MAT 520: Problem Set 6

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Problem 1

Show that $\mathcal{H} := \ell^2(\mathbb{N} \to \mathbb{C})$ is a Hilbert space: define an inner product on it and show that the induced metric is complete.

Solution

Proof. We define the inner product as follows: for any $\varphi, \psi \in \mathcal{H}$, if we let φ_n and ψ_n denote the n^{th} coordinates,

$$\langle \varphi, \psi \rangle = \sum_{n=1}^{\infty} \overline{\varphi_n} \psi_n$$

Note that this is $\mathbb{C}$-linear in the second slot, anti-$\mathbb{C}$-linear in the first slot (and so sesquilinear), and satisfies $\langle \varphi, \psi \rangle = \overline{\langle \psi, \varphi \rangle}$ via the properties of complex conjugation. Furthermore, $\langle \varphi, \varphi \rangle = \sum_{n=1}^{\infty} |\varphi_n|^2$, and so it equals 0 if and only if $\varphi_n = 0 \ \forall n$, or equivalently if $\varphi = 0$. Therefore, it is a valid inner product and turns $\mathcal{H}$ into an inner product space.

To see that it is complete in the norm metric induced by this inner product, let $\{\varphi^{(k)}\}_{k \in \mathbb{N}} \subseteq \mathcal{H}$ be a sequence that is Cauchy w.r.t. the norm (we use upper indices to label the vectors and lower indices for the coordinates of the vectors). Let $\epsilon > 0$. Then, there is some M such that for all $m, k > M$,

$$\epsilon > \|\varphi^{(m)} - \varphi^{(k)}\|^2 = \sum_{n=1}^{\infty} |\varphi_n^{(m)} - \varphi_n^{(k)}|^2,$$

where $\varphi_n^{(k)}$ refers to the n^{th} coordinate of $\varphi^{(k)} \in \mathcal{H}$. Since each element of this sum is nonnegative, we see that for all $n \in \mathbb{N}$,

$$|\varphi_n^{(m)} - \varphi_n^{(k)}|^2 < \epsilon$$

In particular, the sequence $\{\varphi_n^{(k)}\}_k \subseteq \mathbb{C}$ is Cauchy in $\mathbb{C}$, and so converges to some $\varphi_n \in \mathbb{C}$. Construct the vector $\varphi := (\varphi_1, \varphi_2, \ldots)$. We must show that $\varphi \in \ell^2(\mathbb{N} \to \mathbb{C})$ and also that $\varphi^{(k)} \to \varphi$ in the norm on $\mathcal{H}$. Let $M \in \mathbb{N}$ be such that for all $m, k > M$,

$$\|\varphi^{(k)} - \varphi^{(m)}\|^2 = \sum_{n=1}^{\infty} |\varphi_n^{(k)} - \varphi_n^{(m)}|^2 < \epsilon$$

Then, for each $N \in \mathbb{N}$, we certainly have

$$\sum_{n=1}^{N} |\varphi_n^{(k)} - \varphi_n^{(m)}|^2 < \epsilon$$

Since $\varphi_n^{(m)} \to \varphi_n$ as $m \to \infty$ for each n, we may use the continuity of $|\cdot|$ in $\mathbb{C}$ and the linearity of limits to see that

$$\lim_{m \to \infty} \sum_{n=1}^{N} |\varphi_n^{(k)} - \varphi_n^{(m)}|^2 = \sum_{n=1}^{N} |\varphi_n^{(k)} - \varphi_n|^2 < \epsilon$$

Since this holds for every $N \in \mathbb{N}$, it will hold in the limit (the sum increases with N, and so it is monotonic and has a limit). Thus,

$$\|\varphi^{(k)} - \varphi\|^2 = \sum_{n=1}^{\infty} |\varphi_n^{(k)} - \varphi_n|^2 < \epsilon$$

Firstly, this shows by the reverse triangle inequality (which may be applied elementwise) that

$$\Big| \|\varphi\| - \|\varphi^{(k)}\| \Big| \leq \|\varphi^{(k)} - \varphi\| < \sqrt{\epsilon} \implies \|\varphi\| \leq \|\varphi^{(k)}\| + \sqrt{\epsilon} < \infty,$$

and so $\varphi \in \mathcal{H}$. Furthermore, as this holds for all $k > M$, we find that $\varphi^{(k)} \to \varphi$ in the norm on $\mathcal{H}$. So, every Cauchy sequence converges in $\|\cdot\|$ to an element of $\mathcal{H}$. ■

Problem 2

Show that $\mathcal{H} := L^2(\mathbb{R})$ (with the Lebesgue measure) is a Hilbert space: define an inner product on it and show that the induced metric is complete.

Solution

Proof. We define the inner product as follows: for any $f, g \in \mathcal{H}$,

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) \mathrm{d}x$$

Note that this is $\mathbb{C}$-linear in the second slot, anti-$\mathbb{C}$-linear in the first slot (and so sesquilinear), and satisfies $\langle f, g \rangle = \overline{\langle g, f \rangle}$ via the properties of complex conjugation. Furthermore, $\langle f, f \rangle = \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x$, and so it equals 0 if and only if $f(x) = 0$ a.e., or equivalently if f is the 0 element of $\mathcal{H}$. Therefore, it is a valid inner product and turns $\mathcal{H}$ into an inner product space.

To see that it is complete in the norm metric induced by this inner product, let $\{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{H}$ be a sequence that is Cauchy w.r.t. the norm on $\mathcal{H}$. Then, we may inductively construct a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\| \leq 2^{-k}$$

by repeated application of the Cauchy criterion with $\epsilon_k = 2^{-k}$. Let us define

$$f := f_{n_1} + \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})$$

and

$$g := |f_{n_1}| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$$

Then, for each $K \in \mathbb{N}$ we have that

$$\left\| |f_{n_1}| + \sum_{k=1}^{K} |f_{n_{k+1}} - f_{n_k}| \right\| \leq \|f_{n_1}\| + \sum_{k=1}^{K} \|f_{n_{k+1}} - f_{n_k}\| \leq \|f_1\| + \sum_{k=1}^{K} 2^{-k} \leq \|f_{n_1}\| + 1,$$

where the first inequality is the triangle inequality and the second comes from our subsequence selection. Since this holds for all K and the sequence of functions $|f_{n_1}| + \sum_{k=1}^{K} |f_{n_{k+1}} - f_{n_k}|$ is increasing in K and approaches g pointwise, monotone convergence grants that

$$\left\| |f_{n_1}| + \sum_{k=1}^{K} |f_{n_{k+1}} - f_{n_k}| \right\| \to \|g\| \implies \|g\| \leq \|f_{n_1}\| + 1 < \infty,$$

and so $g \in \mathcal{H}$. Since $|f| \leq g$ by the triangle inequality, this means that $\|f\| < \infty$ and so $f \in \mathcal{H}$ as well. Now, we note that by the telescoping sum,

$$f_{n_{K+1}} = f_{n_1} + \sum_{k=1}^{K} (f_{n_{k+1}} - f_{n_k})$$

In other words, $f_{n_{K+1}}$ is the K^{th} partial sum, and so $f_{n_k}(x) \to f(x)$ pointwise a.e. as $k \to \infty$. Furthermore, since

$$|f - f_{n_K}|^2 = \left| \sum_{k=K}^{\infty} (f_{n_{k+1}} - f_{n_k}) \right|^2 \leq g^2$$

Problem 2 continued on next page...

for all K, we can apply dominated convergence since $\int g^2 < \infty$ to see that

$$\lim_{k \to \infty} \int_{\mathbb{R}} |f(x) - f_{n_k}(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} \lim_{k \to \infty} |f(x) - f_{n_k}(x)|^2 \mathrm{d}x = 0$$

since $f_{n_k} \to f$ pointwise a.e.. Thus, f_{n_k} approaches f in the norm. To see that $f_n \to f$ in the norm, let $\epsilon > 0$. By the Cauchy criterion, there is some $N \in \mathbb{N}$ such that for all $n, m > N$, $\|f_n - f_m\| < \frac{\epsilon}{2}$. Let k be large enough that both $n_k > N$ and $\|f_{n_k} - f\| < \frac{\epsilon}{2}$. Then, for all $n > N$ we have by the triangle inequality that

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since such an N exists for all ϵ, we see that $f_n \to f$ in the norm. So, every Cauchy sequence converges in the norm to some element of $\mathcal{H}$, and $\mathcal{H}$ is therefore complete in this norm. ■

Problem 3

Let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of bounded linear operators on $\mathcal{H}$. Show (in a concrete example, e.g., $\mathcal{H} = \mathbb{C}^2$) that $\mathcal{B}(\mathcal{H})$ is not a Hilbert space by showing the operator norm violates the parallelogram law.

Solution

Proof. Let $\mathcal{H} = \mathbb{C}^2$. We wish to find $A, B \in \mathcal{B}(\mathcal{H})$ for which

$$\|A + B\|^2 + \|A - B\|^2 \neq 2\|A\|^2 + 2\|B\|^2,$$

as this would violate the parallelogram law (note that this is the operator norm on $\mathcal{B}(\mathcal{H})$). Let A denote the orthogonal projection onto the first coordinate (i.e. $A(z_1, z_2) = (z_1, 0)$) and B be the similar orthogonal projection onto the second coordinate. Then, $\|A\| = \|B\| = 1$ since there exist unit vectors whose norm is preserved. However, we see that for all $(z_1, z_2) \in \mathcal{H}$,

$$(A + B)(z_1, z_2) = (z_1, 0) + (0, z_2) = (z_1, z_2)$$

and

$$(A - B)(z_1, z_2) = (z_1, 0) - (0, z_2) = (z_1, -z_2)$$

Clearly, $\|A + B\| = \|A - B\| = 1$ then, since they both preserve the norm. Therefore,

$$2 = \|A + B\|^2 + \|A - B\|^2 \neq 2\|A\|^2 + 2\|B\|^2 = 4,$$

and so $\mathcal{B}(\mathcal{H})$ with the operator norm cannot be made into a Hilbert space. ■

Problem 4

Show that if $M \subseteq \mathcal{H}$ is a closed vector subspace of it then $(M^\perp)^\perp = M$.

Solution

Proof. ($\supseteq$) Note first that $(M^\perp)^\perp \supset M$, since for every $\varphi \in M$ we have $\langle \varphi, \psi \rangle = 0$ for all $\psi \in M^\perp$, and so $\varphi \in (M^\perp)^\perp$.

($\subseteq$) We know that $(M^\perp)^\perp$ is closed since orthogonal complements are closed. Now, suppose by way of contradiction that $(M^\perp)^\perp \not\subseteq M$. Then, there must be some element $\varphi \in (M^\perp)^\perp \setminus M$. Since M is a closed subspace, we may decompose $\mathcal{H} = M \oplus M^\perp$, which means that φ decomposes into $\varphi = \varphi_M + \varphi_{M^\perp}$ with $\varphi_M \in M$ and $\varphi_{M^\perp} \in M^\perp$. Then, since $\varphi \in (M^\perp)^\perp$ by assumption and $\varphi_M \in (M^\perp)^\perp$ by the fact that $M \subseteq (M^\perp)^\perp$, we get

$$\varphi_{M^\perp} = \varphi - \varphi_M \implies \varphi_{M^\perp} \in (M^\perp)^\perp$$

since $(M^\perp)^\perp$ is a vector subspace. So, we find that

$$\varphi_{M^\perp} \in M^\perp \cap (M^\perp)^\perp \implies \langle \varphi_{M^\perp}, \varphi_{M^\perp} \rangle = 0 \implies \|\varphi_{M^\perp}\| = 0 \implies \varphi_{M^\perp} = 0$$

Therefore, $\varphi = \varphi_M \implies \varphi \in M$, contradicting our selection of φ. So, $(M^\perp)^\perp \subseteq M$. ■

Problem 5

Show that if $\{\varphi_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal vectors in $\mathcal{H}$, then the following are equivalent:

(a) $\sum_{n\in\mathbb{N}} \varphi_n$ exists in $\|\cdot\|_{\mathcal{H}}$.

(b) $\sum_{n\in\mathbb{N}} \|\varphi_n\|_{\mathcal{H}}^2 < \infty$.

(c) For any $\psi \in \mathcal{H}$, $\sum_{n\in\mathbb{N}} \langle \psi, \varphi_n\rangle_{\mathcal{H}}$ exists.

Solution

Proof. (a $\implies$ c) Suppose that $\sum_{n\in\mathbb{N}} \varphi_n$ exists and is equal to $\varphi \in \mathcal{H}$. Let $\psi \in \mathcal{H}$ be arbitrary. Then, letting $S_n := \sum_{j=1}^n \langle \psi, \varphi_j\rangle \in \mathbb{C}$ denote the partial sums, we have that for all $m > n$,

$$|S_m - S_n| = \left|\sum_{j=n+1}^m \langle \psi, \varphi_j\rangle\right| \le \left|\left\langle \psi, \sum_{j=n+1}^m \varphi_j\right\rangle\right| \le \|\psi\| \left\|\sum_{j=n+1}^m \varphi_j\right\|,$$

where the first inequality is the triangle inequality in $\mathbb{C}$ and the second is Cauchy-Schwartz. Let $\epsilon > 0$. We know that the sequence $\left\{\sum_{j=1}^n \varphi_j\right\}_{n\in\mathbb{N}}$, is Cauchy since it converges to φ, and so there is an N large enough that for all $m > n > N$,

$$\frac{\epsilon}{\|\psi\|} > \left\|\left(\sum_{j=1}^m \varphi_j\right) - \left(\sum_{j=1}^n \varphi_j\right)\right\| = \left\|\sum_{j=n+1}^m \varphi_j\right\|$$

For such $m > n > N$, we therefore have that

$$|S_m - S_n| \le \|\psi\| \frac{\epsilon}{\|\psi\|} = \epsilon,$$

and so the sequence $\{S_n\}_n$ is Cauchy in $\mathbb{C}$. Since $\mathbb{C}$ is complete, this means that the sequence of partial sums converges. Therefore, $\sum_{j=1}^\infty \langle \psi, \varphi_j\rangle$ exists. This holds for all $\psi \in \mathcal{H}$.

(c $\implies$ b) Suppose (c). For each $N \in \mathbb{N}$, define $A_N : \mathcal{H} \to \mathbb{C}$ via $A_N(\psi) = \sum_{n=1}^N \langle \psi, \varphi_n\rangle$. This is certainly linear; since the inner product is continuous w.r.t. the first slot then $A_N \in \mathcal{B}(\mathcal{H} \to \mathbb{C})$. Also, for a fixed $\psi \in \mathcal{H}$ the sequence $\{A_N\psi\}_{N\in\mathbb{N}}$ is convergent by assumption of (c), and so it is bounded. In other words,

$$\sup_{N\in\mathbb{N}} A_N(\psi) < \infty$$

Since this holds for all $\psi \in \mathcal{H}$, we may apply Banach-Steinhaus (uniform boundedness) to see that

$$S := \sup_{N\in\mathbb{N}} \|A_N\|_{\mathcal{B}(\mathcal{H}\to\mathbb{C})} < \infty$$

For each $N \in \mathbb{N}$ we have that

$$A_N\left(\sum_{k=1}^N \varphi_k\right) = \sum_{k,n=1}^N \langle \varphi_k, \varphi_n\rangle = \sum_{n=1}^N \langle \varphi_n, \varphi_n\rangle = \sum_{n=1}^N \|\varphi_n\|^2$$

However, we know that

$$\left|A_N\left(\sum_{k=1}^N \varphi_k\right)\right|^2 \le S^2 \left\|\sum_{k=1}^N \varphi_k\right\|^2 = S^2 \left\langle \sum_{k=1}^N \varphi_k, \sum_{n=1}^N \varphi_n\right\rangle = S^2 \sum_{k=1}^N \|\varphi_k\|^2,$$

where the inequality follows from the bound on operator norm of A_N and the last equality follows from pairwise orthogonality. Combining these two, we see that for all $N \in \mathbb{N}$,

$$\left(\sum_{n=1}^{N} \|\varphi_n\|^2\right)^2 \leq S^2 \left(\sum_{n=1}^{N} \|\varphi_n\|^2\right) \implies \sum_{n=1}^{N} \|\varphi_n\|^2 \leq S^2$$

(Note that if all the φ_n are 0 then the above holds trivially). So, since this holds for all $N \in \mathbb{N}$ and the sequence $\left\{\sum_{n=1}^{N} \|\varphi_n\|^2\right\}_N$ is monotonically increasing and so has a limit, we see that

$$\sum_{n=1}^{\infty} \|\varphi_n\|^2 \leq S^2 < \infty$$

(b $\implies$ a) Suppose now that $\sum_{n=1}^{\infty} \|\varphi_n\|^2 < \infty$. Let $S_n := \sum_{j=1}^{n} \varphi_j \in \mathcal{H}$ denote the partial sums. Then, for all $m > n$,

$$\|S_m - S_n\| = \left\| \sum_{j=n+1}^{m} \varphi_j \right\| = \left\langle \sum_{j=n+1}^{m} \varphi_j, \sum_{j=n+1}^{m} \varphi_j \right\rangle = \sum_{j,k=n+1}^{m} \langle \varphi_j, \varphi_k \rangle ,$$

where for the last equality we used the sesquilinearity of $\langle \cdot, \cdot \rangle$. Since $\{\varphi_j\}$ is pairwise orthogonal, we see that $\langle \varphi_j, \varphi_k \rangle = 0$ unless $j = k$. So,

$$\|S_m - S_n\| = \sum_{j=n+1}^{m} \langle \varphi_j, \varphi_j \rangle = \sum_{j=n+1}^{m} \|\varphi_j\|^2$$

Since the sequence of partial sums of the squared norms $\left\{\sum_{j=1}^{n} \|\varphi_j\|^2\right\}$ converges by assumption, we know that it is Cauchy. Let $\epsilon > 0$. Then, there is an N large enough that for all $m > n > N$,

$$\epsilon > \left| \left(\sum_{j=1}^{m} \|\varphi_j\|^2\right) - \left(\sum_{j=1}^{n} \|\varphi_j\|^2\right) \right| = \sum_{j=n+1}^{m} \|\varphi_j\|^2$$

So, for such $m > n > N$ we have that $\|S_m - S_n\| < \epsilon$. Therefore, $\{S_n\}_n \subseteq \mathcal{H}$ is Cauchy; since $\mathcal{H}$ is complete this means that the sum converges in norm. Therefore, $\sum_{j=1}^{\infty} \varphi_n$ exists. ■

Problem 6

Show that if $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ is an arbitrary sequence of vectors, then (a) implies (c). Find an example where (c) does not imply (a).

Solution

Proof. Note that in the proof of Problem 5, when we showed (a) $\implies$ (c) we made no use of pairwise orthogonality. So, that proof actually holds this more general case.

To find an example where (c) does not imply (a), let $\mathcal{H} := \ell^2(\mathbb{N} \to \mathbb{C})$. Let $\{e_j\}_{j\in\mathbb{N}}$ denote the standard position basis on $\mathcal{H}$. Define a sequence of vectors $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ via

$$\varphi_n := \begin{cases} e_1 & n = 1 \\ e_n - e_{n-1} & n > 1 \end{cases}$$

Then, for each $N \in \mathbb{N}$ we know that

$$\sum_{n=1}^{N} \varphi_n = e_N$$

by a telescoping sum. For any $\psi \in \mathcal{H}$ we then have that

$$\sum_{n=1}^{N} \langle \psi, \varphi_n \rangle = \psi_N \implies \lim_{N\to\infty} \sum_{n=1}^{N} \langle \psi, \varphi_n \rangle = \lim_{N\to\infty} \psi_N = 0,$$

where we used that ψ is square summable and so must have decaying coefficients. So, the sequence $\{\varphi_n\}_n$ satisfies (c).

However, we will show that it does not satisfy (a) by showing that the sequence of partial sums $\{\sum_{k=1}^{n} \varphi_k\}_n$ is not Cauchy in $\mathcal{H}$. In particular, we note that for all $m > n$,

$$\left\| \left(\sum_{k=1}^{m} \varphi_k \right) - \left(\sum_{k=1}^{n} \varphi_k \right) \right\| = \|e_m - e_n\| = \sqrt{2}$$

So, the partial sums always stay a fixed distance away, and cannot converge in norm. Therefore, (a) cannot hold for this sequence. ∎

Problem 7

Let $N \in \mathbb{N}$ and let $\alpha \in \mathbb{C}$ be such that $\alpha^N = 1 \neq \alpha^2$. Show that for all $\varphi, \psi \in \mathcal{H}$,

$$\langle \psi, \varphi \rangle = \frac{1}{N} \sum_{n=1}^{N} \alpha^n \|\varphi + \alpha^n \psi\|^2$$

Show also that

$$\langle \psi, \varphi \rangle = \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{i\theta} \|\varphi + e^{i\theta}\psi\|^2 d\theta$$

Solution

Proof. Firstly, note that $\alpha^N = 1 \implies 1 = |\alpha^N| = |\alpha|^N \implies |\alpha| = 1$. We have that

$$\begin{aligned}\|\varphi + \alpha^n \psi\|^2 &= \|\varphi\|^2 + \|\alpha^n \psi\|^2 + \langle \varphi, \alpha^n \psi \rangle + \langle \alpha^n \psi, \varphi \rangle \\ &= \|\varphi\|^2 + \|\psi\|^2 + \langle \varphi, \alpha^n \psi \rangle + \langle \alpha^n \psi, \varphi \rangle\end{aligned}$$

Plugging this in to the right hand side, we see

$$\begin{aligned}\frac{1}{N} \sum_{n=1}^{N} \alpha^n \|\varphi + \alpha^n \psi\|^2 &= (\|\varphi\|^2 + \|\psi\|^2) \sum_{n=1}^{N} \frac{\alpha^n}{N} + \frac{1}{N} \sum_{n=1}^{N} \alpha^n (\langle \varphi, \alpha^n \psi \rangle + \langle \alpha^n \psi, \varphi \rangle) \\ &= (\|\varphi\|^2 + \|\psi\|^2) \left(\sum_{n=1}^{N} \frac{\alpha^n}{N} \right) + \frac{1}{N} \sum_{n=1}^{N} (\langle \varphi, \alpha^{2n} \psi \rangle + \langle |\alpha^n|^2 \psi, \varphi \rangle)\end{aligned}$$

Since $|\alpha| = 1$, this equals

$$= (\|\varphi\|^2 + \|\psi\|^2) \left(\sum_{n=1}^{N} \frac{\alpha^n}{N} \right) + \langle \psi, \varphi \rangle + \langle \varphi, \psi \rangle \cdot \left(\sum_{n=1}^{N} \frac{\alpha^{2n}}{N} \right)$$

We note that since $\alpha^2 \neq 1$, then $\alpha \neq 1$, and so

$$1 + \sum_{n=1}^{N-1} \alpha^n = 1 + \alpha + \alpha^2 + \ldots + \alpha^{N-1} = \frac{1 - \alpha^N}{1 - \alpha} = 0 \implies \sum_{n=1}^{N-1} \alpha^n = -1$$

So, $\sum_{n=1}^{N} \alpha^n = 0$. Similarly,

$$1 + \sum_{n=1}^{N-1} \alpha^{2n} = 1 + (\alpha^2) + (\alpha^2)^2 + \ldots + (\alpha^2)^{N-1} = \frac{1 - (\alpha^2)^N}{1 - \alpha^2} = \frac{1 - (\alpha^N)^2}{1 - \alpha^2} = 0 \implies \sum_{n=1}^{N-1} \alpha^{2n} = -1$$

Therefore, since $\alpha^{2N} = 1$, we see that $\sum_{n=1}^{N} \alpha^{2n} = 0$ as well. Thus,

$$\frac{1}{N} \sum_{n=1}^{N} \alpha^n \|\varphi + \alpha^n \psi\|^2 = \langle \psi, \varphi \rangle$$

as desired.

The proof of the next part will go similarly. We have

$$\|\varphi + e^{i\theta}\psi\|^2 = \|\varphi\|^2 + \|\psi\|^2 + \langle \varphi, e^{i\theta}\psi \rangle + \langle e^{i\theta}\psi, \varphi \rangle$$

So,

$$\int_{[-\pi,\pi]} e^{i\theta} \|\varphi + e^{i\theta}\psi\|^2 d\theta = (\|\varphi\|^2 + \|\psi\|^2) \int_{[-\pi,\pi]} e^{i\theta} d\theta + \int_{[-\pi,\pi]} \langle \varphi, e^{2i\theta}\psi \rangle \, d\theta + \int_{[-\pi,\pi]} \langle \psi, \varphi \rangle \, d\theta,$$

where we used that $\overline{e^{i\theta}} = e^{-i\theta}$. We note that

$$\int_{[-\pi,\pi]} e^{i\theta} d\theta = \frac{1}{i}[e^{i\theta}]_{-\pi}^{\pi} = \frac{1}{i}(-1-(-1)) = 0$$

Note that $\theta \mapsto \left\langle \varphi, e^{2i\theta}\psi \right\rangle$ is a continuous map, and so it is Riemann integrable. For any partition of this into a finite Riemann sum, we will be able to bring the sum into the inner product. By continuity of the iner product w.r.t. scalar multiplication of one of the vectors, we get that

$$\int_{[-\pi,\pi]} \left\langle \varphi, e^{2i\theta}\psi \right\rangle d\theta = \left\langle \varphi, \left(\int_{[-\pi,\pi]} e^{2i\theta} d\theta \right) \psi \right\rangle$$

We compute

$$\int_{[-\pi,\pi]} e^{2i\theta} d\theta = \frac{1}{2i}[e^{2i\theta}]_{-\pi}^{\pi} = \frac{1}{2i}(1-1) = 0$$

In total, we find that

$$\int_{[-\pi,\pi]} e^{i\theta} \|\varphi + e^{i\theta}\psi\|^2 d\theta = 0 + 0 + \int_{[-\pi,\pi]} \langle \psi, \varphi \rangle = 2\pi \langle \psi, \varphi \rangle$$

Therefore,

$$\langle \psi, \varphi \rangle = \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{i\theta} \|\varphi + e^{i\theta}\psi\|^2 d\theta$$

as desired. ■

Problem 8

Let $\{\varphi_n\}_{n\in\mathbb{N}},\ \{\psi_n\}_{n\in\mathbb{N}} \subseteq \{\xi \in \mathcal{H} :\ \|\xi\| \leq 1\}$ be sequences such that $\langle \varphi_n, \psi_n \rangle \to 1$ in $\mathbb{C}$. Show that

$$\lim_{n\to\infty} \|\varphi_n - \psi_n\| = 0$$

Solution

Proof. We know that

$$\|\varphi_n - \psi_n\|^2 = \langle \varphi_n - \psi_n, \varphi_n - \psi_n \rangle = \|\varphi_n\|^2 + \|\psi_n\|^2 - \langle \varphi_n, \psi_n \rangle - \langle \psi_n, \varphi_n \rangle \leq 2 - \langle \varphi_n, \psi_n \rangle - \langle \psi_n, \varphi_n \rangle$$

We note that since conjugation $z \mapsto \overline{z}$ is continuous in $\mathbb{C}$,

$$\lim_{n\to\infty} \langle \psi, \varphi \rangle = \lim_{n\to\infty} \overline{\langle \varphi, \psi \rangle} = \overline{\lim_{n\to\infty} \langle \varphi, \psi \rangle} = \overline{1} = 1$$

We have that

$$0 \leq \|\varphi_n - \psi_n\|^2 \leq 2 - \langle \varphi_n, \psi_n \rangle - \langle \psi_n, \varphi_n \rangle$$

The right hand side approaches 0 as $n \to \infty$. So, by the squeeze theorem,

$$\lim_{n\to\infty} \|\varphi_n - \psi_n\|^2 = 0 \implies \lim_{n\to\infty} \|\varphi_n - \psi_n\| = 0$$

as desired. ■

Problem 9

Suppose that $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ converges weakly to some $\varphi \in \mathcal{H}$; that is, $\langle \xi, \varphi_n \rangle \to \langle \xi, \varphi \rangle \ \forall \xi \in \mathcal{H}$. Assume further that $\|\varphi_n\| \to \|\varphi\|$ in $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$$

Solution

Proof. We know that

$$\|\varphi_n - \varphi\|^2 = \langle \varphi_n - \varphi, \varphi_n - \varphi \rangle = \|\varphi_n\|^2 + \|\varphi\|^2 - \langle \varphi_n, \varphi \rangle - \langle \varphi, \varphi_n \rangle$$

Letting $\xi = \varphi$, we apply weak convergence to see that

$$\lim_{n\to\infty} \langle \varphi, \varphi_n \rangle = \langle \varphi, \varphi \rangle = \|\varphi\|^2$$

As in Problem 8, continuity of complex conjugation gives that $\langle \varphi_n, \varphi \rangle \to \|\varphi\|^2$ as well. In total, we use the convergence of the norms to see that

$$\lim_{n\to\infty} \left(\|\varphi_n\|^2 + \|\varphi\|^2 - \langle \varphi_n, \varphi \rangle - \langle \varphi, \varphi_n \rangle \right) = \|\varphi\|^2 + \|\varphi\|^2 - \|\varphi\|^2 - \|\varphi\|^2 = 0$$

So, the first equality grants that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|^2 = 0 \implies \lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$$

as desired. ■

Problem 10

Let V be a (not-necessarily-complete) inner product space and let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq V$ be an orthonormal set. Fix a $\psi \in V$ and define the functional $F : \mathbb{C}^N \to \mathbb{R}$ via

$$F(\alpha_1, \dots, \alpha_N) = \left\| \psi - \sum_{n=1}^{N} \alpha_n \varphi_n \right\|$$

Show that F is minimized with the choice of $\alpha_n = \langle \varphi_n, \psi \rangle$.

Solution

Proof. Note that

$$F(\alpha_1, \dots, \alpha_N) = \|\psi\|^2 + \left\| \sum_{n=1}^{N} \alpha_n \varphi_n \right\|^2 - \left\langle \psi, \sum_{n=1}^{N} \alpha_n \varphi_n \right\rangle - \left\langle \sum_{n=1}^{N} \alpha_n \varphi_n, \psi \right\rangle$$

By orthonormality, we know that

$$\left\| \sum_{n=1}^{N} \alpha_n \varphi_n \right\|^2 = \sum_{n=1}^{N} |\alpha_n|^2$$

Expanding the first inner product via linearity, we find that

$$\left\langle \psi, \sum_{n=1}^{N} \alpha_n \varphi_n \right\rangle = \sum_{n=1}^{N} \alpha_n \langle \psi, \varphi_n \rangle$$

Similarly,

$$\left\langle \sum_{n=1}^{N} \alpha_n \varphi_n, \psi \right\rangle = \sum_{n=1}^{N} \overline{\alpha_n \langle \psi, \varphi_n \rangle}$$

Therefore, we have

$$F(\alpha_1, \dots, \alpha_N) = \|\psi\|^2 + \sum_{n=1}^{N} \left(|\alpha_n|^2 - 2\,\mathrm{Re}(\alpha_n \langle \psi, \varphi_n \rangle) \right),$$

where $\mathrm{Re}(\cdot)$ denotes the real component of a complex number. So, to minimize F, we must for each n select the α_n minimizing

$$S_n(\alpha) := |\alpha|^2 - 2\,\mathrm{Re}(\alpha \langle \psi, \varphi_n \rangle)$$

If $\langle \psi, \varphi_n \rangle = 0$, then this is minimized when $\alpha = 0$ as well; so, suppose $\langle \varphi_n, \psi \rangle \neq 0$. Without loss of generality, write $\alpha = z \langle \varphi_n, \psi \rangle$ for some $z \in \mathbb{C}$. Then,

$$S_n(\alpha) = |z|^2 |\langle \varphi_n, \psi \rangle|^2 - 2\,\mathrm{Re}(z \langle \varphi_n, \psi \rangle \langle \psi, \varphi_n \rangle) = |z|^2 |\langle \varphi_n, \psi \rangle|^2 - 2 |\langle \varphi_n, \psi \rangle|^2 \,\mathrm{Re}(z) = |\langle \varphi_n, \psi \rangle|^2 \left(|z|^2 - 2\,\mathrm{Re}(z) \right)$$

Thus, the expression is minimized for the choice of z minimizing $|z|^2 - 2\,\mathrm{Re}(z)$. Clearly, we want z to be real. As such, we seek the $z \in \mathbb{R}$ minimizing $z^2 - 2z$, which we know to be $z = 1$ since it is a convex parabola. The above reasoning tells us that $S_n(\alpha)$ is minimized at $\alpha = \langle \varphi_n, \psi \rangle$. Therefore, since

$$F(\alpha_1, \dots, \alpha_N) = \|\psi\|^2 + \sum_{n=1}^{N} S_n(\alpha_n),$$

we find that F is minimized for the choice of $\alpha_n = \langle \varphi_n, \psi \rangle$. ■