MAT 520: Problem Set 5

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Problem 1

Prove Fekete's lemma: If $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is sub-additive, then $\lim_{n\to\infty} \frac{1}{n}a_n$ exists and equals $\inf_{n\in\mathbb{N}} \frac{1}{n}a_n$.

Solution

Proof. Let $\{a_n\}_n$ be sub-additive. Let $s^* := \inf_{n\in\mathbb{N}} \frac{1}{n}a_n$. We wish to show that $\liminf_{n\to\infty} \frac{1}{n}a_n \geq s^*$ and $\limsup_{n\to\infty} \frac{1}{n}a_n \leq s^*$, as this will complete the proof.

For the first part, let $k \in \mathbb{N}$. Then, it always holds that

$$\inf_{n>k} \frac{1}{n}a_n \geq \inf_{n\in\mathbb{N}} \frac{1}{n}a_n = s^*$$

since the infimum is over a smaller index set. Taking the limit as $k \to \infty$, we see that

$$\liminf_{n\to\infty} \frac{1}{n}a_n \geq s^*$$

For the second part, let $\epsilon > 0$ be arbitrary. By definition of the infimum, there must be some $N \in \mathbb{N}$ such that $\frac{1}{N}a_N < s^* + \epsilon$. For all $n > N$, we may write $n = Nq + r$ for some $q \in \mathbb{N}$ and $r \in \{0, \ldots, N-1\}$. Define

$$A := \max_{r\in\{0,\ldots,N-1\}} a_r < \infty$$

Then, for all $n > N$ we have that

$$\frac{1}{n}a_n = \frac{1}{n}a_{Nq+r} \leq \frac{1}{n}(qa_N + a_r) \leq \frac{1}{n}(Nq(s^* + \epsilon) + A) = \frac{Nq}{n}(s^* + \epsilon) + \frac{A}{n},$$

where we used sub-additivity for the first inequality. Note that $n = Nq + r \implies Nq = n - r \in (n - N, n]$. So, if $s^* > 0$ we find that for all $n > N$,

$$\frac{1}{n}a_n \leq \frac{n}{n}(s^* + \epsilon) + \frac{A}{n} = s^* + \epsilon + \frac{A}{n},$$

whereas if $s^* \leq 0$ we find that

$$\frac{1}{n}a_n \leq \frac{n-N}{n}(s^* + \epsilon) + \frac{A}{n} = s^* + \epsilon + \frac{A - N(s^* + \epsilon)}{n}$$

Since $\frac{A}{n} \to 0$ and $\frac{A-N(s^*+\epsilon)}{n} \to 0$ as $n \to \infty$, this tells us that in either case

$$\limsup_{n\to\infty} \frac{1}{n}a_n \leq s^* + \epsilon$$

Since this holds for all $\epsilon > 0$, we find that

$$\limsup_{n\to\infty} \frac{1}{n}a_n \leq s^*$$

as desired. So, the limit exists and equals s^*, and the proof is complete. ■

Problem 2

Let $R : \mathbb{C} \to \mathbb{C}$ be a rational function, i.e.,

$$R(z) = p(z) + \sum_{k=1}^{n} \sum_{l=1}^{q} c_{k,l}(z - z_k)^{-l}$$

where p is a polynomial, $n \in \mathbb{N}$, and $\{z_k\}_k$, $\{c_{k,l}\}_{k,l} \subseteq \mathbb{C}$. Let now $a \in \mathcal{A}$ such that $\{z_k\}_{k=1}^n \subseteq \rho(a)$. Assume further that we choose some $\sigma(a) \subseteq \Omega \in \text{Open}(\mathbb{C})$ such that R is holomorphic on Ω, and $\gamma_j : [s, t] \to \Omega$, $j = 1, \ldots, m$ a collection of m oriented loops which surround $\sigma(a)$ within Ω, such that

$$\frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} \frac{1}{z - \lambda}\, dz = \begin{cases} 1 & \lambda \in \sigma(a) \\ 0 & \lambda \notin \Omega \end{cases}.$$

Using Lemma 6.26 in the lecture notes (= Lemma 10.24 in Rudin) show that $R(a)$ obeys the Cauchy integral formula, in the sense that

$$p(a) + \sum_{k=1}^{n} \sum_{l=1}^{q} c_{k,l}(a - z_k \mathbb{1})^{-l} = \frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} R(z)(z\mathbb{1} - a)^{-1}\, dz.$$

Solution

Proof. To start, we know that $z_k \notin \Omega$ for all k since R is holomorphic on Ω and so Ω can't contain any poles of R. Let RHS denote the right hand side of the given expression. Substituting in the definition of R,

$$\begin{aligned} \text{RHS} &= \frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} \left[p(z) + \sum_{k=1}^{n} \sum_{l=1}^{q} c_{k,l}(z - z_k)^{-l} \right] (z\mathbb{1} - a)^{-1}\, dz \\ &= \frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} p(z)(z\mathbb{1} - a)^{-1} dz + \sum_{k=1}^{n} \sum_{l=1}^{q} c_{k,l} \frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} (z - z_k)^{-l}(z\mathbb{1} - a)^{-1} dz, \end{aligned}$$

where we used the linearity of the integral. For each k and l, we know that $z_k \in \rho(a)$ and the set of loops surrounds $\sigma(a)$ within $\mathbb{C} \setminus \{z_k\}$ (since $\Omega \subseteq \mathbb{C} \setminus \{z_k\}$). So, for each k and l we may apply the result of Lemma 6.26 from the lecture notes to see that

$$\frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} (z - z_k)^{-l}(z\mathbb{1} - a)^{-1} dz = (a - z_k \mathbb{1})^{-l}$$

Also, we may apply the polynomial functional calculus to see that

$$\frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} p(z)(z\mathbb{1} - a)^{-1} dz = p(a)$$

For the above, we would show this by changing the loops $\{\gamma_j\}_j$ to simply $\partial B_R(0)$ for some $R > \|a\|$ (which wouldn't alter the value of the integral since this new loop still surrounds $\sigma(a)$), after which we may apply equation (7) in the proof of Theorem 10.13 in Rudin. All in all, this tells us that

$$\text{RHS} = p(a) + \sum_{k=1}^{n} \sum_{l=1}^{q} c_{k,l}(a - z_k \mathbb{1})^{-l} \equiv \text{LHS},$$

exactly as desired. ■

Problem 3

Let $\mathcal{A}$ be such that there exists some $a \in \mathcal{A}$ with $\sigma(a)$ not connected. Show that $\mathcal{A}$ then contains some non-trivial idempotent (an element $b \in \mathcal{A}$ with $b^2 = b \notin \{0, \mathbb{1}\}$).

Solution

Proof. Let $a \in \mathcal{A}$ be such that $\sigma(a)$ is not connected. Let $\Omega \in \text{Open}(\mathbb{C})$ contain a connected component of $\sigma(a)$. Let $\widetilde{\Omega} \in \text{Open}(\mathbb{C})$ be such that $\Omega \cap \widetilde{\Omega} = \emptyset$ and $\sigma(a) \subseteq \Omega \sqcup \widetilde{\Omega}$ (i.e. $\widetilde{\Omega}$ contains $\sigma(a) \setminus \Omega$). Then, $\sigma(a) \cap \Omega \neq \emptyset$ and $\sigma(a) \cap \widetilde{\Omega} \neq \emptyset$ since they both contain components of $\sigma(a)$.

Now, define a function $f : \Omega \sqcup \widetilde{\Omega} \to \mathbb{C}$ via

$$f(z) := \begin{cases} 1 & z \in \Omega \\ 0 & z \in \widetilde{\Omega} \end{cases}$$

Then, $f^2 = f$ over $\Omega \sqcup \widetilde{\Omega}$. The map f is holomorphic on $\Omega \sqcup \widetilde{\Omega}$, since it is holomorphic on each of the two disjoint open sets. We may therefore apply the functional calculus and give meaning to $f(a)$. Since the functional calculus produces an algebra homomorphism, we know that $f(a)f(a) = (f^2)(a) = f(a)$. Therefore, $f(a)^2 = f(a)$, and so $f(a)$ is an idempotent. We now wish to show that it is nontrivial.

We know that $f(\lambda) = 0$ for all $\lambda \in \sigma(a) \cap \widetilde{\Omega}$, which is nonempty by construction. Then, by Theorem 10.28(a) we have that $f(a)$ is not invertible in $\mathcal{A}$. So, $f(a) \neq \mathbb{1}$. Note that we may apply the exact same reasoning with the function $g : \Omega \sqcup \widetilde{\Omega} \to \mathbb{C}$ given by $g := 1 - f$ (i.e. g is 1 on $\widetilde{\Omega}$ and 0 on Ω). Then, $g(a)$ is also idempotent and not equal to $\mathbb{1}$, and we now know by the functional calculus that

$$f(a) + g(a) = (f + g)(a) = (z \mapsto 1)(a) = \mathbb{1} \implies f(a) = \mathbb{1} - g(a),$$

where we used that $f + g$ is the constant 1 function. Since $g(a) \neq \mathbb{1}$, we find that $f(a)$ is nonzero. Therefore, $f(a)$ is an idempotent such that $f(a) \notin \{0, \mathbb{1}\}$, as desired. ■

Problem 4

Assume that $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ is a sequence such that $\exists \lim_n a_n =: a \in \mathcal{A}$. Let $\Omega \in \mathrm{Open}(\mathbb{C})$ contain a component of $\sigma(a)$. Show that $\sigma(a_n) \cap \Omega \neq \emptyset$ for all sufficiently large n.

Solution

Proof. We start with the following lemma proving continuity of functions produced by the functional calculus.

Lemma 1. *Let $a \in \mathcal{A}$ and let $\Omega \in \mathrm{Open}(\mathbb{C})$ be such that $\sigma(a) \subseteq \Omega$. Let $f : \Omega \to \mathbb{C}$ be holomorphic and bounded. Let $\epsilon > 0$. Then, there exists a $\delta > 0$ such that for all $b \in \mathcal{A}$ with $\|a-b\| < \delta$, it holds that $\sigma(b) \subseteq \Omega$ and*

$$\|f(a) - f(b)\| \leq \epsilon$$

Proof of Lemma 1. Let $\epsilon > 0$. Let $\{\gamma_1, \ldots, \gamma_m\}$ be a system of simple nonintersecting closed loops that together encircle $\sigma(a)$ within Ω. Denote by $|\gamma_j|$ the length of the j^{th} loop (which is finite), by $\mathrm{int}(\gamma_j) \in \mathrm{Open}(\mathbb{C})$ the interior of the region enclosed by the j^{th} loop, and by $\mathrm{im}(\gamma_j) \subseteq \mathbb{C}$ the image of the j^{th} loop. We know that over the ranges $\mathrm{im}(\gamma_j)$ of these loops (which are compact), the map $z \mapsto \|(z\mathbb{1} - a)^{-1}\|$ is continuous, and so it attains a maximum. Define

$$M := \max_{j\in\{1,\ldots,m\}} \sup_{z\in\mathrm{im}(\gamma_j)} \|(z\mathbb{1} - a)^{-1}\| < \infty$$

to be the resulting max over all the curves. Define $L := \max_{j\in\{1,\ldots,m\}} |\gamma_j|$ to be the max curve length. Write $A = \sup_{z\in\Omega} |f(z)| < \infty$ to be an upper bound for $|f|$ over Ω.

Now, if we let $U := \bigcup_{j=1}^m \mathrm{int}(\gamma_j)$ be the overall region enclosed by the system of loops, we know that U is open and $\sigma(a) \subseteq U \subseteq \Omega$. Let $\eta := \left[\sup_{z\in U^C} \|(a - z\mathbb{1})^{-1}\|\right]^{-1} > 0$. By Theorem 10.20 in Rudin, for all $b \in \mathcal{A}$ with $\|a - b\| < \eta$ we have that $\sigma(b) \subseteq U$ and so the same loops $\{\gamma_j\}_j$ that surround $\sigma(a)$ also surround $\sigma(b)$. We may now define

$$\delta := \min\left\{\eta, \frac{1}{2M}, \frac{2\pi\epsilon}{2mLM^2A}\right\} > 0$$

Let $b \in \mathcal{A}$ be such that $\|a - b\| < \delta$. As mentioned earlier, we know that $\sigma(b)$ is surrounded by the loops $\{\gamma_j\}_j$ in Ω. Furthermore, for all $z \in \bigcup_{j=1}^m im(\gamma_j)$ we have by construction of M that

$$\|(z\mathbb{1} - a) - (z\mathbb{1} - b)\| = \|b - a\| < \min\left\{\frac{1}{2M}, \frac{2\pi\epsilon}{2mLM^2A}\right\} \leq \frac{1}{2\|(z\mathbb{1} - a)^{-1}\|} \min\left\{1, \frac{2\pi\epsilon}{mLA\|(z\mathbb{1} - a)^{-1}\|}\right\}$$

By the bound used in the end of the proof of Claim 6.6 in the lecture notes, we see that for all $z \in \bigcup_{j=1}^m \mathrm{im}(\gamma_j)$ it is true that

$$\|(z\mathbb{1} - a)^{-1} - (z\mathbb{1} - b)^{-1}\| < \frac{2\pi\epsilon}{mLA}$$

So, when $\|a - b\| < \delta$,

$$\begin{aligned}
\|f(a) - f(b)\| &\leq \frac{1}{2\pi} \sum_{j=1}^m \oint_{\gamma_j} |f(z)| \cdot \|(z\mathbb{1} - a)^{-1} - (z\mathbb{1} - b)^{-1}\| \\
&\leq \frac{1}{2\pi} \sum_{j=1}^m |\gamma_j| \cdot \sup_{z\in\mathrm{im}(\gamma_j)} \left\{|f(z)| \cdot \|(z\mathbb{1} - a)^{-1} - (z\mathbb{1} - b)^{-1}\|\right\} \\
&\leq \frac{1}{2\pi} \cdot m \cdot L \cdot A \cdot \frac{2\pi\epsilon}{mLA} = \epsilon,
\end{aligned}$$

where we used the triangle inequality in the first line, the ML lemma (Lemma 6.14 from lecture notes) in the second, and in the third line we applied our earlier definitions. This proves the lemma. ■

Now, let $\widetilde{\Omega} \in \text{Open}(\mathbb{C})$ be such that $\sigma(a) \subseteq \Omega \sqcup \widetilde{\Omega}$ where $\Omega \cap \widetilde{\Omega} = 0$; we may do this because Ω contains a component of, but not all of, $\sigma(a)$. Define a function $f : \Omega \sqcup \widetilde{\Omega} \to \{0, 1\}$ via

$$f(z) := \begin{cases} 1 & z \in \Omega \\ 0 & z \in \widetilde{\Omega} \end{cases}$$

We see that f is holomorphic on $\Omega \sqcup \widetilde{\Omega}$ since for any point in $\Omega \sqcup \widetilde{\Omega}$, there is a neighborhood of that point on which f is constant. We may therefore apply the functional calculus.

Lemma 2. *Let $b \in \mathcal{A}$ be such that $\sigma(b) \subseteq \Omega \sqcup \widetilde{\Omega}$. Then, letting f be defined as above, it holds that*

$$\sigma(b) \cap \Omega = \emptyset \iff f(b) = 0$$

Proof of Lemma 2. ($\Longrightarrow$) Suppose that $\sigma(b) \cap \Omega = \emptyset$. Then, we may encircle $\sigma(b)$ with loops contained entirely within $\widetilde{\Omega}$. Since $f(z) = 0$ for all $z \in \widetilde{\Omega}$, we see that $f(b) = 0$ clearly.

($\Longleftarrow$) Suppose now that $\sigma(b) \cap \Omega \neq \emptyset$. If $\sigma(b) \cap \widetilde{\Omega} = \emptyset$ then $f(b) = \mathbb{1} \neq 0$ by the functional calculus; so, suppose that $\sigma(b) \cap \widetilde{\Omega} \neq \emptyset$. Then, since $f(\lambda) = 0$ for all $\lambda \in \sigma(b) \cap \widetilde{\Omega}$ (which is nonempty), Theorem 10.28(a) in Rudin gives that $f(b)$ is not invertible in $\mathcal{A}$. So, $f(b) \neq \mathbb{1}$. Defining the function $g : \Omega \sqcup \widetilde{\Omega} \to \mathbb{C}$ via $g := 1 - f$, identical logic shows that $g(b) \neq \mathbb{1}$. However, $f(b) + g(b) = (f + g)(b) = \mathbb{1}$ by the functional calculus. In particular, this shows that $f(b) \neq 0$. So, in all cases we see that $\sigma(b) \cap \Omega \neq \emptyset \implies f(b) \neq 0$, proving the lemma. ■

We are now ready to complete the proof of the problem statement. Since $\sigma(a) \cap \Omega \neq \emptyset$ by construction, we know that $f(a) \neq 0$ by Lemma 2. Define $\epsilon := \|f(a)\| > 0$. Then, since f is bounded we may apply Lemma 1 to get $\delta > 0$. Since $a_n \to a$ in norm, there is a $N \in \mathbb{N}$ such that for all $n > N$,

$$\|a - a_n\| < \delta \implies \|f(a)\| = \epsilon > \|f(a) - f(a_n)\| \geq \|f(a)\| - \|f(a_n)\| \implies \|f(a_n)\| > 0 \implies f(a_n) \neq 0,$$

where we used the reverse triangle inequality. Applying Lemma 2 again shows that for all $n > N$, $\sigma(a_n) \cap \Omega \neq \emptyset$. We are done. ■

Problem 5

Let X, Y be two Banach spaces and A, B be two bounded linear operators on X, Y respectively. Let $T \in \mathcal{B}(X \to Y)$. Show that the following two assertions are equivalent:

(a) $TA = BT$.

(b) $Tf(A) = f(B)T$ for any $f : \mathbb{C} \to \mathbb{C}$ holomorphic in some open set U which contains $\sigma(A) \cup \sigma(B)$.

Solution

Proof. ($\implies$) Suppose that $TA = BT$. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic in some open set U containing $\sigma(A) \cup \sigma(B)$. Let $\{\gamma_1, \ldots, \gamma_m\}$ be a system of simple nonintersecting closed loops that together encircle $\sigma(a) \cup \sigma(b)$ within U. Then, applying the holomorphic functional calculus, linearity of T, and the discussion in Remark 10.22 of Rudin,

$$Tf(A) = \frac{1}{2\pi i} \sum_{j=1}^{m} T \oint_{\gamma_j} f(z)(z\mathbb{1}_X - A)^{-1} dz = \frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} f(z) T (z\mathbb{1}_X - A)^{-1} dz$$

Let $z \notin \sigma(A) \cup \sigma(B) \iff z \in \rho(A) \cap \rho(B)$ be arbitrary. Then,

$$\begin{aligned}
T &= T(z\mathbb{1}_X - A)(z\mathbb{1}_X - A)^{-1} \\
&= zT(z\mathbb{1}_X - A)^{-1} - TA(z\mathbb{1}_X - A)^{-1} \\
&= zT(z\mathbb{1}_X - A)^{-1} - BT(z\mathbb{1}_X - A)^{-1} \\
&= (z\mathbb{1}_Y - B)T(z\mathbb{1}_X - A)^{-1}
\end{aligned}$$

Left multiplying by $(z\mathbb{1}_Y - B)^{-1}$, we find that

$$(z\mathbb{1}_Y - B)^{-1} T = T(z\mathbb{1}_X - A)^{-1}$$

We may apply this identity inside our earlier integral to see that

$$Tf(A) = \frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} f(z)(z\mathbb{1}_Y - B)^{-1} T dz$$

Again using the discussion in Remark 10.22 of Rudin, we can factor out the T on the right to get

$$Tf(A) = \left(\frac{1}{2\pi i} \sum_{j=1}^{m} \oint_{\gamma_j} f(z)(z\mathbb{1}_Y - B)^{-1} dz \right) T = f(B)T,$$

where we know that the above integral evaluates to $f(B)$ since the loops γ_j encircle $\sigma(B)$ by assumption.

($\impliedby$) Suppose (b). If we let f be the identity map sending $z \mapsto z$, then f is holomorphic on all of $\mathbb{C}$, which is open and certainly contains $\sigma(A) \cup \sigma(B)$. Furthermore, we know $f(A) = A$ and $f(B) = B$. Thus,

$$Tf(A) = f(B)T \implies TA = BT$$

as desired. ■