MAT 520: Problem Set 3

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Problem 1

Prove the $\mathbb{C}$-Hahn-Banach theorem using the $\mathbb{R}$-Hahn-Banach theorem. In particular, you have to set up the forgetful functor which maps a $\mathbb{C}$-vector space to its underlying $\mathbb{R}$-vector space to show:
Let X be a $\mathbb{C}$-vector space, $p : X \to \mathbb{R}$ be given such that

$$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) \quad (x, y \in X; \alpha, \beta \in \mathbb{C} : |\alpha| + |\beta| = 1).$$

Let $\lambda : Y \to \mathbb{C}$ linear where $Y \subseteq X$ is a subspace, and such that

$$|\lambda(x)| \leq p(x) \quad (x \in Y).$$

Then there exists $\Lambda : X \to \mathbb{C}$ linear such that $\Lambda|_Y = \lambda$ and such that

$$|\Lambda(x)| \leq p(x) \quad (x \in X).$$

Solution

Proof. Let $F : X \to \tilde{X}$ be the forgetful functor that sends X as a $\mathbb{C}$-vector space to X as a $\mathbb{R}$-vector space (which we call $\tilde{X}$ for clarity). Define a functional $\tilde{\lambda} : Y \to \mathbb{R}$ (where Y is viewed as a subspace of $\tilde{X}$) via

$$\tilde{\lambda}(y) := \operatorname{Re}(\lambda(y)) \quad (y \in Y)$$

Then, it certainly holds that for all $y \in Y$, $\tilde{\lambda}(y) = \operatorname{Re}(\lambda(y)) \leq |\lambda(y)| \leq p(y)$. Furthermore, we may view $p : \tilde{X} \to \mathbb{R}$ as a convex function in the sense that for all $t \in [0, 1]$ and all $x, y \in \tilde{X}$ we have

$$p(tx + (1-t)y) \leq tp(x) + (1-t)p(y),$$

where the above holds by applying our original hypothesis on p with $\alpha = t$ and $\beta = 1 - t$. We are now all set up to use the $\mathbb{R}$-Hahn-Banach theorem, which produces a linear map $\tilde{\Lambda} : \tilde{X} \to \mathbb{R}$ such that $\tilde{\Lambda}|_Y = \tilde{\lambda}$ and $\tilde{\Lambda} \leq p$ on $\tilde{X}$.

From this, we may define the functional $\Lambda : X \to \mathbb{C}$ via

$$\Lambda(x) = \tilde{\Lambda}(F(x)) - i\tilde{\Lambda}(F(ix))$$

Λ is certainly linear since $\tilde{\Lambda}$ is linear and F is a linear map. Furthermore, for any $y \in Y$ we have that

$$\Lambda(y) = \tilde{\Lambda}(y) - i\tilde{\Lambda}(iy) = \operatorname{Re}(\lambda(y)) - i \operatorname{Re}(\lambda(iy)) = \operatorname{Re}(\lambda(y)) + i \operatorname{Im}(\lambda(y)) = \lambda(y)$$

The last thing we wish to show is that $|\Lambda| \leq p$. To this end, let $x \in X$ be arbitrary. There is some $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha\Lambda(x) = |\Lambda(x)|$. Since $p(\alpha x) = |\alpha| p(x) = p(x)$ by assumption, we may observe that

$$|\Lambda(x)| = \alpha\Lambda(x) = \Lambda(\alpha x) = \operatorname{Re}(\Lambda(\alpha x)) = \tilde{\Lambda}(F(\alpha x)) \leq p(\alpha x) = p(x),$$

where we were able to say that $\Lambda(\alpha x) = \operatorname{Re}(\Lambda(\alpha x))$ because $\Lambda(\alpha x) = |\Lambda(x)| \in \mathbb{R}$. Since this holds for all $x \in X$, it stands that $|\Lambda| \leq p$ over X. We see that Λ statisfies all our desired properties, and so we are done.
■

Problem 2

A Banach space is called reflexive iff $X \cong X^{**}$. Show that a Banach space X is reflexive iff X^* is reflexive.

Solution

Proof. ($\implies$) Suppose first that X is reflexive. Let $J : X \to X^{**}$ be the linear isometric injection sending points in X to evaluation functionals at those points; by reflexitivity, J is surjective. Now, let $J^* : X^* \to X^{***}$ be the similar injection. We want to show that J^* is surjective. To this end, let $\Lambda \in X^{***}$ be arbitrary. Then, Λ is a bounded and linear map from $X^{**} \to \mathbb{C}$ by definition. Thus, $\Lambda \circ J : X \to \mathbb{C}$ is a bounded and linear map since J is isometric and linear. So, $\Lambda \circ J \in X^*$. We claim that $J^*(\Lambda \circ J) = \Lambda$, which would prove surjectivity of J^*.

To see this, let $\lambda \in X^{**}$ be arbitrary. Then, $J^*(\Lambda \circ J)(\lambda)$ is equal to $\lambda(\Lambda \circ J)$ by definition. Furthermore, since $\lambda \in X^{**}$ and $J : X \to X^{**}$ is surjective, $\lambda = J(x)$ for some $x \in X$. Thus,

$$J^*(\Lambda \circ J)(\lambda) = \lambda(\Lambda \circ J) = J(x)(\Lambda \circ J) = (\Lambda \circ J)(x) = \Lambda(J(x)) = \Lambda(\lambda),$$

where for the third equality we used the definition of J, and for the second and last equalities we used that $\lambda = J(x)$. Since $J^*(\Lambda \circ J)(\lambda) = \Lambda(\lambda)$ for all $\lambda \in X^{**}$, we find that $\Lambda \in \text{range}(J^*)$. Since this holds for all $\Lambda \in X^{***}$, we see that J^* is surjective, which means that X^* is reflexive.

($\impliedby$) Suppose now that X^* is reflexive. Let $J : X \to X^{**}$ be the linear isometric injection sending points in X to evaluation functionals at those points; we wish to show that J is surjective. Let $J^* : X^* \to X^{***}$ be the similar point evaluation map; we know that J^* is an isometric bijection by reflexivity of X^*. Suppose by way of contradiction that J were not surjective, or equivalently that $J(X) \subsetneq X^{**}$. Let $\Lambda_0 \in X^{**} \setminus J(X)$; clearly, Λ_0 is nonzero. Let $Y := J(X) + \mathbb{C}\Lambda_0$ denote the set of linear combinations of elements of $J(X)$ with Λ_0. Then, $Y \subseteq X^{**}$ is a subspace of X^{**} since $J(X)$ is a subspace by linearity. Define a functional $\Gamma : Y \to \mathbb{C}$ via

$$\Gamma(\lambda + \alpha\Lambda_0) = \alpha \quad \forall \lambda \in J(X) \text{ and } \forall \alpha \in \mathbb{C}$$

Note that Γ is linear, since for all $y_1 := \lambda_1 + \alpha_1\Lambda_0 \in Y$ and $y_2 := \lambda_2 + \alpha_2\Lambda_0 \in Y$ and $\beta_1, \beta_2 \in \mathbb{C}$, we have

$$\Gamma(\beta_1 y_1 + \beta_2 y_2) = \Gamma((\beta_1\lambda_1 + \beta_2\lambda_2) + (\beta_1\alpha_1 + \beta_2\alpha_2)) = \beta_1\alpha_1 + \beta_2\alpha_2 = \beta_1\Gamma(y_1) + \beta_2\Gamma(y_2)$$

Γ is also clearly continuous, as for any $\delta > 0$ and $\alpha \in \mathbb{C}$ we know

$$\Gamma^{-1}(B_\delta(\alpha)) = J(X) + \{\gamma\Lambda_0 : \ \gamma \in B_\delta(\alpha)\}$$

The right element of the above sum is open since scalar multiplication is a homeomorphism, and so the preimages through Γ of basic open sets are open. Thus, $\Gamma \in Y^*$. By the Hahn-Banach theorem, we are then able to extend it to some $\Lambda \in X^{***}$ with the same operator norm that agrees with Γ on Y. Thus, $\Lambda_{J(X)} = 0$ by construction, and so $\Lambda(J(x)) = 0$ for all $x \in X$. By surjectivity of J^*, there is some nonzero $\lambda \in X^*$ such that $\Lambda = J^*(\lambda)$ (note that λ must be nonzero since Γ was nonzero, which means Λ is nonzero). Thus, for all $x \in X$ it must be that

$$\Lambda(J(x)) = 0 \implies J^*(\lambda)(J(x)) = 0 \implies J(x)(\lambda) = 0 \implies \lambda(x) = 0,$$

where for the first implication we used the definition of λ, for the second we used the definition of J^*, and for the last implication we used the definition of J. Since $\lambda(x) = 0$ for all $x \in X$, we see that λ is the zero functional, a contradiction. Therefore, X must be reflexive. ■

Problem 3

A pair of Banach spaces are called strictly dual iff $\exists$ map $f : X \to Y^*$ which is isometric, so that the induced map $f^* : Y \to X^*$ is also isometric. Prove that if X and Y are strictly dual and X is reflexive, then $Y = X^*$ and $X = Y^*$ using the Hahn-Banach theorem.

Solution

Proof. Let $f : X \to Y^*$ be the isometric map witnessing the strict duality between X and Y. Then, for every $x \in X$ and every $y \in Y$ we know that $f(x) \in Y^*$. We define the induced map $f^* : Y \to X^*$ via

$$f^*(y)(x) = f(x)(y) \qquad \forall x \in X \text{ and } \forall y \in Y$$

Note that f must be linear to ensure that $f^*(y) \in X^*$ for all y. By assumption, f^* is also isometric, and we see that it too must be linear. Lastly, let $J : X \to X^{**}$ be the isometric linear map for point evaluation, which we know to be a bijection since X is reflexive.

Note that it suffices to prove that $Y = X^*$; if we are able to do so, then applying the result of Problem 2 we find that Y is also reflexive, and from there we would be able to apply symmetric logic with X and Y switched to see that $X = Y^*$. To prove that $Y = X^*$, it suffices to show that f^* is bijective, as this will show that it is an isometric vector space isomorphism, which would immediately give that Y and X^* are equal as Banach spaces. So, we proceed in trying to show that f^* is bijective.

f^* is clearly injective, as linear isometries are always injective (if $f^*(x) = 0$ for some $y \in Y$, then $0 = \|f^*(y)\| = \|y\| \implies y = 0$, and so f^* has trivial kernel). Suppose by way of contradiction that f^* is not surjective, and so $f^*(Y) \subsetneq X^*$. Let $\Lambda_0 \in X^* \setminus f^*(Y)$. Define the space $E := f^*(Y) + \mathbb{C}\Lambda_0 \subseteq X^*$; then, E is a vector subspace of X^* since $f^*(Y)$ is (by linearity of f^*) and since $\mathbb{C}\Lambda_0$ is (trivially). We perform the exact same construction as we did in the second half of Problem 2, where we construct a functional $\Gamma : E \to \mathbb{C}$ such that

$$\Gamma(\lambda + \alpha\Lambda_0) = \alpha \quad \forall \lambda \in f^*(Y) \text{ and } \forall \alpha \in \mathbb{C}$$

By the exact same logic as in Problem 2, Γ is linear and continuous, and it is not the zero functional. So, by the Hahn-Banach theorem, we are able to extend it to some nonzero $\Lambda \in X^{**}$ such that $\Lambda_{f^*(Y)} = 0$. By surjectivity of J, there is some nonzero $x \in X$ such that $\Lambda = J(x)$. Thus, we find that for all $y \in Y$,

$$0 = \Lambda(f^*(y)) = J(x)(f^*(y)) = f^*(y)(x) = f(x)(y)$$

Since $f(x)(y) = 0$ for all $y \in Y$, we see that $f(x)$ is the zero functional in Y^*. This is a contradiction since x was nonzero and f is isometric. Thus, f^* must be surjective, and so it is an isometric vector space isomorphism. Therefore, $Y = X^*$.

This result shows that Y is reflexive. Since the definition of strict duality is symmetric, we may apply the above logic with X and Y switched (and f and f^* switched) to see that $X = Y^*$. ∎

Problem 4

Let $S \subseteq L^1([0,1] \to \mathbb{C})$ be a closed linear subspace. Suppose that S is such that $f \in S$ implies $f \in L^p([0,1] \to \mathbb{C})$ for some $p > 1$ (we will call this the S-condition). Show that $S \subseteq L^p([0,1] \to \mathbb{C})$ for some $p > 1$.

Solution

Proof. Let L^p denote $L^p([0,1] \to \mathbb{C})$ for $p \geq 1$ for notation. We note that $L^p \subseteq L^q$ whenever $q \leq p$.

Firstly, we note that since L^1 is a Banach space and $S \subseteq L^1$ is a closed subspace, then S equipped with the L^1 norm is itself a Banach space. For each $N \in \mathbb{N}$, define

$$E_N := \{f \in S : \quad f \in L^{1+\frac{1}{N}} \text{ and } \|f\|_{L^{1+\frac{1}{N}}} \leq N\}$$

We claim that E_N is closed in S for each N. To see this, let $\{f_n\}_{n \in \mathbb{N}} \subseteq E_N$ be a sequence of elements of E_N such that $f_n \to f$ in the L^1 norm for some $f \in S$; we may suppose without loss of generality that $f_n \geq 0$ for all n by dealing with the positive and negative sides separately, as one does in the usual construction of the Lebesgue integral. Then, we know that there is some subsequence $\{f_{n_k}\}_{k \in \mathbb{N}} \subseteq E_N$ such that f_{n_k} converges to f pointwise a.e.. Thus, $|f_{n_k}|^{1+\frac{1}{N}}$ converges pointwise a.e. to $|f|^{1+\frac{1}{N}}$, and so

$$\int_{[0,1]} |f|^{1+\frac{1}{N}} = \int_{[0,1]} \liminf_{k \to \infty} |f_{n_k}|^{1+\frac{1}{N}} \leq \liminf_{k \to \infty} \int_{[0,1]} |f_{n_k}|^{1+\frac{1}{N}} \leq N^{\frac{N+1}{N}},$$

where the equality comes from the pointwise a.e. convergence of the subsequence, the first inequality is Fatou's lemma, and the last inequality uses that $\|f_{n_k}\|_{L^{1+\frac{1}{N}}} = \left(\int_{[0,1]} |f_{n_k}|^{1+\frac{1}{N}}\right)^{\frac{N}{N+1}} \leq N$ by construction of E_N. So, $\|f\|_{L^{1+\frac{1}{N}}} \leq N$, which means that $f \in E_n$ and so E_n is closed in S (with the L^1 norm).

Now, note that we may express

$$S = \bigcup_{N \in \mathbb{N}} E_N$$

To see this, let $f \in S$ be arbitrary. There is some $p > 1$ for which $f \in L^p$ by the S-condition. Let $N \in \mathbb{N}$ be large enough that $(\|f\|_{L^1} + \|f\|_{L^p}^p)^{\frac{N}{N+1}} \leq N$ (which can be done since the LHS is eventually almost constant). Then,

$$\begin{aligned}
\|f\|_{L^{1+\frac{1}{N}}}^{\frac{N+1}{N}} &= \int_{[0,1]} |f|^{1+\frac{1}{N}} = \int_{[0,1] \cap \{|f| \leq 1\}} |f|^{1+\frac{1}{N}} + \int_{[0,1] \cap \{|f| > 1\}} |f|^{1+\frac{1}{N}} \\
&\leq \int_{[0,1] \cap \{|f| \leq 1\}} |f| + \int_{[0,1] \cap \{|f| > 1\}} |f|^p \\
&\leq \int_{[0,1]} |f| + \int_{[0,1]} |f|^p \\
&= \|f\|_{L^1} + \|f\|_{L^p}^p,
\end{aligned}$$

where the first inequality is because decreasing the exponent when $|f| \leq 1$ and increasing it when $|f| > 1$ increases the value, and the second inequality is because the integrands are nonnegative. So

$$\|f\|_{L^{1+\frac{1}{N}}} \leq (\|f\|_{L^1} + \|f\|_{L^p}^p)^{\frac{N}{N+1}} \leq N$$

So, $f \in E_n$, and therefore $S \subseteq \bigcup_{N \in \mathbb{N}} E_N$ as desired. Since S is a Banach space, by the Baire Category Theorem we must have some E_N that is not nowhere dense. Since each E_N is closed in S, that means that E_N has nonempty interior in S for some N. So, the set $S \cap L^{1+\frac{1}{N}}$ also has nonempty interior in S. However, we know $S \cap L^{1+\frac{1}{N}}$ to be a linear subspace of S clearly. Since proper subspaces always have empty interior (I proved this lemma on the last problem set), we see that $S \cap L^{1+\frac{1}{N}}$ cannot be a proper subspace, and so $S = S \cap L^{1+\frac{1}{N}} \implies S \subseteq L^{1+\frac{1}{N}}$. This completes the proof. ■

Problem 5

Let L be the (unilateral) left shift operator on $\ell^\infty(\mathbb{N} \to \mathbb{R})$:

$$(L\psi)(n) \equiv \psi(n+1) \quad (n \in \mathbb{N})$$

Prove that there exists a Banach limit, i.e. some $\Lambda : \ell^\infty(\mathbb{N} \to \mathbb{R}) \to \mathbb{R}$ linear such that: (a) $\Lambda L = \Lambda$, (b)

$$\liminf_n \psi(n) \leq \Lambda\psi \leq \limsup_n \psi(n) \quad (\psi \in \ell^\infty)$$

Suggestion: Define the functional Λ_n via $\Lambda_n \psi := \frac{1}{n}\sum_{j=1}^n \psi(n)$, the space $M := \{\psi \in \ell^\infty \mid (\lim_{n\to\infty} \Lambda_n \psi) \text{ exists}\}$, and the convex function $p(\psi) := \limsup_n \Lambda_n \psi$.

Solution

Proof. Let $X := \ell^\infty(\mathbb{N} \to \mathbb{R})$ for notation. As the hint suggests, for each $n \in \mathbb{N}$ we define the averages $\Lambda_n : X \to \mathbb{R}$ via

$$\Lambda_n(\psi) := \frac{1}{n}\sum_{j=1}^n \psi(j)$$

and define

$$M := \{\psi \in X : \lim_{n\to\infty} \Lambda_n \psi \text{ exists}\} \subseteq X,$$

which we know to be a vector subspace by linearity of the limit (i.e. if the limit exists for ψ and ϕ, it certainly exists for $\psi + \phi$ and $\alpha\psi$). Next, define the function $p : X \to \mathbb{R}$ via

$$p(\psi) := \limsup_{n\to\infty} \Lambda_n(\psi)$$

Note that p is convex since each Λ_n is convex (in fact it is linear), the supremum of a collection of convex functions is convex, and the limit of a sequence of convex functions is also convex. Lastly, define the functional $\lambda : M \to \mathbb{R}$ via $\lambda(\psi) = \lim_{n\to\infty} \Lambda_n\psi$, where we know the limit always exists by construction of M. Clearly, λ is linear by linearity of the limit. Furthermore, $\lambda = p$ over M, and so $\lambda \leq p|_M$ trivially. So, we may apply the Hahn-Banach theorem to get a linear functional $\Lambda : X \to \mathbb{R}$ such that $\Lambda \leq p$ over X and $\Lambda|_M = \lambda$. We claim that Λ satisfies the desired properties and is the Banach limit we are looking for.

(a) We wish to show that $\Lambda(L\psi) = \Lambda(\psi)$ for all $\psi \in X$. So, let $\psi \in X$ be arbitrary. We know that

$$\Lambda(L\psi) - \Lambda(\psi) = \Lambda(L\psi - \psi) \leq p(L\psi - \psi) = \limsup_{n\to\infty} \Lambda_n(L\psi - \psi)$$

by linearity of Λ and the fact that $\Lambda \leq p$. We may note that

$$|\Lambda_n(L\psi - \psi)| = \frac{1}{n}\left|\sum_{j=1}^n L\psi(j) - \psi(j)\right| = \frac{1}{n}|\psi(n+1) - \psi(1)| \leq \frac{2\|\psi\|_X}{n}$$

via a telescoping sum and the fact that $\|psi\|_X$ is a uniform upper bound on elements ψ by definition of ℓ^∞. So,

$$\Lambda(L\psi) - \Lambda(\psi) \leq \limsup_{n\to\infty} \Lambda_n(L\psi - \psi) \leq \limsup_{n\to\infty} |\Lambda_n(L\psi - \psi)| = 0$$

Similarly, we may find that

$$\Lambda(\psi) - \Lambda(L\psi) = \Lambda(\psi - L\psi) \leq p(\psi - L\psi) = \limsup_{n\to\infty} \Lambda_n(\psi - L\psi) \leq \limsup_{n\to\infty} |\Lambda_n(\psi - L\psi)| = 0$$

Thus, $\Lambda(L\psi) = \Lambda(\psi)$. Since this holds for all $\psi \in X$, (a) is proven.

(b) First, note that $p(\psi) = \limsup_{n\to\infty}\left(\frac{1}{n}\sum_{j=1}^{n}\psi(j)\right) \leq \limsup_{n\to\infty}\psi(n)$; this follows from the general form for the Stolz-Cesaro theorem, or equivalently from the observation that the limsup of averages cannot be larger than the limsup of elements in a sequence. Thus, since $\Lambda \leq p$, we know that $\Lambda(\psi) \leq \limsup_{n\to\infty}\psi(n)$ for all $\psi \in X$. To see the other direction, let $\psi \in X$ be arbitrary. Then, $-\psi$ is also in X, and so we may apply the recent result to see that

$$\Lambda(-\psi) \leq \limsup_{n\to\infty} -\psi(n) \implies -\Lambda(\psi) \leq -\liminf_{n\to\infty}\psi(n) \implies \liminf_{n\to\infty}\psi(n) \leq \Lambda(\psi)$$

So, for all $\psi \in X$ we know

$$\liminf_{n\to\infty}\psi(n) \leq \Lambda(\psi) \leq \limsup_{n\to\infty}\psi(n),$$

proving (b). ■

Problem 6

Prove that the closed unit ball of an infinite-dimensional Banach space is not compact.

Solution

Proof. Let X be an infinite-dimensional Banach space, and let $B_1 := \{x \in X : \|x\| < 1\}$ be the open unit ball about the origin. Then, $B_1 \in \text{Nbhd}(0_X)$. Suppose by way of contradiction that $\overline{B_1}$ is compact. Therefore, X is locally compact since the origin has a neighborhood whose closure is compact. Since X is a locally compact TVS, by Theorem 1.22 in Rudin we see that X has finite dimension. This contradicts the infinite-dimensionality of X, and so $\overline{B_1}$ cannot be compact. ■

Problem 7

Prove that an infinite-dimensional Banach space cannot be spanned, as a vector space, by a countable subset.

Solution

Proof. Let X be an infinite-dimensional Banach space. Suppose by way of contradiction there were some $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $X = \text{span}\{x_1, x_2, \dots\}$. Without loss of generality we may say that $\|x_n\| = 1$, since dividing each x_n by its norm will not change the span. For each $n \in \mathbb{N}$ define the set

$$E_n := \text{span}\{x_1, x_2, \dots x_n\}$$

Then, each E_n is a finite-dimensional subspace of X. Furthermore, for any $x \in X$, we know that $x = \sum_{k=1}^{N} \alpha_k x_{n_k}$ (i.e. x is a linear combination of finitely many elements of the span), and so

$$x \in E_{n_N} \implies x \in \bigcup_{n\in\mathbb{N}} E_n$$

Since this holds for all $x \in X$, we find that

$$X \subseteq \bigcup_{n\in\mathbb{N}} E_n \implies X = \bigcup_{n\in\mathbb{N}} E_n$$

Therefore, X is an infinite-dimensional TVS that is a countable union of finite-dimensional subspaces; by Problem 15 on problem set 2, we find that X is meagre. This is a contradiction since X is a Banach space and so the Baire Category Theorem applies. ■