# MAT 520: Problem Set 4

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# Problem 1

Prove that any norm-closed convex bounded subset of a reflexive Banach space is weakly compact.

## Solution

**Proof.** Let $X$ be a reflexive Banach space, and let $K \subseteq X$ be norm-closed, convex, and norm-bounded. We start with the following lemma.

**Lemma 1.** *Let $C \subseteq X$ be a norm-closed and convex subset of a normed vector space $X$. Then, $C$ is weakly-closed (closed in the weak topology on $X$).*

**Proof of Lemma 1.** Note that norm-open balls are clearly convex by the triangle inequality. Since the norm topology on $X$ is generated by norm-open balls, it follows that $X$ with the norm topology is locally-convex. By Theorem 3.12 in Rudin, since $C$ is convex we know that the norm-closure of $C$ equals its weak-closure. Norm-closure of $C$ then guarantees that $C$ is also weakly-closed, as desired. ■

Certainly, Lemma 1 gives that $K$ is weakly-closed. Now, norm-boundedness gives some $M < \infty$ such that $\|x\| \leq M$ for all $x \in K$. Write

$$B_r := \{x \in X : \|x\| \leq r\}$$

to be the norm-closed ball of radius $r$ in $X$ (and so $K \subseteq B_M$), and

$$B_r^{**} := \{\Lambda \in X^{**} : \|\Lambda\| \leq r\}$$

to be the norm-closed ball of radius $r$ in $X^{**}$. Recall the definition of the weak-* topology on $X^{**}$ to be the initial topology generated by $J^*(X^*)$, where $J^* : X^* \to X^{***}$ is the canonical point evaluation map. Also, the weak topology on $X^{**}$ is the initial topology generated by $X^{***}$. Since $X$ is reflexive, we know by Problem 2 on the Problem Set 3 that $X^*$ is reflexive, and so $J^*(X^*) = X^{***}$. What this means is that the weak and weak-* topologies on $X^{**}$ must be equivalent.

Now, the Banach-Alaoglu theorem tells us that $B_1^{**}$ is compact in the weak-* topology on $X^{**}$. Since $X^{**}$ equipped with the weak-* topology is a TVS and so scaling is a weak-* homeomorphism, we find that $B_M^{**}$ is also weak-* compact. By the previous discussion, reflexivity guarantees that $B_M^{**}$ is also compact in the weak topology on $X^{**}$. Now, let $J : X \to X^{**}$ be the canonical point evaluation map. Since it is linear and isometric (and therefore bounded), we know that $J$ is continuous with respect to the norms on $X$ and $X^{**}$. Therefore, for all $\lambda \in X^{***}$ we have that $\lambda \circ J : X \to \mathbb{C}$ is continuous w.r.t. the norm topology on $X$, and so $\lambda \circ J \in X^*$. Since bounded linear functionals are continuous in the weak topology, this means that $\lambda \circ J$ is continuous w.r.t. the weak topology on $X$. Since this holds for all $\lambda \in X^{***}$, we see that $J : X \to X^{**}$ is continuous w.r.t. the weak topologies on both $X$ and $X^{**}$. Since $B_M^{**}$ is weakly-compact in $X^{**}$, its preimage $J^{-1}(B_M^{**})$ is therefore weakly-compact in $X$. However, since $J$ is isometric and invertible (by reflexivity), we have that $J^{-1}(B_M^{**}) = B_M$. Thus, $B_M$ is compact in the weak topology on $X$.

So, we have seen that $K$ is weakly-closed, $B_M$ is weakly-compact, and $K \subseteq B_M$. Since closed subsets of compact sets are compact, this reveals that $K$ is weakly-compact, and the proof is complete. ■

# Problem 5

Let $X$ be an infinite-dimensional Banach space, and define

$$S := \{x \in X \mid \|x\| = 1\}.$$

Show that the weak-closure of $S$ is

$$B := \{x \in X \mid \|x\| \leq 1\}.$$

## Solution

**Proof.** ($\subseteq$) We first show that $B \subseteq \text{cl}_w(S)$. Let $x \in B$ be arbitrary. Let $U \in \text{Nbhd}_w(x)$ (i.e. $U$ is a weak neighborhood of $x$). Then, for some $\epsilon > 0$ and some $\lambda_1, \dots, \lambda_m \in X^*$ we know that

$$x \subseteq x + \bigcap_{j=1}^{m} \lambda_j^{-1}(B_\epsilon(0_{\mathbb{C}})) \subseteq U$$

since the sets $\{\lambda^{-1}(B_\delta(0_{\mathbb{C}}))\}_{\substack{\lambda \in X^*, \\ \delta > 0}}$ form a local subbasis at $0_X$ for the weak topology. Define

$$V := \bigcap_{j=1}^{m} \lambda_j^{-1}(B_\epsilon(0_{\mathbb{C}}))$$

Certainly, $V$ is open in the weak topology on $X$, as it is an intersection of continuous, linear preimages of open sets in $\mathbb{C}$; therefore, $x + V$ is also weakly-open. Since $x + V$ is a nonempty, weakly-open subset of an infinite-dimensional Banach space, it is not bounded in norm; in particular, there exists some $y \in x + V$ such that $\|y\| > 1$.

Next, we claim that $V$ is convex. To this end, let $a, b \in V$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \leq 1$. Then, for all $j = 1, \dots, m$ we know that

$$\begin{aligned} |\lambda_j(\alpha a + \beta b)| &= |\alpha \lambda_j(a) + \beta \lambda_j(b)| \leq |\alpha| \cdot |\lambda_j(a)| + |\beta| \cdot |\lambda_j(b)| \\ &< |\alpha|\epsilon + |\beta|\epsilon = \epsilon(|\alpha| + |\beta|)) \leq \epsilon, \end{aligned}$$

where we used the triangle inequality and the fact that $a, b \in V \implies |\lambda_j(a)|, |\lambda_j(b)| < \epsilon$ by construction. So, $|\lambda_j(\alpha a + \beta b)| < \epsilon$ for all $j$, and so $\alpha a + \beta b \in V$. Since this holds for all such selections of $a, b, \alpha$, and $\beta$, we see that $V$ is convex. Certainly, this also means that $x + V$ is convex.

So, we have that $x, y \in x + V \subseteq U$, where $\|x\| \leq 1$, $\|y\| > 1$, and $x + V$ is convex. So, the line segment connecting $x$ and $y$ must pass through $S$ (the segment is a connected set originating in $B$ and ending outside the norm-closure of $B$, and so it must intersect the norm-boundary of $B$, which is $S$). By convexity we know that this line lies in the set $x + V$. So, there must exists some $w \in (x + V) \cap S \subseteq U \cap S$. In particular, $U \cap S$ is nonempty. Since this holds for all $U \in \text{Nbhd}_w(x)$ for all $x \in B$, we find that $B \subseteq \text{cl}_w(S)$.

($\supseteq$) We now wish to show that $\text{cl}_w(S) \subseteq B$. Note that $B$ is clearly convex by the triangle inequality. By Lemma 1, norm-closure of $B$ implies that $B$ is weakly-closed. Since $B \supseteq S$ is weakly-closed, we know by the topological definition of a closure as the smallest closed superset that

$$\text{cl}_w(S) \subseteq B$$

This proves the reverse direction. ■

# Problem 7

Use $(xy)^n = x(yx)^{n-1}y$ to show that $r(xy) = r(yx)$.

## Solution

**Proof.** We will simply show that $r(xy) \leq r(yx)$, as the reverse direction will then follow from a symmetric argument. We apply Gelfand's formula and submultiplicativity of the norm to see that

$$r(xy) = \lim_{n\to\infty} \|(xy)^n\|^{1/n} = \lim_{n\to\infty} \|x(yx)^{n-1}y\|^{1/n} \leq \lim_{n\to\infty} \left(\|x\|^{1/n} \cdot \|y\|^{1/n} \cdot \|(yx)^{n-1}\|^{1/n}\right),$$

Now, the limits $\lim_{n\to\infty} \|x\|^{1/n}$ and $\lim_{n\to\infty} \|y\|^{1/n}$ both exist and equal 1, and so we can use multiplicativity of the limit to see that

$$\begin{aligned}\lim_{n\to\infty} \left(\|x\|^{1/n} \cdot \|y\|^{1/n} \cdot \|(yx)^{n-1}\|^{1/n}\right) &= \lim_{n\to\infty} (\|x\|^{1/n}) \cdot \lim_{n\to\infty} (\|y\|^{1/n}) \cdot \lim_{n\to\infty} \left(\|(yx)^{n-1}\|^{1/n}\right) \\ &= \lim_{n\to\infty} \left(\|(yx)^{n-1}\|^{1/n}\right) \\ &= \lim_{n\to\infty} \left(\left(\|(yx)^{n-1}\|^{1/(n-1)}\right)^{(n-1)/n}\right)\end{aligned}$$

We know that the limit of a composition of functions is the composition of limits when the outer function is continuous at the inner limit. Since $(\cdot)^{(n-1)/n}$ is continuous for all $n$, we see that

$$\begin{aligned}\lim_{n\to\infty} \left(\left(\|(yx)^{n-1}\|^{1/(n-1)}\right)^{(n-1)/n}\right) &= \lim_{k\to\infty} \left(\lim_{n\to\infty} \left(\|(yx)^{n-1}\|^{1/(n-1)}\right)^{(k-1)/k}\right) \\ &= \lim_{k\to\infty} (r(yx))^{(k-1)/k} \\ &= r(yx),\end{aligned}$$

where we used Gelfand's formula again to get to the second line. So, $r(xy) \leq r(yx)$ and, symmetrically, $r(yx) \leq r(xy)$. This proves the desired result. ■

# Problem 8

Show that if $x, xy \in \mathcal{G}_{\mathcal{A}}$, then $y \in \mathcal{G}_{\mathcal{A}}$.

## Solution

**Proof.** We know by definition of an inverse and the associative identity that

$$(xy)^{-1}(xy) = \mathbb{1} \implies \left((xy)^{-1}x\right) y = \mathbb{1},$$

and so $y$ has a left inverse (namely, $(xy)^{-1}x$). Next, we may see that

$$(xy)(xy)^{-1}x = x \implies x^{-1}xy(xy)^{-1}x = \mathbb{1} \implies y(xy)^{-1}x = \mathbb{1},$$

where we used the invertibility of $x$. So, $y$ has a right inverse (namely, $(xy)^{-1}x$ again). Thus, $y$ is invertible.
■

# Problem 9

Show that if $xy, yx \in \mathcal{G}_{\mathcal{A}}$ then $x, y \in \mathcal{G}_{\mathcal{A}}$.

## Solution

**Proof.** We need only show that $x \in \mathcal{G}_{\mathcal{A}}$, since then we would be able to use Problem 8 to see that $y \in \mathcal{G}_{\mathcal{A}}$ as well. To this end, observe that

$$(xy)(xy)^{-1} = \mathbb{1} \implies x(y(xy)^{-1}) = \mathbb{1}$$

So, $y(xy)^{-1}$ is a right inverse for $x$. Similarly,

$$(yx)^{-1}(yx) = \mathbb{1} \implies ((yx)^{-1}y)x = \mathbb{1},$$

and so $(yx)^{-1}y$ is a left inverse for $x$. Therefore, $x$ is invertible, and the proof is complete. ■

# Problem 10

On the Banach space $X := \ell^2(\mathbb{N} \to \mathbb{C})$, define the right shift operator $R \in \mathcal{B}(X)$ by

$$(Ra)_n := \begin{cases} a_{n-1} & n \geq 2 \\ 0 & n = 1 \end{cases}$$

and the left shift operator $L \in \mathcal{B}(X)$ by

$$(La)_n = a_{n+1} \quad \forall n \in \mathbb{N}$$

Calculate $RL$ and $LR$. Conclude that one may have $xy = \mathbb{1}$ but $yx \neq \mathbb{1}$ in a Banach algebra.

## Solution

**Proof.** Let $a \in X$ be arbitrary. Then, for all $n \in \mathbb{N}$ we have that

$$((RL)a)_n = (R(La))_n = \begin{cases} a_n & n \geq 2 \\ 0 & n = 1 \end{cases}$$

since for all $n \geq 2$ we have that $(La)_n = a_{n+1}$, and so $((RL)a)_n = a_n$. However, we find that $((RL)a)_1 = 0$ by definition of $R$, leading to the above calculation.

By contrast, for all $n \in \mathbb{N}$ we have that

$$((LR)a)_n = (L(Ra))_n = (Ra)_{n+1} = a_n,$$

since for all $n$ it holds that $(Ra)_{n+1} = a_n$. So, $LR$ is the identity map, but $RL$ is not.

Viewing $\mathcal{B}(X)$ as a Banach algebra and letting $x = L$ and $y = R$, we see that $xy = \mathbb{1}$ but $yx \neq \mathbb{1}$. ■

# Problem 12

Show that if $z \in \mathbb{C} \setminus \{0\}$, then $z \in \sigma(xy)$ if and only if $z \in \sigma(yx)$. i.e.,

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}.$$

Find an example where $\sigma(xy) \neq \sigma(yx)$.

## Solution

**Proof.** Suppose that $z \in \mathbb{C} \setminus \{0\}$ and $z \in \sigma(xy)$. Then, $xy - z\mathbb{1} \notin \mathcal{G}_\mathcal{A}$ by definition of the spectrum. Clearly, this also means that $\mathbb{1} - \frac{x}{z}y \notin \mathcal{G}_\mathcal{A}$ since $\mathcal{G}_\mathcal{A}$ is closed under nonzero scalar multiplication. The result from Problem 11 tells us that $\mathbb{1} - y\frac{x}{z} \notin \mathcal{G}_\mathcal{A}$, and so $yx - z \notin \mathcal{G}_\mathcal{A}$. Thus, $z \in \sigma(yx)$. We may apply symmetric with $x$ and $y$ switched to prove the converse, and so we see that for $z \in \mathbb{C} \setminus \{0\}$ we have that $z \in \sigma(xy) \iff z \in \sigma(yx)$.

To find an example where $\sigma(xy) \neq \sigma(yx)$, let $x = L$ and $y = R$ be the left and right shift elements of the Banach algebra $\mathcal{A} := \mathcal{B}(\ell^2(\mathbb{N} \to \mathbb{C}))$ from Problem 10. The result of that problem shows that $LR = \mathbb{1} \implies LR \in \mathcal{G}_\mathcal{A}$, yet $RL \neq \mathbb{1}$. In fact, the computations we made there revealed that $RL \notin \mathcal{G}_\mathcal{A}$ since it is not injective (it sends both $(0, 0, \dots)$ and $(1, 0, \dots)$ to the zero element). Thus, $0 \in \sigma(RL)$ but $0 \notin \sigma(LR)$. The spectra of $LR$ and $RL$ are therefore not equal. ∎

# Problem 13

Define $\mathcal{A} := C^2([0,1] \to \mathbb{C})$, the space of functions with continuous second derivative. Define, for $a, b > 0$,

$$\|f\| := \|f\|_\infty + a\|f'\|_\infty + b\|f''\|_\infty.$$

Show that $\mathcal{A}$ is a Banach space. Show that $\mathcal{A}$ is a Banach algebra (with pointwise multiplication) iff $a^2 \geq 2b$. You may consider the functions $x \mapsto x$ and $x \mapsto x^2$.

## Solution

**Proof.** Clearly, $\mathcal{A}$ is a vector space. We verify that the defined norm is indeed a norm.

1. We note that $\|f\| \geq \|f\|_\infty$ since the other terms are nonnegative, and so if $\|f\| = 0$ then $\|f\|_\infty = 0 \implies f = 0$. Thus, $\|\cdot\|$ is positive definite.

2. Let $\alpha \in \mathbb{C}$. Then, for all $f \in \mathcal{A}$,

$$\|\alpha f\| = \|\alpha f\|_\infty + a\|(\alpha f)'\|_\infty + b\|(\alpha f)''\|_\infty = |\alpha|\|f\|_\infty + |\alpha| \cdot a\|f'\|_\infty + |\alpha| \cdot b\|f''\|_\infty = |\alpha|\|f\|$$

   by linearity of the derivative and homogeneity of $\|\cdot\|_\infty$, and so $\|\cdot\|$ is homogenous.

3. For all $f, g \in \mathcal{A}$,

$$\begin{aligned}\|f+g\| &= \|f+g\|_\infty + a\|(f+g)'\|_\infty + b\|(f+g)''\|_\infty \\ &= \|f+g\|_\infty + a\|f'+g'\|_\infty + b\|f''+g''\|_\infty \\ &\leq \|f\|_\infty + \|g\|_\infty + a\|f'\|_\infty + a\|g'\|_\infty + b\|f''\|_\infty + b\|g''\|_\infty \\ &= \|f\| + \|g\|\end{aligned}$$

   by the linearity of the derivative and the triangle inequality on $\|\cdot\|_\infty$, and so $\|\cdot\|$ satisfies the triangle inequality.

Now, we wish to show that $\mathcal{A}$ is complete in the $\|\cdot\|$ norm. To this end, let $\{f_n\}_n \subseteq \mathcal{A}$ be Cauchy in this norm. Since $\|g\| \geq \|g\|_\infty$, $\|g\| \geq a\|g'\|_\infty$, and $\|g\| \geq b\|g''\|_\infty$ for all $g \in \mathcal{A}$, we see that the sequences $\{f_n\}_n, \{f_n'\}_n$, and $\{f_n''\}_n$ are also Cauchy in the $\|\cdot\|_\infty$ norm when viewed as elements of $C^0([0,1] \to \mathbb{C})$. We know that $C^0([0,1] \to \mathbb{C})$ is complete in the $\|\cdot\|_\infty$ norm, and so we find that $f_n \to f$, $f_n' \to g$, and $f_n'' \to h$ for some $f, g, h \in C^0([0,1] \to \mathbb{C})$, where the aforementioned convergences are with respect to the $\|\cdot\|_\infty$ norm. Since $f_n \to f$ uniformly and $f_n' \to g$ uniformly, we see that $f' = g$ by a well-known result about uniform convergence of derivatives. Similarly, $f'' = g' = h$. So, for any $\epsilon > 0$ we may find a $N_0$ large enough that $\|f_n - f\|_\infty < \frac{\epsilon}{3}$ $\forall n > N_0$, a $N_1$ large enough that $\|f_n' - f'\|_\infty < \frac{\epsilon}{3a}$ $\forall n > N_1$, and a $N_2$ large enough that $\|f_n'' - f''\|_\infty < \frac{\epsilon}{3b}$ $\forall n > N_2$. Letting $N := \max\{N_0, N_1, N_2\}$, we see that for all $n > N$,

$$\|f_n - f\| = \|f_n - f\|_\infty + a\|f_n' - f'\|_\infty + b\|f_n'' - f''\|_\infty < \epsilon$$

Therefore, $f_n \to f$ in the $\|\cdot\|$ norm for some $f$ with continuous second derivative, and the space with this norm is therefore complete.

Now, the only extra property necessary for $\mathcal{A}$ with pointwise multiplication to be a Banach algebra is for submultiplicativity of the norm. We show that $\mathcal{A}$ has this property iff $a^2 \geq 2b$.

( $\implies$ ) Suppose that $a^2 < 2b$. Then, letting $f(x) = x$ be an element of $\mathcal{A}$, we see that

$$\|f\| = \|(x \mapsto x)\| = \|(x \mapsto x)\|_\infty + a\|(x \mapsto 1)\|_\infty + b\|(x \mapsto 0)\|_\infty = 1 + a$$

and

$$\|f^2\| = \|(x \mapsto x^2)\| = \|(x \mapsto x^2)\|_\infty + a\|(x \mapsto 2x)\|_\infty + b\|(x \mapsto 2)\|_\infty = 1 + 2a + 2b$$

So,

$$\|f\|^2 = (1+a)^2 = 1 + 2a + a^2 < 1 + 2a + 2b = \|f^2\|$$

If submultiplicativity of the norm had held, we would have had $\|f^2\| \leq \|f\|^2$; so, $\mathcal{A}$ is not a Banach algebra in the case that $a^2 < 2b$.

( $\Longleftarrow$ ) Suppose now that $a^2 \geq 2b$. Then, for all $f, g \in \mathcal{A}$, the product rule grants that

$$\|fg\| = \|fg\|_\infty + a\|fg' + f'g\|_\infty + b\|fg'' + 2f'g' + f''g\|_\infty$$

We note that the $\|\cdot\|_\infty$ norm is submultiplicative, since for $h_1, h_2 \in \mathcal{A}$ we have $\|h_1 h_2\|_\infty = \sup_{[0,1]}\{|h_1|\cdot|h_2|\} \leq \sup_{[0,1]}\{|h_1|\} \cdot \sup_{[0,1]}\{|h_2|\} = \|h_1\|_\infty \cdot \|h_2\|_\infty$. For notation, let $F_j := \|f^{(j)}\|_\infty$ and $G_j := \|g^{(j)}\|_\infty$ for $j = 0, 1, 2$. Then,

$$\begin{aligned}
\|fg\| &\leq F_0G_0 + aF_0G_1 + aF_1G_0 + bF_0G_2 + 2bF_1G_1 + bF_2G_0 \\
&\leq F_0G_0 + aF_0G_1 + aF_1G_0 + bF_0G_2 + a^2F_1G_1 + bF_2G_0 \\
&\leq F_0G_0 + aF_0G_1 + aF_1G_0 + bF_0G_2 + a^2F_1G_1 + bF_2G_0 + abF_1G_2 + abF_2G_1 + b^2F_2G_2 \\
&= (F_0 + aF_1 + bF_2)(G_0 + aG_1 + bG_2) \\
&= \|f\| \cdot \|g\|,
\end{aligned}$$

where the first line applies submultiplicativity of $\|\cdot\|_\infty$, the second line comes from the $2b \leq a^2$ assumption, and the third line follows because we added nonnegative terms, and the last lines are algebraic manipulation. Since this holds for all $f, g \in \mathcal{A}$, the norm is submultiplicative and $\mathcal{A}$ is a Banach algebra. ■

# Problem 14

Show that if $z \in \partial\sigma(x)$ then $x - z\mathbb{1} \in \partial\mathcal{G}_{\mathcal{A}}$.

## Solution

**Proof.** Suppose that $z \in \partial\sigma(x)$, and so $z \in \partial\rho(x)$ since $\rho(x) = \sigma(x)^C$ is open. Then, there exists $\{z_n\}_{n\in\mathbb{N}} \subseteq \rho(x)$ such that $z_n \to z$ in $\mathbb{C}$. For each $n \in \mathbb{N}$ define

$$y_n := x - z_n\mathbb{1}$$

Since $z_n \in \rho(x)$, we know that $y_n \in \mathcal{G}_{\mathcal{A}}$ for each $n$. Since $\sigma(x)$ is closed and contains its boundary, $z \in \sigma(x)$ and so $y := x - z\mathbb{1} \notin \mathcal{G}_{\mathcal{A}}$. However, we have that

$$\|y - y_n\| = \|(x - z\mathbb{1}) - (x - z_n\mathbb{1})\| = \|(z_n - z)\mathbb{1}\| = |z_n - z|,$$

which goes to 0 as $n \to \infty$. Thus, $y_n \to y$ in norm. So, any open neighborhood of $y$ must intersect both $\mathcal{G}_{\mathcal{A}}^C$ (since $y \in \mathcal{G}_{\mathcal{A}}^C$) and $\mathcal{G}_{\mathcal{A}}$ (since a sequence of elements of $\mathcal{G}_{\mathcal{A}}$ converges to $y$). Thus, $y \in \partial\mathcal{G}_{\mathcal{A}}$ as we wanted to show. ■

# Problem 15

Let $x \in \partial \mathcal{G}_{\mathcal{A}}$. Show there exists some $\{y_n\}_n \subseteq \mathcal{A}$ with $\|y_n\| = 1$ and

$$\lim_{n\to\infty} xy_n = \lim_{n\to\infty} y_n x = 0.$$

Try to characterize the type of Banach algebras in which there are such elements $x$ (which are called topological divisors of zero).

## Solution

**Proof.** Let $x \in \partial \mathcal{G}_{\mathcal{A}}$. Then, there exists a sequence $\{x_n\}_n \subseteq \mathcal{G}_{\mathcal{A}}$ such that $x_n \to x$ in the norm. For each $n \in \mathbb{N}$, define

$$y_n := \frac{x_n^{-1}}{\|x_n^{-1}\|}$$

Clearly, each $y_n$ is unit norm. Furthermore, we know that for all $n$

$$\begin{aligned}
\|xy_n\| = \|x_n y_n + (x - x_n)y_n\| &= \left\| \frac{1}{\|x_n^{-1}\|}\mathbb{1} + (x - x_n)y_n \right\| \\
&\leq \frac{1}{\|x_n^{-1}\|} + \|(x - x_n)y_n\| \\
&\leq \frac{1}{\|x_n^{-1}\|} + \|x - x_n\|,
\end{aligned}$$

where for the first line we used the definition of $y_n$, for the second line we used the triangle inequality, and for the third line we used submultiplicativity of the norm and the fact that $\|y_n\| = 1$. By Lemma 10.17 in Rudin, we know that $\|x_n^{-1}\| \to \infty$ as $n \to \infty$. By the fact that $x_n \to x$ in the norm, we know that $\|x - x_n\| \to 0$ as $n \to \infty$. Together, these and the above reveal that

$$\lim_{n\to\infty} \|xy_n\| = 0 \implies \lim_{n\to\infty} xy_n = 0$$

Similar logic shows that

$$\begin{aligned}
\|y_n x\| = \|y_n x_n + y_n(x - x_n)\| &= \left\| \frac{1}{\|x_n^{-1}\|}\mathbb{1} + y_n(x - x_n) \right\| \\
&\leq \frac{1}{\|x_n^{-1}\|} + \|x - x_n\| \to 0,
\end{aligned}$$

and so

$$\lim_{n\to\infty} xy_n = \lim_{n\to\infty} y_n x = 0$$

In the general case, any divisor of 0 is automatically a topological divisor of 0. The only way for there to be no more topological divisors of 0 is if $\partial \mathcal{G}_{\mathcal{A}} \setminus \{\text{divisors of } 0\}$ is empty. This happens if and only iff all elements that don't divide 0 are invertible, or equivalently if $\mathcal{A} \setminus \{\text{divisors of } 0\} \subseteq \mathcal{G}_{\mathcal{A}}$. However, the Gelfand-Mazur theorem tells us that such Banach algebras are isometrically isomorphic to $\mathbb{C}$. ∎

# Problem 17

Show that if $x \in \mathcal{A}$ is nilpotent (i.e., $\exists n \in \mathbb{N}$ with $x^n = 0$) then $\sigma(x) = \{0\}$.

## Solution

**Proof.** Let $x \in \mathcal{A}$ be nilpotent with exponent $n$. We first show that $x$ cannot be invertible. To this end, suppose by way of contradiction that $x \in \mathcal{G}_{\mathcal{A}}$. Then, $x^{-1}x = xx^{-1} = \mathbb{1}$. We will prove by induction that this means that $x^k \in \mathcal{G}_{\mathcal{A}}$ for all $k \in \mathbb{N}$. The base case holds for $k = 1$; so, suppose by way of induction that $x^k \in \mathcal{G}_{\mathcal{A}}$ and let $y = (x^k)^{-1}$ be its inverse. Then,

$$x^{k+1}(x^{-1}y) = x^k(xx^{-1})y = x^k y = \mathbb{1}$$

and

$$(yx^{-1})x^{k+1} = y(x^{-1}x)x^k = yx^k = \mathbb{1}$$

So, $x^{k+1}$ has a left and right inverse, and so $x^{k+1} \in \mathcal{G}_{\mathcal{A}}$. By induction, we see that $x^k \in \mathcal{G}_{\mathcal{A}}$ for all $k \in \mathbb{N}$, which in particular means that $x^n \in \mathcal{G}_{\mathcal{A}}$. However, we know $x^n = 0 \notin \mathcal{G}_{\mathcal{A}}$ since 0 cannot be invertible. This yields a contradiction, and so we find that $x \notin \mathcal{G}_{\mathcal{A}}$. Therefore, $x - 0\mathbb{1} \notin \mathcal{G}_{\mathcal{A}} \implies 0 \in \sigma(x)$.

Now, note that Gelfand's formula gives that

$$r(x) = \inf_{k \in \mathbb{N}} \|x^k\|^{1/k}$$

Since $\|x^k\| = 0$ for $k \geq n$ and $\|x^k\| \geq 0$ for all $k$ in general, we see that this infimum must equal 0. So,

$$r(x) = \sup_{z \in \sigma(x)} |z| = 0,$$

by definition of the spectral radius, which necessarily means that there are no nonzero elements in the spectrum of $x$. Thus, $\sigma(x) = \{0\}$. ■

# Problem 18

Show that $r$ is upper semicontinuous.

## Solution

**Proof.** Let $x_0 \in \mathcal{A}$ be arbitrary, and let $p \in \mathbb{R}$ be such that $p > r(x_0)$. We want to show that there is a $U \in \text{Nbhd}(x_0)$ such that $r(x) < p$ for all $x \in U$, as that will show that $r$ is upper semicontinuous at $x_0$.

Let $q \in \mathbb{R}$ be such that $r(x_0) < q < p$. Define $B_q(0_\mathbb{C}) \subseteq \mathbb{C}$ to be the open ball of radius $q$ in the complex plane; then, $\sigma(x_0) \subseteq B_q(0_\mathbb{C})$ since $r(x_0) = \sup_{z \in \sigma(x_0)} |z| < q$. Define

$$L := \sup_{z \notin B_q(0_\mathbb{C})} \|(x_0 - z\mathbb{1})^{-1}\|$$

and

$$U := B_L(x_0) \equiv \{x \in \mathcal{A} : \quad \|x - x_0\| < L\} \subseteq \mathcal{A}$$

to be the open ball of radius $L$ around $x_0$ (in particular, it is open in the norm topology on $\mathcal{A}$). By Theorem 10.20 in Rudin (with $\Omega = B_q(0_\mathbb{C})$) we see that $\sigma(x) \subseteq B_q(0_\mathbb{C})$ for all $x \in U$. Thus, for all $x \in U$ we know that $r(x) \leq q < p$. To reiterate, there is a neighborhood $U$ of $x_0$ such that $r(x) < p$ for all $x \in U$, and so $r$. Since this can be done for all $p > r(x_0)$, we find that $r$ is upper semicontinuous at $x_0$. Since this holds for an arbitrary $x_0 \in \mathcal{A}$, the result is proven. ■