# MAT 520: Problem Set 2

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# Problem 1

Let a normed vector space $(X, \|\cdot\|)$ be given. Show that there exists some sesquilinear inner product $\langle\cdot,\cdot\rangle : X^2 \to \mathbb{C}$ which is compatible with the norm, in the sense that

$$\|x\| = \sqrt{\langle x, x\rangle} \quad (x \in X)$$

if and only if the norm satisfies (any one of the equivalent) parallelogram identity. Here is one version which I like:

$$\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \quad (x, y \in X).$$

(This is phrased as an inequality rather than equality since the other direction of the inequality is always true, so there is nothing to verify. Show this first).

## Solution

**Proof.** I have done this proof before. Thank you for giving us the option to not have to do it again. :) ■

# Problem 2

Let an inner-product vector space (X, $\langle\cdot,\cdot\rangle$) be given. Prove the Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

## Solution

**Proof.** I have done this proof before. Thank you for giving us the option to not have to do it again. :) ■

# Problem 3

Two norms $\|\cdot\|_1$ , $\|\cdot\|_2$ on a normed space $X$ are called equivalent iff $\exists\, a, b \in (0, \infty)$ such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \qquad (\forall x \in X)$$

Show that all norms on $\mathbb{C}^n$ are equivalent.

## Solution

**Proof.** Clearly, norm equivalence is an equivalence relation, and so it suffices to show that all norms are equivalent to one fixed norm. Let $\|\cdot\|_1$ be the 1-norm on $\mathbb{C}^n$, and let $\|\cdot\|$ be some arbitrary other norm; we want to show that they are equivalent. Let $e_1, \ldots, e_n$ denote any basis of $\mathbb{C}^n$, and let $b := \max_{1\leq i\leq n} \|e_i\|$. Then, for every $z = \sum_{i=1}^n z_i e_i \in \mathbb{C}^n$ we know by the triangle inequality and homogeneity that

$$\|z\| = \left\|\sum_{i=1}^n z_i e_i\right\| \leq \sum_{i=1}^n \|z_i e_i\| = \sum_{i=1}^n |z_i| \cdot \|e_i\| \leq b \sum_{i=1}^n |z_i| = b\|z\|_1$$

From here, we may see that for all $z, w \in \mathbb{C}^n$,

$$|\|z\| - \|w\|| \leq \|z - w\| \leq b\|z - w\|_1,$$

and so the norm function $\|\cdot\|$ is $b$-Lipschitz w.r.t. the $\|\cdot\|_1$ norm and is therefore continuous in the $\|\cdot\|_1$ norm topology. Also, the unit sphere $S := \{x \in \mathbb{C}^n : \quad \|x\|_1 = 1\}$ is compact in the $\|\cdot\|_1$ norm topology, and so the $\|\cdot\|$ function attains its minimum. Then, there is some $x \in S$ such that $a := \|x\| \leq \|y\|$ for all $y \in S$. We know $a > 0$ since $x \neq 0$ (this is because $0 \notin S$). Lastly, for any $z \in \mathbb{C}^n$ we know by homogeneity that

$$\|z\| = \|z\|_1 \cdot \left\|\frac{z}{\|z\|_1}\right\| \geq a\|z\|_1,$$

where we used that $\frac{z}{\|z\|_1} \in S \implies \left\|\frac{z}{\|z\|_1}\right\| \geq a$. So, for all $z \in \mathbb{C}^n$ we have that

$$a\|z\|_1 \leq \|z\| \leq b\|z\|_1,$$

and thus $\|\cdot\|$ is equivalent to $\|\cdot\|_1$. Therefore, all norms on $\mathbb{C}^n$ are equivalent. ■

# Problem 4

Let $X$ be a Banach space which is Banach w.r.t. two different norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume that $\|\cdot\|_1 \leq C\|\cdot\|_2$ for some $C \in (0, \infty)$. Show that there exists a $D \in (0, \infty)$ such that $\|\cdot\|_2 \leq D\|\cdot\|_1$.

## Solution

**Proof.** Let $X_1$ denote the Banach space $(X, \|\cdot\|_1)$ and $X_2$ the Banach space $(X, \|\cdot\|_2)$. Consider the identity map $i : X_2 \to X_1$ sending $x \mapsto x$; it is certainly bijective and linear. We show that $i$ is also continuous. Let $x \in X_2$ be arbitrary and let $\epsilon > 0$. Then, letting $\delta := \frac{\epsilon}{C}$, we have that for all $y \in X_2$ with $\|x - y\|_2 < \delta$,

$$\|i(x) - i(y)\|_1 = \|x - y\|_1 \leq C\|x - y\|_2 < C\delta = \epsilon$$

So, the identity map from $X_2 \to X_1$ is continuous, which means that $i \in \mathcal{B}(X_2 \to X_1)$. Since $i$ is surjective, we may apply the Open Mapping Theorem to see that $i$ is also an open map. Since $i$ is a bijective, continuous, and open map, it is a homeomorphism, which means that $i^{-1} \in \mathcal{B}(X_1 \to X_2)$. Letting $D$ denote the operator norm of $i^{-1}$ in $\mathcal{B}(X_1 \to X_2)$, we find by the result of Problem 5 that for all $x \in X_1$,

$$\|i^{-1}(x)\|_2 \leq D\|x\|_1 \implies \|x\|_2 \leq D\|x\|_1$$

as desired.

In words, since the identity map is continuous, the topology on $X_1$ refines the topology on $X_2$. Since the identity map from $X_1 \to X_2$ (i.e. $i^{-1}$) is also continuous, the topology on $X_2$ refines that of $X_1$; so, they must be equivalent. ■

# Problem 5

Show that if $A : X \to Y$ is a bounded linear map between Banach spaces then

$$\|Ax\|_Y \leq \|A\|_{\text{op}} \|x\|_X \quad (x \in X).$$

## Solution

**Proof.** We know by definition of the operator norm as a supremum that for all $z \in \overline{B_1(0_X)}$ (the closed unit ball),

$$\|Az\|_Y \leq \|A\|_{\text{op}}$$

Let $x \in X$ be arbitrary. Then,

$$\|Ax\|_Y = \|x\|_X \cdot \left\| \frac{1}{\|x\|_X} Ax \right\|_Y = \|x\|_X \cdot \left\| A\left( \frac{x}{\|x\|_X} \right) \right\|_Y \leq \|x\|_X \|A\|_{\text{op}},$$

where for the first equality we used homogeneity, for the second we used linearity of $A$, and for the third we used that $\frac{x}{\|x\|_X} \in \overline{B_1(0_X)}$. ■

# Problem 6

Show that if $A, B : X \to X$ are bounded linear maps on a Banach space, then

$$\|AB\|_{\text{op}} \leq \|A\|_{\text{op}} \|B\|_{\text{op}}$$

## Solution

**Proof.** Let $x \in X$ be an arbitrary vector with $\|x\| \leq 1$. Then, by definition of the operator norm as a supremum we know that

$$\|Bx\| \leq \|B\|_{\text{op}} \implies \left\| \frac{Bx}{\|B\|_{\text{op}}} \right\| \leq 1$$

So, we may apply the definition of the operator norm of $A$ to see that

$$\left\| A \left( \frac{Bx}{\|B\|_{\text{op}}} \right) \right\| \leq \|A\|_{\text{op}} \implies \left\| \frac{1}{\|B\|_{\text{op}}} ABx \right\| \leq \|A\|_{\text{op}} \implies \|ABx\| \leq \|A\|_{\text{op}} \|B\|_{\text{op}}$$

Since this holds for all $x$ with $\|x\| \leq 1$, we may take a supremum over such $x$ to find that

$$\|AB\|_{\text{op}} \leq \|A\|_{\text{op}} \|B\|_{\text{op}}$$

as desired. ∎

# Problem 7

Let $d : X^2 \to [0, \infty)$ be a homogeneous metric on a TVS $X$. Show that $S \subseteq X$ is bounded (in the TVS sense: for any $N \in \text{Nbhd}(0_X)$ one has $S \subseteq tN$ for all $t > 0$ large) if and only if $S$ is bounded in the metric sense:

$$\sup_{x \in S} d(x, 0_X) < \infty$$

## Solution

**Proof.** For this problem, let $B_r := \{z \in X : \ d(z, 0_X) < r\}$ be the open ball of radius $r$ around the origin; we know by definition of the topology induced by $d$ that $B_r \in \text{Nbhd}(0_X)$ for all $r > 0$.

( $\implies$ ) Suppose first that $S$ is bounded in the TVS sense, and let $B_1 \in \text{Nbhd}(0_X)$ be the open unit ball. By boundedness, we know that there is some $t > 0$ large enough that $S \subseteq tB_1$. Let $x \in S$ be arbitrary; then, $x \in tB_1$. So, $\frac{x}{t} \in B_1$, which means that

$$d\left(\frac{x}{t}, 0_X\right) < 1 \implies d(x, 0_X) < t$$

by homogeneity of the metric and the fact that $t0_X = 0_X$. Since this holds for all $x \in S$, we find that

$$\sup_{x \in S} d(x, 0_X) \leq t < \infty,$$

and so $S$ is bounded in the metric sense.

( $\impliedby$ ) Suppose now that $S$ is bounded in the metric sense. If $S \subseteq \{0_X\}$ then the result holds trivially; so, suppose $S$ contains a nonzero element. Then, there is some $s := \sup_{x \in S} d(x, 0_X) > 0$ such that for all $x \in S$,

$$d(x, 0_X) \leq s$$

Now, let $N \in \text{Nbhd}(0_X)$ be arbitrary; we may select a $U \subseteq N$ that is a balanced and open neighborhood of $0_X$. Let $\left\{B_{\frac{1}{n}}\right\}_{n \in \mathbb{N}}$ be the open balls of radius $\frac{1}{n}$ about the origin. Since these form a local basis by definition of the topology induced by $d$, we have that there is some $M \in \mathbb{N}$ large enough that $B_{\frac{1}{M}} \subseteq U$. Now, let $x \in S$. Then, $d(x, 0_X) < s$, and so by homogeneity

$$d\left(\frac{x}{sM}, 0_X\right) < \frac{1}{M} \implies \frac{x}{sM} \in B_{\frac{1}{M}} \subseteq U$$

So, for all $t > sM$ we know that

$$\left|\frac{sM}{t}\right| \leq 1 \implies \left(\frac{sM}{t}\right) \frac{x}{sM} = \frac{x}{t} \in U$$

since $U$ is balanced. Thus, for all $t > sM$ we have

$$x \in tU \subseteq tN$$

Since this holds for all $x \in S$, we find that $S \subseteq tN$ for all $t > sM$. This proves TVS boundedness of $S$. ■

# Problem 8

Show that a linear map $A : X \to Y$ between two Banach spaces maps bounded sets of $X$ to bounded sets of $Y$ iff $\|A\|_{\text{op}} < \infty$.

## Solution

**Proof.** ( $\Longrightarrow$ ) Suppose that $A$ maps bounded sets of $X$ to bounded sets of $Y$. Since $B_1(0_X)$ is a bounded set (see Problem 7), then $A(B_1(0_X))$ is also bounded. So, for all $x_0 \in X$ with $\|x_0\|_X \leq 1$, we have

$$\|Ax_0\|_Y \leq \sup_{x \in B_1(0_X)} \|Ax\|_Y = \sup_{y \in A(B_1(0_X))} \|y\|_Y =: M < \infty,$$

where we know that $M < \infty$ since $A(B_1(0_X))$ is bounded. Thus, $\|A\|_{\text{op}} \leq M < \infty$.

( $\Longleftarrow$ ) Suppose now that $\|A\|_{\text{op}} < \infty$. Let $E \subseteq X$ be any bounded set. Then, there is some $M < \infty$ such that $\|x\|_X \leq M$ for every $x \in E$. By Problem 5,

$$\forall x \in E \quad \|Ax\|_Y \leq \|A\|_{\text{op}} \|x\|_X \leq M\|A\|_{\text{op}} \implies \sup_{x \in E} \|Ax\|_Y \leq M\|A\|_{\text{op}} < \infty$$

Therefore, we get that

$$\sup_{y \in A(E)} \|y\|_Y \leq M\|A\|_{\text{op}} < \infty,$$

and so $A(E)$ is bounded. This means that $A$ maps bounded sets of $X$ to bounded sets of $Y$. ∎

# Problem 9

Show that a linear map $A : X \to Y$ between two Banach spaces is bounded iff it is continuous iff it is continuous at zero (in showing continuity implies boundedness please do not use the TVS theorem from Chapter 1 but rather do this directly in the context of Banach spaces).

## Solution

**Proof.** (bounded $\implies$ continuous) First, suppose that $A$ is bounded. Then, for any $x, z \in X$ we know by linearity and Problem 5 that

$$\|Ax - Az\|_Y = \|A(x - z)\|_Y \leq \|A\|_{\text{op}} \|x - z\|_X$$

Let $x \in X$ be arbitrary, and let $\epsilon > 0$. Then, for any $z \in X$ such that $\|x - z\|_X < \delta := \frac{\epsilon}{\|A\|_{\text{op}}}$, we have that

$$\|Ax - Az\|_Y \leq \|A\|_{\text{op}} \|x - z\|_X < \|A\|_{\text{op}} \delta = \epsilon,$$

and so $A$ is continuous at $x$. Since this holds for all $x \in X$, the map $A$ is continuous.

(continuous $\implies$ continuous at 0) This comes immediately from the definition of continuity.

(continuous at 0 $\implies$ bounded) Suppose that $A$ is continuous at $0_X$. Let $\epsilon > 0$. Then, by definition of continuity in norm topologies, there exists some $\delta > 0$ such that whenever $\|x - 0_X\|_X = \|x\|_X \leq \delta$, we have that

$$\|Ax - A0_X\|_Y < \epsilon \implies \|Ax\|_Y < \epsilon,$$

where the implication follows since $A$ sends $0_X$ to $0_Y$. So, $\|x\|_X \leq \delta \implies \|Ax\|_Y < \epsilon$. Let $z \in X$ be arbitrary such that $\|z\|_X \leq 1$. Then, by homogeneity of the norm, $\|\delta z\|_X \leq \delta$, and so by continuity this tells us that

$$\|A(\delta z)\|_Y < \epsilon \implies \delta \|Az\|_Y < \epsilon \implies \|Az\|_Y < \frac{\epsilon}{\delta},$$

where we used linearity and homogeneity for the first implication. Since this holds for all $z \in X$ with $\|z\|_X \leq 1$, taking a supremum tells us that

$$\sup_{\substack{z \in X \\ \|z\|_X \leq 1}} \|Az\|_Y \leq \frac{\epsilon}{\delta} < \infty$$

Therefore, $A$ is bounded and $\|A\|_{\text{op}} \leq \frac{\epsilon}{\delta}$. ■

# Problem 10

Show that $L^\infty(\mathbb{R} \to \mathbb{C})$ is a Banach space.

## Solution

**Proof.** Define the essential supremum

$$\|f\|_\infty := \inf\{M \in \mathbb{R} : \quad |f(x)| \leq M \text{ for a.e. } x \in \mathbb{R}\},$$

and so $L^\infty(\mathbb{R} \to \mathbb{C}) \equiv \{f : \mathbb{R} \to \mathbb{C} : \ \|f\|_\infty < \infty\}$ is the set of all functions from $\mathbb{R}$ to $\mathbb{C}$ that are bounded a.e. (actually, it is this space modulo the equivalence relation of functions differing only on a set of measure 0). Let $f, g : \mathbb{R} \to \mathbb{C}$ and $\alpha \in \mathbb{C}$ be arbitrary. Then we know that for a.e. $x \in \mathbb{R}$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty,$$

and so it certainly holds that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Next,

$$\begin{aligned}
\|\alpha f\|_\infty &= \inf\{M \in \mathbb{R} : \quad |\alpha f(x)| \leq M \text{ for a.e. } x \in \mathbb{R}\} \\
&= \inf\{M \in \mathbb{R} : \quad |\alpha||f(x)| \leq M \text{ for a.e. } x \in \mathbb{R}\} \\
&= \inf\left\{M \in \mathbb{R} : \quad |f(x)| \leq \frac{M}{|\alpha|} \text{ for a.e. } x \in \mathbb{R}\right\} \\
&= |\alpha| \inf\left\{\frac{M}{|\alpha|} \in \mathbb{R} : \quad |f(x)| \leq \frac{M}{|\alpha|} \text{ for a.e. } x \in \mathbb{R}\right\} \\
&= |\alpha|\|f\|_\infty
\end{aligned}$$

Lastly, if $\|f\|_\infty = 0$ then $|f(x)| \leq 0$ for a.e. $x \in \mathbb{R}$, which means that $f = 0$ almost everywhere. So, $f$ is in the equivalence class of the constant 0 function, which means that $f$ is the 0 element in $L^\infty(\mathbb{R} \to \mathbb{C})$. From the above, we also see that if $f, g \in L^\infty(\mathbb{R} \to \mathbb{C})$, then $f + g, \ \alpha f \in L^\infty(\mathbb{R} \to \mathbb{C})$ too, and so $L^\infty(\mathbb{R} \to \mathbb{C})$ is a vector space. Furthermore, we have shown that $\| \cdot \|_\infty$ on this vector space is homogenous, positive, and satisfies the triangle inequality, and so this means that $L^\infty(\mathbb{R} \to \mathbb{C})$ is a normed vector space. All that remains to be shown is that it is complete in this norm.

To this end, let $\{f_n\}_n \subseteq L^\infty(\mathbb{R} \to \mathbb{C})$ be a Cauchy sequence with respect to the $\| \cdot \|_\infty$ norm. Firstly, this tells us that the sequence is bounded. To see this, pick any $\delta > 0$, and then there is some $N \in \mathbb{N}$ such that for all $n > N$ we have

$$\delta > \|f_N - f_n\|_\infty \geq |\|f_N\|_\infty - \|f_n\|_\infty| \implies \|f_n\|_\infty \leq \|f_N\|_\infty + \delta$$

So, for any $m \in \mathbb{N}$ we know that $\|f_m\|_\infty \leq \max\{\|f_1\|_\infty, ..., \|f_N\|_\infty, \|f_N\|_\infty + \delta\} =: K < \infty$, and so the sequence is bounded. Then, for a.e. $x \in \mathbb{R}$ and any $n, m \in \mathbb{N}$ we have that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty,$$

and so the sequence $\{f_n(x)\}_n$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, this means that $f_n(x)$ converges to some point in $\mathbb{R}$, which we call $f(x)$. Since this holds for a.e. $x \in \mathbb{R}$, we see that $f_n \to f$ pointwise a.e. for some function $f : \mathbb{R} \to \mathbb{C}$. It remains to be shown that $\|f\|_\infty < \infty$ and that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

For the first result, let $\epsilon > 0$. Then, for a.e. $x \in \mathbb{R}$ there exists an $N_x \in \mathbb{N}$ such that $|f(x) - f_{N_x}(x)| < \epsilon$ by definition of pointwise convergence. So, by the reverse triangle inequality,

$$||f(x)| - |f_{N_x}(x)|| \leq |f(x) - f_{N_x}(x)| < \epsilon \implies |f(x)| \leq |f_{N_x}(x)| + \epsilon \leq \|f_{N_x}\|_\infty + \epsilon \leq K + \epsilon,$$

where we used the definition of the essential supremum and the fact that the sequence is bounded (note that the above doesn't hold everywhere, but only a.e.). So, $|f(x)| \leq K + \epsilon$ for a.e. $x \in \mathbb{R}$, and so $\|f\|_\infty \leq K + \epsilon < \infty$; thus, $f \in L^\infty(\mathbb{R} \to \mathbb{C})$.

Now, we must show that $\|f_n - f\|_\infty \to 0$. Let $\epsilon > 0$ be arbitrary. Select a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}} \subseteq \{f_n\}_{n\in\mathbb{N}}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq \frac{\epsilon}{2^k} \qquad \forall k \in \mathbb{N},$$

which we may do by the Cauchy criterion (pick $n_k = N$ for the $N$ such that for all $n, m \geq N$ we know $\|f_n - f_m\|_\infty \leq \epsilon 2^{-k}$). Since $f_n \to f$ pointwise a.e. and so $f_{n_k} \to f$ pointwise a.e., we may express

$$f(x) = f_{n_k}(x) + \sum_{m=k}^{\infty} (f_{n_{m+1}}(x) - f_{n_m}(x))$$

for any $k \in \mathbb{N}$ via a telescoping sum. So, for $k \in \mathbb{N}$ we have

$$f(x) - f_{n_k}(x) = \sum_{m=k}^{\infty} (f_{n_{m+1}}(x) - f_{n_m}(x))$$

Since this holds for a.e. $x \in \mathbb{R}$, we have that

$$\|f - f_{n_k}\|_\infty = \left\| \sum_{m=k}^{\infty} (f_{n_{m+1}} - f_{n_m}) \right\|_\infty \leq \sum_{m=k}^{\infty} \|f_{n_{m+1}} - f_{n_m}\|_\infty \leq \sum_{m=k}^{\infty} \frac{\epsilon}{2^m} \leq \sum_{m=1}^{\infty} \frac{\epsilon}{2^m} = \epsilon,$$

where the first inequality comes from the infinite triangle inequality (which we can apply because $|\cdot|$ is continuous and thus so is $\|\cdot\|_\infty$), the second inequality comes from our selection of the subsequence, and the third inequality is because it is a sum of nonnegative terms. Thus, $f_{n_k} \to f$ in the $\|\cdot\|_\infty$ norm. Now, let $\delta > 0$. Then, there is some $N \in \mathbb{N}$ such that for all $n, m > N$ we have $\|f_n - f_m\|_\infty < \frac{\delta}{2}$ by the Cauchy criterion. By the fact that $\|f_{n_k} - f\|_\infty \to 0$, there is some $M \in \mathbb{N}$ such that for all $k$ with $n_k > M$, we know $\|f_{n_k} - f\|_\infty < \frac{\delta}{2}$. So, for any $n > N$, we know that

$$\|f_n - f\|_\infty \leq \|f_n - f_{n_k}\|_\infty + \|f_{n_k} - f\|_\infty < \frac{\delta}{2} + \frac{\delta}{2} = \delta \qquad \forall k \text{ s.t. } n_k > \max\{N, M\}$$

Since such an $N$ exists for any $\delta > 0$, we find that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Since this holds for all Cauchy sequences, the space is complete. ∎

# Problem 11

Let a normed vector space $(X, \|\cdot\|)$ be given. Show that $(X, \|\cdot\|)$ is complete iff, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$,

$$\left(\sum_{n\in\mathbb{N}} \|x_n\| < \infty\right) \implies \left(\lim_{N\to\infty} \sum_{n=1}^{N} x_n \text{ exists and equals some } x \in X\right)$$

## Solution

**Proof.** ( $\implies$ ) Suppose first that $(X, \|\cdot\|)$ is complete. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be a sequence such that $\sum_{n\in\mathbb{N}} \|x_n\| < \infty$. Define $z_k := \sum_{n=1}^{k} x_n \in X$; we want to show that the sequence $\{z_k\}_{k\in\mathbb{N}}$ is Cauchy. So, let $\epsilon > 0$. Let $N$ be large enough that $\sum_{n=N}^{\infty} \|x_n\| < \epsilon$, which can be done since infinite sums with finite value have arbitrarily small tails. Then, for any $k, l > N$ with $k < l$ we have that

$$\|z_k - z_l\| = \left\|\sum_{n=k}^{l} x_n\right\| \leq \sum_{n=k}^{l} \|x_n\| \leq \sum_{n=N}^{\infty} \|x_n\| < \epsilon,$$

where the first equality is by definition of the $z_k$'s, the first inequality uses the triangle inequality, the second inequality uses that $k > N$ and $l < \infty$ and that we are summing nonnegative terms, and the last inequality uses our selection of $N$. Since such an $N$ exists for all $\epsilon$, we find that $\{z_k\}_k$ is Cauchy. So, by completeness of $X$ the sequence converges to some $x \in X$. In other words,

$$\lim_{N\to\infty} \sum_{n=1}^{N} x_n = \lim_{N\to\infty} z_N = x$$

for some $x \in X$, which is what we wanted to show.

( $\impliedby$ ) Suppose that the big implication at the top of the page holds for all sequences. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence; we want to show that $x_n \to x$ for some $x \in X$ to show completeness. For each $k \in \mathbb{N}$, let $N_k$ be such that for all $n, m \geq N_k$ we have $\|x_n - x_m\| \leq \frac{1}{2^k}$. Suppose without loss of generality that $\{N_k\}_k$ is strictly monotonically increasing. For each $k \in \mathbb{N}$ define

$$d_k := x_{N_{k+1}} - x_{N_k}$$

Then, by construction we have that $\|d_k\| \leq \frac{1}{2^k}$ for all $k$. So,

$$\sum_{k\in\mathbb{N}} \|d_k\| \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty$$

So, we may apply the implication to the sequence $\{d_k\}_k$ and see that

$$\lim_{m\to\infty} \sum_{k=1}^{m} d_k = x$$

for some $x \in X$, where the above limit is taken with respect to the norm $\|\cdot\|$. Now, for every $m \in \mathbb{N}$ we have that

$$\sum_{k=1}^{m} d_k = \sum_{k=1}^{m} (x_{N_{k+1}} - x_{N_k}) = x_{N_{m+1}} - x_{N_1},$$

and so the existence of the limit means that

$$x + x_{N_1} = \lim_{m\to\infty} x_{N_{m+1}} = \lim_{n\to\infty} x_n,$$

where for the last equality we used the monotonicity of $\{N_k\}_k$. In partuclar, we find that there is some $\tilde{x} := x + x_{N_1} \in X$ for which $\tilde{x} = \lim_{n\to\infty} x_n$. Since such a result is true for every Cauchy sequence, we find that $(X, \|\cdot\|)$ is complete. ■

# Problem 12

Show that the $\frac{1}{3}$-Cantor set is nowhere dense.

## Solution

**Proof.** Let $\mathcal{C} \subseteq [0,1]$ denote the $\frac{1}{3}$-Cantor set. We can write

$$\mathcal{C}_n := \bigcup_{k=0}^{3^{n-1}-1} \left( \left[ \frac{3k}{3^n}, \frac{3k+1}{3^n} \right] \cup \left[ \frac{3k+2}{3^n}, \frac{3k+3}{3^n} \right] \right), \qquad \mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n$$

Each $\mathcal{C}_n = \bigcup_{k=0}^{3^n-1} \left( \left[ \frac{3k}{3^n}, \frac{3k+1}{3^n} \right] \cup \left[ \frac{3k+2}{3^n}, \frac{3k+3}{3^n} \right] \right)$ is a finite union of closed sets and is therefore closed, and so $\mathcal{C}$ is an intersection of closed sets and is itself closed. So, to show that $\mathcal{C}$ is nowhere dense it suffices to show that it has empty interior. To this end, suppose by way of contradiction that $x \in \text{int}(\mathcal{C})$; clearly, $x \notin \{0,1\}$. Then, there is some $\delta > 0$ for which $(x-\delta, x+\delta) \subseteq \mathcal{C}$. Suppose without loss of generality that $\delta = 3^{-n}$ for some $n \in \mathbb{N}$, which we can do because we can shrink $\delta$ arbitrarily. We want to show that $(x-\delta, x+\delta) \cap ([0,1] \setminus \mathcal{C}_{n+1})$ is nonempty, as this will show that $(x-\delta, x+\delta) \nsubseteq \mathcal{C}_{n+1}$ and therefore $(x-\delta, x+\delta) \nsubseteq \mathcal{C}$. We note that

$$[0,1] \setminus \mathcal{C}_{n+1} = [0,1] \setminus \left( \bigcup_{k=0}^{3^n-1} \left( \left[ \frac{3k}{3^{n+1}}, \frac{3k+1}{3^{n+1}} \right] \cup \left[ \frac{3k+2}{3^{n+1}}, \frac{3k+3}{3^{n+1}} \right] \right) \right)$$

So, for any $k < 3^n$ it will hold that

$$\left( \frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right) \subseteq [0,1] \setminus \mathcal{C}_{n+1}$$

In words, the above states that any element of one of the removed middle thirds will no longer be in $\mathcal{C}_{n+1}$. Plugging in the relation between $n$ and $\delta$, we see that for every $k < \frac{1}{\delta}$ we have

$$\left( k\delta + \frac{\delta}{3}, k\delta + \frac{2\delta}{3} \right) \subseteq [0,1] \setminus \mathcal{C}_{n+1}$$

Note that

$$[0,1) = \bigsqcup_{k=0}^{\frac{1}{\delta}-1} [k\delta, (k+1)\delta),$$

where the above is a disjoint union. Select the $k < \frac{1}{\delta}$ such that $x \in [k\delta + (k+1)\delta) = [k\delta, k\delta + \delta)$. Then, it certainly holds that

$$[k\delta, k\delta + \delta) \subseteq (x-\delta, x+\delta)$$

In particular, $(x-\delta, x+\delta) \cap \left( k\delta + \frac{\delta}{3}, k\delta + \frac{2\delta}{3} \right)$ must be nonempty. Therefore, $(x-\delta, x+\delta) \nsubseteq \mathcal{C}_{n+1}$, and so $(x-\delta, x+\delta) \nsubseteq \mathcal{C}$. This is a contradiction, and thus $\mathcal{C}$ has empty interior. Since it is closed, we get that the $\frac{1}{3}$-Cantor set is nowhere dense in $[0,1]$. ■

# Problem 13

Prove that if $X$ is a locally compact Hausdorff space and $\{V_j\}_{j\in\mathbb{N}}$ are open dense sets then $V := \bigcap_{j\in\mathbb{N}} V_j$ is also dense in $X$.

## Solution

**Proof.** We start with a useful lemma.

**Lemma 1.** *Let $X$ be a locally compact Hausdorff space. For any nonempty $U \in \text{Open}(X)$, there is a nonempty $W \in \text{Open}(X)$ such that $\overline{W}$ is compact and $\overline{W} \subseteq U$.*

**Proof of Lemma 1.** Let $x \in U$. Then, by local compactness there is a $V \in \text{Open}(X)$ for which $\overline{V}$ is compact and $x \in V$. Define $K := \overline{V} \cap U^C$; then, $K$ is a closed (since $U$ is open) subset of a compact set and is itself compact. Also, $x \notin K$ since $x \in U$. Now, we may apply the Hausdorff condition between $x$ and every point of $K$ as follows: for every $k \in K$ we may find two open sets $A_k, B_k \in \text{Open}(X)$ such that $x \in A_k$, $k \in B_k$, and $A_k \cap B_k = \emptyset \implies A_k \subseteq B_k^C \implies \overline{A_k} \subseteq B_k^C \implies \overline{A_k} \cap B_k = \emptyset$ (here, we used that the closure is the smallest closed superset, and $B_k^C$ is a closed superset). We note that $K \subseteq \bigcup_{k\in K} B_k$ is an open cover of a compact set, and so there exists a finite subcover; i.e. there are some $k_1, \ldots, k_N \in K$ such that $K \subseteq \bigcup_{n=1}^N B_{k_n}$. Now, define $A := \bigcap_{n=1}^N A_{k_n}$. Certainly, $A$ is open and nonempty (since $A$ is a finite intersection of open sets and $x \in A$). Furthermore, for all $n = 1, \ldots, N$ we have

$$\overline{A} \cap B_{k_n} \subseteq \overline{A_{k_n}} \cap B_{k_n} = \emptyset,$$

and so $\overline{A} \cap K = \emptyset \implies \overline{A} \cap \overline{V} \cap U^C = \emptyset \implies \overline{A} \cap \overline{V} \subseteq U$. Letting $W := A \cap V$, we find that $W$ is open (since $A$ and $V$ are), $W$ is nonempty (since $x \in A$ and $x \in V$), and $\overline{W} = \overline{A \cap V} \subseteq \overline{A} \cap \overline{V} \subseteq U$. Lastly, since $\overline{W} \subseteq \overline{V}$, we have that $\overline{W}$ is a closed subset of a compact set and is thus compact. So, $W$ satisfies all the desired properties, and the lemma is proved. ∎

We may now proceed with proving the main result. Let $U \in \text{Open}(X)$ be arbitrary. We want to show that $U \cap V \neq \emptyset$. Note that $U \cap V_1$ is open since $V_1$ is open and nonempty since $V_1$ is dense; so, we may apply Lemma 1 to find a nonempty $W_1 \in \text{Open}(X)$ such that $\overline{W_1} \subseteq U \cap V_1$ is compact. Similarly, since $W_1$ is open and $V_2$ is open and dense, there is a nonempty $W_2 \in \text{Open}(X)$ for which $\overline{W_2} \subseteq W_1 \cap V_2$ is compact. Repeating, we find a sequence of nonempty open sets $\{W_n\}_n \subseteq \text{Open}(X)$ where for all $n \in \mathbb{N}$, $\overline{W_n}$ is compact and

$$\overline{W_{n+1}} \subseteq W_n \cap V_{n+1} \subseteq U \cap \bigcap_{j=1}^{n+1} V_j$$

Define

$$W := \bigcap_{n\in\mathbb{N}} \overline{W_n}$$

Then, we have that $W \subseteq U \cap \bigcap_{j\in\mathbb{N}} V_j = U \cap V$, and so all we must show is that $W$ is nonempty. Clearly, $W$ is compact, since it is a closed subset of a compact set. Suppose by way of contradiction that $W$ were empty. Then, for all $x \in \overline{W_1}$ there must be some $N_x > 1$ such that $x \notin \overline{W_{N_x}} \implies x \in \overline{W_{N_x}}^C$. Thus, we have that

$$\overline{W_1} \subseteq \bigcup_{\substack{n\in\mathbb{N}\\ n>1}} \overline{W_n}^C$$

Since each $\overline{W_n}^C$ is open (as $\overline{W_n}$ is closed), this is an open cover of a compact set, and so it yields a finite subcover $\{\overline{W_{n_k}}^C\}_{k=1}^N$. So, DeMorgan's laws give that

$$\overline{W_1} \subseteq \bigcup_{k=1}^N \overline{W_{n_k}}^C \implies \overline{W_1}^C \supseteq \bigcap_{k=1}^N \overline{W_{n_k}} = \overline{W_{n_N}},$$

where the last equality uses that $\overline{W_{n+1}} \subseteq \overline{W_n}$ for all $n \in \mathbb{N}$, and so the sequence is decreasing. However, because the sequence is decreasing we know that $\overline{W_{n_N}} \subseteq \overline{W_1}$, a contradiction ($\overline{W_{n_N}}$ cannot be a subset of a set and its complement at the same time). So, $W$ must be nonempty. Thus, $U \cap V$ is nonempty. Since this holds for all $U \in \text{Open}(X)$, we see that $V$ is dense as desired. ■

# Problem 15

Show that if $X$ is an infinite-dimensional TVS which is the union of countably many finite-dimensional subspaces, then $X$ is of Baire's first category. Conclude that no infinite-dimensional Banach space has a countable Hamel basis.

## Solution

**Proof.** We start with the following lemma.

**Lemma 2.** *If $X$ is a TVS and $E \subsetneq X$ is a proper vector subspace, then $E$ has empty interior.*

**Proof of Lemma 2.** Suppose by way of contradiction that there were some $x \in \text{int}(E)$. Then, there must exist some $U \in \text{Nbhd}(x)$ such that $U \subseteq E$. Since $E$ is closed under vector addition, then $V = U - \{x\} \in \text{Nbhd}(0_X)$ and $V \subseteq E$. Let $g \in X \setminus E$, which can be done because $E \subsetneq X$. We know by Theorem 1.15(a) in Rudin that $g \in rV$ for some $r > 0$, and so $\frac{1}{r}g \in V$. Therefore, $\frac{1}{r}g \in E$ since $V \subseteq E$. Since $E$ is closed under scalar multiplication, this means that $g \in E$, contradicting our choice of $g$. So, there can be no $x \in \text{int}(E)$. ■

Let

$$X = \bigcup_{n \in \mathbb{N}} E_n$$

be as described, where each $E_n$ is a finite-dimensional subspace. By Theorem 1.21(b) in Rudin, each $E_n$ is closed. Furthermore, since each $E_n$ is finite-dimensional but $X$ isn't, we know that each $E_n$ is a proper subspace. By Lemma 2, each $E_n$ has empty interior, which means that each $E_n$ is nowhere dense (as $\text{int}(\overline{E_n}) = \text{int}(E_n) = \emptyset$). Thus, $X$ is a countable union of nowhere dense sets, and it meagre as desired.

We now turn to the last result. Suppose by way of contradiction that $X$ is an infinite-dimensional Banach space with a countable Hamel basis $\{x_k\}_{k \in \mathbb{N}}$. Let $\{I_n\}_{n \in \mathbb{N}} \subseteq 2^{\mathbb{N}}$ denote the set of all *finite* subsets of $\mathbb{N}$ (we know this to be countable). For each $n \in \mathbb{N}$, define

$$E_n := \text{span}\{x_k : \quad k \in I_n\}$$

Then, each $E_n$ is a finite-dimensional vector subspace of $X$, since it is the span of finitely many elements of $X$. Furthermore, for any $x \in X$, we know by definition of a Hamel basis that $x = \sum_{k \in I_n} \alpha_k x_k$ for some $I_n$ (i.e. $x$ lies in the span of some finite subset of the Hamel basis). So, $x \in E_n$, and we see

$$X \subseteq \bigcup_{n \in \mathbb{N}} E_n \implies X = \bigcup_{n \in \mathbb{N}} E_n$$

This means that $X$ is an infinite-dimensional TVS who can be written as a countable union of finite-dimensional subspaces; the first result of this problem then gives that $X$ is of Baire's first category. However, $X$ is a Banach space, which means it is of Baire's second category by the Baire Category Theorem. This is a contradiction, and so $X$ cannot have a countable Hamel basis. ■

# Problem 16

Find a subset $S \subseteq [0,1]$ which is of Baire's first category but whose Lebesgue measure equals 1.

## Solution

**Proof.** Let $m(\cdot)$ denote the Lebesgue measure. We start by creating a monotonic family of dense subsets of $[0,1]$ whose Lebesgue measure decreases to 0. To do so, let $\{q_n\}_{n\in\mathbb{N}}$ be an enumeration of the rationals in $\mathbb{Q}\cap(0,1)$ and for each $\epsilon > 0$ define

$$E_\epsilon := (0,1) \cap \left( \bigcup_{n\in\mathbb{N}} B_{\epsilon/2^{n+1}}(q_n) \right) \subseteq [0,1],$$

where $B_{\epsilon/2^{n+1}}(q_n) = \left(q_n - \frac{\epsilon}{2^{n+1}}, q_n + \frac{\epsilon}{2^{n+1}}\right)$ is exactly what you expect. Also, define

$$F_\epsilon := [0,1] \setminus E_\epsilon$$

for notation. We have the following properties:

1. Each $E_\epsilon$ is open. So, each $F_\epsilon$ is closed. Furthermore, each $F_\epsilon$ has empty interior. To see this, suppose by way of contradiction that $x \in \text{int}(F_\epsilon)$. Then, there must be some neighborhood of $x$ contained in $F_\epsilon$, which by density of the rationals means that $q_n \in F_\epsilon$ for some $n$. However, $q_n \in E_\epsilon = [0,1]\setminus F_\epsilon$ for all $n$, a contradiction. Thus, each $F_\epsilon$ is closed with empty interior, and so is nowhere dense.

2. If $0 < \delta < \epsilon$, then clearly $B_{\delta/2^{n+1}}(q_n) \subseteq B_{\epsilon/2^{n+1}}(q_n)$ for all $n$, and so $E_\delta \subseteq E_\epsilon$. Then, $F_\epsilon \subseteq F_\delta$.

3. Each $E_\epsilon$ and therefore each $F_\epsilon$ is obviously measurable. By countable subadditivity of the measure, we have for all $\epsilon > 0$ that

$$m(E_\epsilon) \leq m\left( \bigcup_{n\in\mathbb{N}} B_{\epsilon/2^{n+1}}(q_n) \right) \leq \sum_{n\in\mathbb{N}} m(B_{\epsilon/2^{n+1}}(q_n)) = \sum_{n\in\mathbb{N}} \frac{\epsilon}{2^n} = \epsilon$$

   So,

$$1 - \epsilon \leq 1 - m(E_\epsilon) = m(F_\epsilon) \leq 1$$

Now, define the set

$$S := \bigcup_{k\in\mathbb{N}} F_{\frac{1}{k}} \subseteq [0,1]$$

It is certainly measurable since it is a countable union of measurable sets. By property 1, $S$ is a countable union of nowhere dense sets, and so it is meagre. By property 2, the family $\{F_{\frac{1}{k}}\}_k$ is a monotonically increasing family of sets, and so the continuity of measures from below guarantees that

$$m(S) = \lim_{k\to\infty} m(F_{\frac{1}{k}}) \geq \lim_{k\to\infty}\left(1 - \frac{1}{k}\right) = 1,$$

where the inequality above comes from property 3. Since $S \subseteq [0,1] \implies m(S) \leq 1$, we get that $S$ has Lebesgue measure 1. ■

# Problem 17

For any $f \in L^2(\mathbb{S}^1)$, let $\hat{f} : \mathbb{Z} \to \mathbb{C}$ be given by

$$\hat{f}(n) := \frac{1}{2\pi} \int_{\theta \in [-\pi,\pi]} f(e^{i\theta}) e^{-in\theta} d\theta \qquad (n \in \mathbb{Z})$$

For each $n \in \mathbb{N}$, define $\Lambda_n : L^2(\mathbb{S}^1) \to \mathbb{C}$ via

$$\Lambda_n f := \sum_{k=-n}^{n} \hat{f}(k) \qquad (f \in L^2(\mathbb{S}^1))$$

Show that

$$D := \left\{ f \in L^2(\mathbb{S}^1) : \lim_{n\to\infty} \Lambda_n f \text{ exists} \right\}$$

is a dense subspace of $L^2(\mathbb{S}^1)$ of Baire's first category.

## Solution

**Proof.** We start by showing that $D$ is a subspace. We know that $0 \in D$ since $\lambda_n 0 = 0$ for all $n$. Next, for any $f, g \in D$ and any $\alpha \in \mathbb{C}$, it holds that for all $n \in \mathbb{Z}$,

$$\begin{aligned}
(\widehat{f+\alpha g})(n) &= \frac{1}{2\pi} \int_{\theta \in [-\pi,\pi]} (f+\alpha g)(e^{i\theta}) e^{-in\theta} d\theta \\
&= \frac{1}{2\pi} \int_{\theta \in [-\pi,\pi]} f(e^{i\theta}) e^{-in\theta} d\theta + \frac{1}{2\pi} \int_{\theta \in [-\pi,\pi]} \alpha g(e^{i\theta}) e^{-in\theta} d\theta \\
&= \hat{f}(n) + \alpha \hat{g}(n),
\end{aligned}$$

and so $\Lambda_n(f + \alpha g) = \Lambda_n f + \alpha \Lambda_n g$ (note that this proves that $\Lambda_n$ is linear for each $n$). By linearity of the limit, since $\lim_{n\to\infty} \Lambda_n f$ and $\lim_{n\to\infty} \Lambda_n g$ both exist, so too does $\lim_{n\to\infty} \Lambda_n(f + \alpha g)$, and so $f + \alpha g \in D$.

To show that $D$ is dense, we will show that it has a dense subset. Let

$$\mathcal{S} := \left\{ f \in L^2(\mathbb{S}^1) : \ f \text{ is a Schwartz function} \right\}$$

denote the Schwartz space, where we say $f$ is a Schwartz function if $f$ is smooth (infinitely differentiable) s.t. for all multi-indices $\alpha, \beta$, we know $x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta f$ is bounded. We know that $\mathcal{S}$ is dense in $L^2(\mathbb{S}^1)$ (see Lemma 1.2 from Chapter 5 of Stein III). Furthermore, because $f$ is differentiable at $e^{i0}$ (since it is differentiable everywhere), we know that the Fourier series at $e^{i0}$ converges (by Theorem 2.1 in Chapter 3 of Stein I). Namely, we know that for all $f \in \mathcal{S}$,

$$\lim_{n\to\infty} \Lambda_n f = \lim_{n\to\infty} \sum_{k=-n}^{n} \hat{f}(k) = \lim_{n\to\infty} \sum_{k=-n}^{n} \hat{f}(k) e^{ik0} \text{ exists}$$

This result is the pointwise convergence of the Fourier series at points of differentiability. Thus, $\mathcal{S} \subseteq D$. Since $D$ contains a subset that is dense in $L^2(\mathbb{S}^1)$, so too is $D$. Note that we could have used a larger class of functions (such as $C^1(\mathbb{S}^1) \subseteq L^2(\mathbb{S}^1)$ also being dense and a subset of $D$), but I found a reference in Stein for Schwartz functions.

Now, we wish to show that $D$ is meagre. To this end, write

$$D = \bigcup_{R \in \mathbb{N}} D_R,$$

where each $D_R \subseteq D$ is defined by

$$D_R := \left\{ f \in L^2(\mathbb{S}^1) : \ |\Lambda_n f| \leq R \text{ for all } n \right\}$$

We may do this since an infinite series in $\mathbb{C}$ converges if and only if the sequence of partial sums is bounded. We claim that each $D_R$ is closed with empty interior, and so is nowhere dense.

We first show closure; to this end, let $R \in \mathbb{N}$ be arbitrary and let $\{f_j\}_{j\in\mathbb{N}} \subseteq D_R$ be a sequence such that $f_j \to f$ in the $L^2$ norm for some $f \in L^2(\mathbb{S}^1)$. We wish to show that $f \in D_R$ as well. Since $f_j \to f$ in the $L^2$ norm, by Holder's inequality and the fact that the domain is finite measure we may say that $f_j \to f$ in the $L^1$ norm as well. Let $n \in \mathbb{N}$ be arbitrary, and let $\epsilon > 0$. Then, we may find a $N \in \mathbb{N}$ large enough that $\|f - f_N\|_{L^1} < \frac{\epsilon}{2n}$, and so by linearity of $\Lambda_n$ and linearity of the integral,

$$\begin{aligned}
|\Lambda_n f - \Lambda_n f_N| = |\Lambda_n(f - f_N)| &= \frac{1}{2\pi}\int_{\theta\in[-\pi,\pi]} \left|f(e^{i\theta}) - f_N(e^{i\theta})\right| \left|\sum_{k=-n}^{n} e^{ik\theta}\right| d\theta \\
&\leq \frac{1}{2\pi}\int_{\theta\in[-\pi,\pi]} \left|f(e^{i\theta}) - f_N(e^{i\theta})\right| \cdot 2n d\theta = 2n\|f - f_N\|_{L^1} < \epsilon
\end{aligned}$$

So,

$$|\Lambda_n f| = |\Lambda_n f_N + \Lambda_n(f - f_N)| \leq |\Lambda_n f_N| + |\Lambda_n(f - f_N)| \leq R + \epsilon,$$

where we used that $|\Lambda_n f_N| \leq R$ since $f_N \in D_R$. Since this holds for all $\epsilon > 0$, we know that $|\Lambda_n f| \leq R$. Since this holds for all $n$, we find that $f \in D_R$, and so $D_R$ is closed.

We now wish to show that $D_R$ has empty interior. Write $I := \text{int}(D_R)$ to be the interior of $D_R$ for notation, and suppose by way of contradiction that $I \neq \emptyset$. Let $f \in I$. Define $g \in L^2(\mathbb{S}^1)$ to be the function with Fourier coefficients of the form $\hat{g}(k) = \left|\frac{1}{k}\right|$. Then, clearly $g \notin D$ since the harmonic sum doesn't converge, and so $g \notin I$. We claim that $f + \frac{1}{j}g \notin I$ for all $j \in \mathbb{N}$. To this end, note that for all $n \in \mathbb{N}$, we have

$$\left|\Lambda_n\left(f + \frac{1}{j}g\right)\right| = \left|\Lambda_n f + \frac{1}{j}\Lambda_n g\right| \geq \frac{1}{j}|\Lambda_n g| - |\Lambda_n f| \geq \frac{1}{j}2\ln(n) - |\Lambda_n f| \geq \frac{2}{j}\ln(n) - R,$$

where the first inequality is the reverse triangle inequality, the second uses the lower bound $\sum_{k=1}^{n} \frac{1}{k} \geq \ln(n)$, and the last inequality uses that $|\Lambda_n f| \leq R$. So, for each $j \in \mathbb{N}$, if we select an $n > e^{Rj}$ we find that

$$\left|\Lambda_n\left(f + \frac{1}{j}g\right)\right| > \frac{2}{j}Rj - R = R,$$

and so it must be that $f + \frac{1}{j}g \notin D_R$. However, we know that

$$\left\|\left(f + \frac{1}{j}g\right) - f\right\|_{L^2} = \frac{1}{j}\|g\|_{L^2} \to 0,$$

and so $f + \frac{1}{j}g \to f$ in $L^2$. Thus, we have a sequence of elements $\left\{f + \frac{1}{j}g\right\}_{j\in\mathbb{N}} \subseteq I^C$ that converges in $L^2$ to an element $f$, and so $f \in I^C$ since $I^C$ is closed (as $I$ is open). This is a contradiction, and so $D_R$ must have empty interior. Thus, $D_R$ is nowhere dense. Since $D$ can be written as a countable union of nowhere dense sets, it is meagre. ■

# Problem 18

Let $X$ be a Banach space and $Y$ a subspace of $X$ whose complement is of Baire's first category. Show that $Y = X$.

## Solution

**Proof.** By definition of sets of Baire's first category, write

$$X \setminus Y = \bigcup_{n \in \mathbb{N}} E_n$$

where each $E_n$ is nowhere dense. We will show that $Y$ must have nonempty interior. To this end, suppose by way of contradiction that $\text{int}(Y) = \emptyset$. Then, for every $U \in \text{Open}(X)$ we have that $U \cap (X \setminus Y) \neq \emptyset$ (if this weren't the case, then we would find a $U \subseteq Y$ and $Y$ would have nonempty interior). In particular, this means that $X \setminus Y$ is dense in $X$, and so $\overline{X \setminus Y} = X$. Since the closure of a countable union is the countable union of the closures, we find that

$$X = \overline{X \setminus Y} = \bigcup_{n \in \mathbb{N}} \overline{E_n}$$

Clearly, since each $E_n$ is nowhere dense it holds that $\overline{E_n}$ is also nowhere dense. So, $X$ is a countable union of nowhere dense sets, and is therefore meagre. However, the Baire Category Theorem tells us that since $X$ is a Banach space, it cannot be meagre. This is a contradiction, and so $Y$ has nonempty interior. From here, we simply apply the contrapositive of Lemma 2 and get that $Y$ cannot be a proper subspace of $X$. Thus, $Y = X$. ■

# Problem 19

Let $X, K$ be metric spaces with $K$ compact. Assume that $f : X \to K$ is a map with $\Gamma(f) \in \text{Closed}(X \times K)$. Show that $f$ is continuous.

## Solution

**Proof.** Note that $f$ is continuous if, for every $A \subseteq X$, we know that

$$f(\overline{A}) \subseteq \overline{f(A)}$$

So, let $A \subseteq X$ and let $x \in \overline{A}$. Then, there is some sequence $\{x_n\}_n \subseteq A$ for which $x_n \to x$. Note that $K$ is a compact metric space, and so it is sequentially compact. As $\{f(x_n)\}_n \subseteq K$ is a sequence of points in a sequentially compact space, it therefore has a convergent subsequence $\{f(x_{n_k})\}_{k\in\mathbb{N}}$ such that $f(x_{n_k}) \to y \in K$ for some $y \in K$ as $k \to \infty$. Since $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n = x$, we get that

$$\lim_{k\to\infty} (x_{n_k}, f(x_{n_k})) = (x, y) \in X \times K,$$

where the above convergence is in the product topology on $X \times K$ (we have coordinatewise convergence on both coordinates). However, each element $(x_{n_k}, f(x_{n_k}))$ of the above sequence is an element of $\Gamma(f)$, which by closure of $\Gamma(f)$ means that its limit point $(x, y) \in \Gamma(f)$. This necessarily means that $y = f(x)$. So, $f_{n_k} \to f(x)$ in $K$ as $k \to \infty$, and so $f(x)$ is a limit point of a sequence of elements of $f(A)$. Thus, $f(x) \in \overline{f(A)}$. We have shown that for every $x \in \overline{A}$, $f(x) \in \overline{f(A)}$. Therefore, for all $A \subseteq X$,

$$f(\overline{A}) \subseteq \overline{f(A)}$$

So, $f$ is continuous as desired. ■