right\rangle$; since this holds for all $\varphi, \psi \in \ell^2(\mathbb{Z})$, they are indeed adjoints.

(b) We may now compute $|\hat{R}|^2 = \hat{L}\hat{R}$ and $|\hat{R}^*|^2 = \hat{R}\hat{L}$. For any $j \in \mathbb{Z}$, we have that

$$\hat{L}\hat{R}e_j = \hat{L}e_{j+1} = e_j \quad \text{and} \quad \hat{R}\hat{L}e_j = \hat{R}e_{j-1} = e_j$$

Since they are both the identity on an orthonormal basis, then $|\hat{R}|^2 = |\hat{R}^*|^2 = \mathbb{1}$.

(c) The condition that $|\hat{R}|^2 = |\hat{R}^*|^2 = \mathbb{1}$ is the definition of unitary, and so $\hat{R}$ is clearly unitary.

(d) Since $\hat{R}$ is unitary, its spectrum lies on the circle $\mathbb{S}^1$. We will show that the spectrum is the entire circle. Suppose by way of contradiction there were some $|\lambda| = 1$ for which $\lambda\mathbb{1} - \hat{R}$ is invertible. Then, it must be surjective, and so there must be some vector $\varphi = (\dots, \varphi_{-1}, \varphi_0, \varphi_1, \dots)$ that gets mapped to e_0. So, $\lambda\varphi_0 - \varphi_{-1} = 1$. From the other coordinates, $\lambda\varphi_j = \varphi_{j-1}$. Together, these two facts tell us that

$$\varphi_j = \lambda^{-j}\varphi_0 \quad (j \geq 0)$$

As such, we find that

$$\|\varphi\|^2 = \sum_{j\in\mathbb{N}} |\varphi_j|^2 \geq \sum_{j\geq 0} |\varphi_j|^2 = \sum_{j\geq 0} |\lambda^{-j}\varphi_0|^2 = \sum_{j\geq 0} |\varphi_0|^2 = \infty$$

This is a contradiction, and so $\lambda\mathbb{1} - \hat{R}$ cannot be invertible. Thus, $\sigma(\hat{R}) = \mathbb{S}^1$. By the continuous functional calculus, $\sigma(\hat{R}^*) = \mathbb{S}^1$ as well. Lastly, since $|\hat{R}|^2 = |\hat{R}^*|^2 = \mathbb{1}$, we see that $\sigma\left(|\hat{R}|^2\right) = \sigma\left(|\hat{R}^*|^2\right) = \{1\}$. ■

Problem 20

Let $\frac{1}{X} \in \mathcal{B}(\ell^2(\mathbb{N}))$ be given by

$$\frac{1}{X}e_j := \frac{1}{j}e_j \quad (j \in \mathbb{N})$$

and extend linearly.

(a) Calculate $\left(\frac{1}{X}\right)^*$.

(b) Calculate $\sigma\left(\frac{1}{X}\right)$.

(c) Show that $\frac{1}{X}$ does not have closed range.

Solution

Proof. (a) Let $\varphi, \psi \in \ell^2(\mathbb{N})$ be arbitrary. We may therefore express

$$\varphi \equiv \sum_{j\in\mathbb{N}} \varphi_j e_j \quad \text{and} \quad \psi \equiv \sum_{j\in\mathbb{N}} \psi_j e_j$$

Note that

$$\left\langle \frac{1}{X}\varphi, \psi \right\rangle = \sum_{j\in\mathbb{N}} \overline{\frac{1}{j}\varphi_j}\psi_j = \sum_{j\in\mathbb{N}} \overline{\varphi_j}\frac{1}{j}\psi_j = \left\langle \varphi, \frac{1}{X}\psi \right\rangle$$

So, $\frac{1}{X}$ is self-adjoint, i.e. $\left(\frac{1}{X}\right)^* = \frac{1}{X}$.

(b) We know that since $\frac{1}{X}$ is self-adjoint, it must have real spectrum. We note that clearly $\frac{1}{j} \in \sigma\left(\frac{1}{X}\right)$ for all $j \in \mathbb{N}$. However, by the fact that the spectrum is closed, this also tells us that $0 \in \sigma\left(\frac{1}{X}\right)$ since it is the limit point of a sequence inside the spectrum. So,

$$\sigma\left(\frac{1}{X}\right) \supset \{0\} \cup \left\{\frac{1}{j} : j \in \mathbb{N}\right\}$$

We claim that this is the entire spectrum. Let $\lambda \neq 0$ be such that $\lambda \neq \frac{1}{j}$ for all $j \in \mathbb{N}$. Then, there must be some $\delta > 0$ such that $\left|\lambda - \frac{1}{j}\right| > \delta$ for all $j \in \mathbb{N}$. Then, for all $\varphi \in \ell^2(\mathbb{N})$, we see that

$$\left\langle \varphi, \left((\lambda - \delta)\mathbb{1} - \frac{1}{X}\right)\varphi \right\rangle \geq 0$$

since the bound holds coordinate-wise. So, $(\lambda-\delta)\mathbb{1} - \frac{1}{X} \geq 0 \implies \lambda\mathbb{1} - \frac{1}{X} \geq \delta\mathbb{1}$. Since $\lambda\mathbb{1} - \frac{1}{X}$ is self-adjoint, we may apply Corollary 8.18 to see that $\lambda\mathbb{1} - \frac{1}{X}$ is invertible, and so $\lambda \notin \sigma\left(\frac{1}{X}\right)$. The above shows that

$$\sigma\left(\frac{1}{X}\right) = \{0\} \cup \left\{\frac{1}{j} : j \in \mathbb{N}\right\}$$

(c) To see that it does not have closed range, observe that for each $N \in \mathbb{N}$ we have that $\varphi_N := (1, 1, \ldots, 1, 0, 0, \ldots) \in \ell^2(\mathbb{N})$ (the vector with N ones and the rest zeros) and therefore that $\frac{1}{X}\varphi_N = (1, \ldots, \frac{1}{N}, 0, 0, \ldots)$. The sequence $\left\{\frac{1}{X}\varphi_N\right\}_{N\in\mathbb{N}}$ clearly approaches the limit of $\psi := (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ going infinitely. This sequence is square summable and therefore $\psi \in \ell^2(\mathbb{N})$, and so $\frac{1}{X}\varphi_N \to \psi$ as $N \to \infty$. However, we claim that ψ cannot be in the range of $\frac{1}{X}$, as the only sequence that could map to ψ under $\frac{1}{X}$ is the sequence of all ones, which does not exist in $\ell^2(\mathbb{N})$. Since there is a sequence in $im(\frac{1}{X})$ that converges to an element in $\ell^2(\mathbb{N})$ that is not in $im(\frac{1}{X})$, we see that $\frac{1}{X}$ does not have closed range. ■

Problem 21

Show that if M is a closed linear subspace and $P_M : \mathcal{H} \to \mathcal{H}$ is given by

$$P_M \psi := a$$

where $\psi = a + b$ in the unique decomposition $\mathcal{H} = M \oplus M^\perp$, then P_M is a self-adjoint projection, i.e., show that $P_M = P_M^* = P_M^2$. Conversely, given any self-adjoint projection $P \in \mathcal{B}(\mathcal{H})$, find a closed linear subspace M such that $P = P_M$.

Solution

Proof. ($\implies$) Let M be a closed linear subspace and P_M be as described. Then, for any $\psi = a + b \in \mathcal{H}$, we have that

$$P_M^2(\psi) = P_M(a) = a = P_M(\psi),$$

where the second equality holds since $a = a + 0$ is the unique decomposition under $\mathcal{H} = M \oplus M^\perp$. So, $P_M^2 = P_M$. Now, let $\varphi = a + b, \psi = c + d \in \mathcal{H}$ be arbitrary, where $a, c \in M$ and $b, d \in M^\perp$. We have that

$$\langle P_M(\varphi), \psi \rangle = \langle a, c + d \rangle = \langle a, c \rangle + \langle a, d \rangle = \langle a, c \rangle$$

and

$$\langle \varphi, P_M(\psi) \rangle = \langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle = \langle a, c \rangle \,,$$

where we used that $\langle a, d \rangle = \langle b, c \rangle = 0$ by orthogonality. So, $\langle P_M(\varphi), \psi \rangle = \langle \varphi, P_M(\psi) \rangle$ for all $\varphi, \psi \in \mathcal{H}$, and so $P_M^* = P_M$. Therefore, P_M is a self-adjoint projection.

($\impliedby$) Now, let $P \in \mathcal{B}(\mathcal{H})$ be any self-adjoint projection. Define

$$M := \overline{im(P)} = \overline{\{a \in \mathcal{H} : \; P(\varphi) = a \text{ for some } \varphi \in \mathcal{H}\}}$$

It is clear that M is a closed linear subspace since P is linear, and so it suffices to show that $P = P_M$. To this end, let $\varphi \in \mathcal{H}$ be arbitrary, and write $\varphi = a + b$ for some $a \in M$ and $b \in M^\perp$ uniquely (as $\mathcal{H} = M \oplus M^\perp$). Since $a \in M$ we know that there is some sequence $\{\psi_n\}_n \subseteq \mathcal{H}$ such that $P(\psi_n) \to a$. By continuity of P, we see that $P(P(\psi_n)) \to P(a)$. Since $P^2 = P$, we know that $P(\psi_n) \to P(a)$; therefore, $P(a) = a$ since limits are unique in Hilbert space. We claim that $P(b) = 0$. To see this, note that

$$\|P(b)\|^2 = \langle P(b), P(b) \rangle = \langle P^* P(b), b \rangle = \langle P(b), b \rangle$$

where we used that $P^* = P \implies P^* P = P^2 = P$. However, by construction $P(b) \in M$ and $b \in M^\perp$, and so $\langle P(b), b \rangle = 0$. Thus, $P(b) = 0$. By linearity of P, we then have that

$$P(\psi) = P(a + b) = P(a) + P(b) = a + 0 = a = P_M(\psi)$$

Since this holds for all $\psi \in \mathcal{H}$, we see that $P = P_M$. ∎

Problem 23

For any $t > 0$, let $T_t \in \mathcal{B}(L^2(\mathbb{R}))$ be given by

$$T_t\varphi := \varphi(\cdot + t), \quad (\varphi \in L^2(\mathbb{R})).$$

(a) Calculate $\|T_t\|$.

(b) Find a limit to which T_t converges as $t \to \infty$ (in which operator topology?).

Solution

Proof. (a) For any $\varphi \in L^2(\mathbb{R})$, we have that

$$\|T_t\varphi\|_{L^2} = \int_{\mathbb{R}} |T_t\varphi(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} |\varphi(x+t)|^2 \mathrm{d}x = \int_{\mathbb{R}} |\varphi(x)|^2 \mathrm{d}x = \|\varphi\|_{L^2},$$

where the second to last inequality used the translation invariance of the Lebesgue integral. So, T_t preserves the norm, which automatically means it has operator norm 1. Thus, $\|T_t\| = 1$.

(b) We claim that $T_t \to 0$ in the weak operator topology. That is, we claim that for all $\varphi, \psi \in L^2(\mathbb{R})$, it holds that $\langle T_t\varphi, \psi\rangle_{L^2} \to 0$ as $t \to \infty$. To see this, first recall that the set of smooth functions of compact support is dense in $L^2(\mathbb{R})$. Let $\epsilon > 0$. Then, there exists a $\phi \in C_C^\infty(\mathbb{R})$ such that $\|\varphi - \phi\|_{L^2} < \frac{\epsilon}{2\|\psi\|_{L^2}}$. We have that

$$\langle T_t\varphi, \psi\rangle_{L^2} = \langle T_t\phi, \psi\rangle_{L^2} + \langle T_t(\varphi - \phi), \psi\rangle_{L^2}$$

By the triangle inequality and Cauchy-Schwartz, we then see

$$|\langle T_t\varphi, \psi\rangle_{L^2}| \le |\langle T_t\phi, \psi\rangle_{L^2}| + \|T_t(\varphi - \phi)\|_{L^2}\|\psi\|_{L^2} = |\langle T_t\phi, \psi\rangle_{L^2}| + \|\varphi - \phi\|_{L^2}\|\psi\|_{L^2} \le |\langle T_t\phi, \psi\rangle_{L^2}| + \frac{\epsilon}{2},$$

where the equality is because we know T_t preserves the norm. Next, again by density we know that there is an $\alpha \in C_C^\infty(\mathbb{R})$ such that $\|\psi - \alpha\|_{L^2} < \frac{\epsilon}{2\|\phi\|_{L^2}}$, and so

$$|\langle T_t\phi, \psi\rangle_{L^2}| \le |\langle T_t\phi, \alpha\rangle_{L^2}| + |\langle T_t\phi, \psi - \alpha\rangle_{L^2}|$$

We know by Cauchy-Schwartz and the fact that T_t preserves norms that

$$|\langle T_t\phi, \psi - \alpha\rangle_{L^2}| \le \|T_t\phi\|_{L^2}\|\psi - \alpha\|_{L^2} = \|\phi\|_{L^2}\|\psi - \alpha\|_{L^2} \le \frac{\epsilon}{2}$$

In total, we see that

$$|\langle T_t\varphi, \psi\rangle_{L^2}| \le |\langle T_t\phi, \alpha\rangle_{L^2}| + \epsilon$$

Now, since ϕ has compact support, there is some interval $[-M, M] \subseteq \mathbb{R}$ such that ϕ vanishes outside $[-M, M]$. Similarly, there is some $[-N, N] \subseteq \mathbb{R}$ such that α vanishes outside $[-N, N]$ (note that M, N depend on ϵ). So, we see that for $t > M + N$,

$$\langle T_t\phi, \alpha\rangle_{L^2} = \int_{\mathbb{R}} \overline{\phi(x+t)}\alpha(x)\mathrm{d}x = 0$$

To see why this equals 0, note that the only way for $\overline{\phi(x+t)}\alpha(x)$ to be nonzero is if $x + t \in [-M, M]$ and $x \in [-N, N]$. The second condition implies that $x + t \in [-N + t, N + t]$. For $t > M + N$, we see that $[-M, M] \cap [-N + t, N + t] = \emptyset$, and so there are no values of x such that $\overline{\phi(x+t)}\alpha(x)$ is nonzero. Thus, for all $t > M + N$,

$$|\langle T_t\varphi, \psi\rangle_{L^2}| \le 0 + \epsilon = \epsilon$$

Since for all ϵ there exist M, N such that this holds, this tells us that

$$\lim_{t\to\infty} \langle T_t\varphi, \psi\rangle_{L^2} = 0$$

So, T_t converges weakly to the 0 operator. ■

Problem 25

Let $A_n \to A$ and $B_n \to B$ in the strong operator topology. Show that $A_nB_n \to AB$ in the strong operator topology.

Solution

Proof. Let $\varphi \in \mathcal{H}$ be arbitrary. Let $\epsilon > 0$ be arbitrary. Since $A_n \to A$ strongly, we know that

$$A_n\psi \to A\psi \quad (\psi \in \mathcal{H})$$

Since the norm is continuous w.r.t. the norm topology on $\mathcal{H}$, this means that

$$\|A_n\psi\| \to \|A\psi\| \implies \sup_{n\in\mathbb{N}} \|A_n\psi\| < \infty \quad (\psi \in \mathcal{H}),$$

where the implication follows since convergent sequences of real numbers are bounded. So, by Banach-Steinhaus (uniform boundedness), the family of operators $\{A_n\}_n$ is uniformly bounded in operator norm. In other words, there is some $R < \infty$ such that $\|A_n\|_{\text{op}} \leq R$ for all n. Now, since $B_n \to B$ strongly, that means that there is some $N \in \mathbb{N}$ such that for all $n > N$,

$$\|B_n\varphi - B\varphi\| \leq \frac{\epsilon}{2R}$$

Similarly, since $A_n \to A$ strongly, there is some $M \in \mathbb{N}$ such that for all $n > M$,

$$\|A_n(B\varphi) - A(B\varphi)\| \leq \frac{\epsilon}{2}$$

Now, we may apply the triangle inequality with these estimates to see that for all $n > \max\{N, M\}$,

$$\begin{aligned}\|A_nB_n\varphi - AB\varphi\| &\leq \|A_nB_n\varphi - A_nB\varphi\| + \|A_nB\varphi - AB\varphi\| \\ &\leq \|A_n\|_{\text{op}}\|B_n\varphi - B\varphi\| + \|A_nB\varphi - AB\varphi\| \\ &\leq R\frac{\epsilon}{2R} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

So, we see that $A_nB_n\varphi \to AB\varphi$ in the norm topology on $\mathcal{H}$. Since this holds for all φ, we get that $A_nB_n \to AB$ in the strong operator topology. ■

Problem 26

Let $A_n \to A, B_n \to B$ in the weak operator topology. Find a counterexample for $A_n B_n \to AB$ in the weak operator topology.

Solution

Proof. Let R be the unilateral shift operator on $\ell^2(\mathbb{N})$ from Problem 18. Define $A_n := (R^*)^n$ and $B_n := R^n$ (which means that $B_n^* = (R^n)^* = (R^*)^n = A_n$). Then, for every $n > 1$ we have that

$$A_n B_n = (R^*)^{n-1}(R^* R)R^{n-1} = (R^*)^{n-1}|R|^2 R^{n-1}$$

Since we saw in Problem 18 that $|R|^2 = \mathbb{1}$, we see that

$$A_n B_n = (R^*)^{n-1} R^{n-1} = A_{n-1}B_{n-1} \quad (n > 1)$$

Since $A_1 B_1 = R^* R = |R|^2 = \mathbb{1}$, we find by induction on n that

$$A_n B_n = \mathbb{1} \quad (n \in \mathbb{N})$$

However, we have already seen that B_n converges weakly to 0 (see Example 9.11 in the lecture notes). Furthermore, since the adjoint operation is continuous in the weak operator topology (this follows from the definition and the fact that complex conjugation is continuous), we see that $A_n = B_n^*$ converges weakly to 0 as well. However, it cannot be that $A_n B_n \to 0$ weakly since $A_n B_n = \mathbb{1}$ for all n. So, this a valid counterexample. ∎

Problem 27

Show that for $A \in \mathcal{B}(\mathcal{H})$,

$$\|A\|_{\text{op}} = \sup\{|\langle \varphi, A\psi\rangle| : \|\varphi\| = \|\psi\| = 1\}$$

and if $A = A^*$ then

$$\|A\|_{\text{op}} = \sup\{|\langle \varphi, A\varphi\rangle| : \|\varphi\| = 1\}.$$

Solution

Proof. We have the definition

$$\|A\|_{\text{op}} = \sup\{\|A\varphi\| : \|\varphi\| = 1\}$$

So, for any unit vectors φ and ψ, by Cauchy-Schwartz $|\langle \varphi, A\psi\rangle| \leq \|\varphi\|\|A\psi\| = \|A\psi\| \leq \|A\|_{\text{op}}$. Therefore,

$$\sup\{|\langle \varphi, A\psi\rangle| : \|\varphi\| = \|\psi\| = 1\} \leq \|A\|_{\text{op}}$$

To prove the other direction, let $\epsilon > 0$ be arbitrary. Let $\psi \in \mathcal{H}$ be such that $\|\psi\| = 1$ and

$$\|A\psi\| \geq \|A\|_{\text{op}} - \epsilon,$$

which exists by definition of the supremum. Let $\varphi := \frac{A\psi}{\|A\psi\|}$, which means that $\|\varphi\| = 1$. Also,

$$|\langle \varphi, A\psi\rangle| = \frac{1}{\|A\psi\|}|\langle A\psi, A\psi\rangle| = \|A\psi\| \geq \|A\|_{\text{op}} - \epsilon$$

Since such a pair of unit vectors ψ, φ exists for any $\epsilon > 0$, we see that

$$\sup\{|\langle \varphi, A\psi\rangle| : \|\varphi\| = \|\psi\| = 1\} \geq \|A\|_{\text{op}}$$

So, the first result is proven.

Suppose now that $A = A^*$. For any unit vector φ, by Cauchy-Schwartz $|\langle \varphi, A\varphi\rangle| \leq \|\varphi\|\|A\varphi\| = \|A\varphi\| \leq \|A\|_{\text{op}}$. Therefore,

$$\sup\{|\langle \varphi, A\varphi\rangle| : \|\varphi\| = 1\} \leq \|A\|_{\text{op}}$$

To prove the other direction, let $\epsilon > 0$ be arbitrary. Let $\varphi, \psi \in \mathcal{H}$ be such that $\|\varphi\| = \|\psi\| = 1$ and

$$|\langle \varphi, A\psi\rangle| \geq \|A\|_{\text{op}} - \epsilon,$$

which exists by definition of the supremum and our earlier result. We may suppose without loss of generality that $\langle \varphi, A\psi\rangle \in \mathbb{R}$. If we let $\alpha := \varphi + \psi$ and $\beta := \varphi - \psi$, then

$$\langle \alpha, A\alpha\rangle = \langle \varphi, A\varphi\rangle + \langle \varphi, A\psi\rangle + \langle \psi, A\varphi\rangle + \langle \psi, A\psi\rangle$$

Since $\langle \psi, A\varphi\rangle = \langle A\varphi, \psi\rangle = \langle \varphi, A\psi\rangle$ by the facts that $\langle \varphi, A\psi\rangle \in \mathbb{R}$ and A is self-adjoint, we have that

$$\langle \alpha, A\alpha\rangle = \langle \varphi, A\varphi\rangle + \langle \psi, A\psi\rangle + 2\langle \varphi, A\psi\rangle$$

Similarly,

$$\langle \beta, A\beta\rangle = \langle \varphi, A\varphi\rangle + \langle \psi, A\psi\rangle - 2\langle \varphi, A\psi\rangle$$

So, we see that

$$\langle \varphi, A\psi\rangle = \frac{\langle \alpha, A\alpha\rangle - \langle \beta, A\beta\rangle}{4} \implies |\langle \varphi, A\psi\rangle| \leq \frac{|\langle \alpha, A\alpha\rangle| + |\langle \beta, A\beta\rangle|}{4}$$

Problem 27 continued on next page. . .

Letting $R := \sup\{|\langle\varphi, A\varphi\rangle| : \|\varphi\| = 1\}$ for notation, we know that $\frac{\alpha}{\|\alpha\|}$ is a unit vector, and so $\left|\left\langle \frac{\alpha}{\|\alpha\|}, A\frac{\alpha}{\|\alpha\|}\right\rangle\right| \leq R \implies |\langle\alpha, A\alpha\rangle| \leq \|\alpha\|^2 R$. Similarly, $|\langle\beta, A\beta\rangle| \leq \|\beta\|^2 R$. So,

$$|\langle\varphi, A\psi\rangle| \leq R\frac{\|\alpha\|^2 + \|\beta\|^2}{4}$$

By the parallelogram law, however, we know that $2 = \|\varphi\|^2 + \|\psi\|^2 = \frac{\|\alpha\|^2+\|\beta\|^2}{2}$. Thus,

$$|\langle\varphi, A\psi\rangle| \leq R$$

By our choice of φ and ψ, we conclude that

$$\|A\|_{\text{op}} - \epsilon \leq R$$

Since this holds for all $\epsilon > 0$, we see that

$$\|A\|_{\text{op}} \leq R = \sup\{|\langle\varphi, A\varphi\rangle| : \|\varphi\| = 1\},$$

and so they are equal. ■

Problem 28

Show that if $A_n \geq 0$, $A_n \to A$ in norm (resp. strongly) then $\sqrt{A_n} \to \sqrt{A}$ in norm (resp. strongly).

Solution

Proof. Let $A_n \geq 0$ for all $n \in \mathbb{N}$. We first note that if $A_n \to A$ strongly, then $A \geq 0$ as well. To see this, consider any $\varphi \in \mathcal{H}$. Then, $A_n\varphi \to A\varphi$ in $\|\cdot\|_{\mathcal{H}}$ by strong operator convergence. So, by continuity of the inner product, $\langle \varphi, A_n\varphi\rangle \to \langle \varphi, A\varphi\rangle$. Since $\langle \varphi, A_n\varphi\rangle \geq 0$ for all n by positivity, we see that $\langle \varphi, A\varphi\rangle \geq 0$. Since this holds for all $\varphi \in \mathcal{H}$, we see $A \geq 0$ and so $\sqrt{A}$ exists.

Furthermore, we have already seen in the proof of Problem 25 that when $A_n \to A$ strongly, there is a uniform bound $\|A_n\| \leq M < \infty$. So, if we divide all of A_n and A by $\max\{M, \|A\|\}$, we may suppose without loss of generality that $\|A_n\|, \|A\| \leq 1$.

Now, note that for any $B \in \mathcal{B}(\mathcal{H})$ with $B \geq 0$ and $\|B\| \leq 1$, we know that $\sigma(B) \subseteq [0, \|B\|] \subseteq [0,1]$, and so $\sigma(\mathbb{1} - B) \subseteq [0,1] \implies r(\mathbb{1} - B) \leq 1$. Since B is self-adjoint, this means that $\|\mathbb{1} - B\| \leq 1$, which means that we have that the following absolutely norm convergent series

$$\sqrt{B} = \mathbb{1} - \sum_{k=1}^{\infty} \frac{4^{-k}}{2k-1}\binom{2k}{k}(\mathbb{1} - B)^k$$

Writing $c_k := \frac{4^{-k}}{2k-1}\binom{2k}{k}$, we know that $\sum_{k\in\mathbb{N}} c_k = 1$, which can be seen by plugging $z = 1$ into the power series $\sqrt{1-z} = 1 - \sum_{k\in\mathbb{N}} c_k z^k$ (which converges for all $|z| \leq 1$). The above logic applied to both A_n and A means that

$$\sqrt{A_n} - \sqrt{A} = \sum_{k\in\mathbb{N}} c_k \left((\mathbb{1} - A_n)^k - (\mathbb{1} - A)^k\right)$$

We note that by a telescoping series,

$$\begin{aligned}
(\mathbb{1} - A_n)^k - (\mathbb{1} - A)^k &= \sum_{j=0}^{k-1} (\mathbb{1} - A_n)^{k-j}(\mathbb{1} - A)^j - (\mathbb{1} - A_n)^{k-(j+1)}(\mathbb{1} - A)^{j+1} \\
&= \sum_{j=0}^{k-1} (\mathbb{1} - A_n)^{k-(j+1)}\left[(\mathbb{1} - A_n) - (\mathbb{1} - A)\right](\mathbb{1} - A)^j \\
&= \sum_{j=0}^{k-1} (\mathbb{1} - A_n)^{k-(j+1)}(A - A_n)(\mathbb{1} - A)^j,
\end{aligned}$$

and so

$$\sqrt{A_n} - \sqrt{A} = \sum_{k\in\mathbb{N}} c_k \sum_{j=0}^{k-1} (\mathbb{1} - A_n)^{k-(j+1)}(A - A_n)(\mathbb{1} - A)^j$$

Thus, for all $\psi \in \mathcal{H}$ we see by the triangle inequality that

$$\begin{aligned}
\|\sqrt{A_n}\psi - \sqrt{A}\psi\| &\leq \sum_{k\in\mathbb{N}} c_k \sum_{j=0}^{k-1} \|\mathbb{1} - A_n\|_{\text{op}}^{k-(j+1)} \|\mathbb{1} - A\|_{\text{op}}^j \|A\psi - A_n\psi\| \\
&= \|A\psi - A_n\psi\| \cdot \sum_{k\in\mathbb{N}} c_k \sum_{j=0}^{k-1} \|\mathbb{1} - A_n\|_{\text{op}}^{k-(j+1)} \|\mathbb{1} - A\|_{\text{op}}^j \\
&\leq \|A\psi - A_n\psi\| \cdot \sum_{k\in\mathbb{N}} c_k \\
&= \|A\psi - A_n\psi\|
\end{aligned}$$

Problem 28 continued on next page. . .

From the above we see that if $A_n \to A$ strongly then $\sqrt{A_n} \to \sqrt{A}$ and if $A_n \to A$ in norm then $\sqrt{A_n} \to \sqrt{A}$ in norm. ■

Problem 29

Show that if $A_n \to A$ in norm then $|A_n| \to |A|$ in norm.

Solution

Proof. First, we will show that multiplication is jointly norm-continuous in the following sense:

Lemma 1. *Let $S_n \to S$ in norm and $T_n \to T$ in norm. Then, $S_nT_n \to ST$ in norm.*

Proof of Lemma 1. To see this, let $\epsilon > 0$ and note that

$$\|ST - S_nT_n\| \leq \|ST - S_nT\| + \|S_nT - S_nT_n\| \leq \|S - S_n\|\|T\| + \|S_n\|\|T - T_n\|$$

Since $S_n \to S$ in norm and the norm is norm-continuous, we see that $\|S_n\| \to \|S\|$ in $\mathbb{R}$; in particular, the sequence $\{\|S_n\|\}_n \subseteq \mathbb{R}$ is bounded in the sense that there is some $M < \infty$ such that $\|S_n\| \leq M$ for all n. So, for all n large enough that $\|S - S_n\| \leq \frac{\epsilon}{2\|T\|}$ and $\|T - T_n\| \leq \frac{\epsilon}{2M}$ (both of which are eventually guaranteed by norm convergence), then

$$\|ST - S_nT_n\| \leq \frac{\epsilon}{2\|T\|}\|T\| + M\frac{\epsilon}{2M} = \epsilon$$

Since this can be done for all $\epsilon > 0$, we see that $S_nT_n \to ST$ in norm. ■

Now, we may tackle the problem at hand. We are given that $A_n \to A$ in norm. Since the adjoint operation is norm-continuous, we also see that $A_n^* \to A^*$ in norm. Applying Lemma 1 with $S_n = A_n^*$ and $T_n = A_n$, we find that $A_n^*A_n \to A^*A$ in norm. In other words,

$$|A_n|^2 \to |A|^2$$

in norm. Since $|A_n|^2 \geq 0$ for all n, we may apply Problem 28 to see that

$$\sqrt{|A_n|^2} \to \sqrt{|A|^2}$$

in norm. This is precisely equivalent to

$$|A_n| \to |A|$$

in norm, which is the desired result. ■