MAT 520: Problem Set 11

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Problem 1

Let X be the position operator on $\mathcal{H} := L^2(\mathbb{R})$. Show that

$$\mathcal{D}(X) := \left\{ \psi \in \mathcal{H} : \int_{\mathbb{R}} x^2 |\psi(x)|^2 \mathrm{d}x < \infty \right\}$$

is the largest vector space V such that for each $\psi \in V$, $X\psi \in L^2(\mathbb{R})$.

Solution

Proof. Suppose by way of contradiction that there were some $\psi \in \mathcal{H}$ such that $\psi \notin \mathcal{D}(X)$ yet $X\psi \in L^2(\mathbb{R})$. We note that

$$\|X\psi\|^2 = \int_{\mathbb{R}} |x\psi(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} x^2 |\psi(x)|^2 \mathrm{d}x = \infty,$$

where we know that this equals ∞ because $\psi \notin \mathcal{D}(X)$. This is a contradiction since $X\psi \in L^2(\mathbb{R}) \implies \|X\psi\|^2 < \infty$. ■

Problem 2

Let $\mathcal{H} := L^2([0,1])$. Define

$$\mathcal{A} := \{\psi \in \mathcal{H} : \ \psi \text{ is absolutely continuous and } \psi' \in \mathcal{H}\}$$

Let A_1 and A_2 both be defined as $\psi \mapsto -i\psi'$ on the respective domains

$$\mathcal{D}(A_1) := \mathcal{A}$$
$$\mathcal{D}(A_2) := \{\psi \in \mathcal{A} : \ \psi(0) = 0\}$$

Show that both domains are dense in $\mathcal{H}$ and that A_1, A_2 are closed. Finally, show that

$$\sigma(A_1) = \mathbb{C}$$
$$\sigma(A_2) = \emptyset$$

Solution

Proof. We will show that $\mathcal{D}(A_2)$ is dense in $\mathcal{H}$, from which density of $\mathcal{D}(A_1)$ will follow. Note that the set $C^\infty([0,1])$ of smooth functions on $[0,1]$ is dense in $\mathcal{H}$; so, if we can approximate an arbitrary $f \in C^\infty([0,1])$ with elements of $\mathcal{D}(A_2)$ then density follows by a standard $\frac{\varepsilon}{3}$ argument. Consider the sequence of functions $f_n : [0,1] \to \mathbb{C}$ given by

$$f_n(x) := \begin{cases} e^{-1/(nx)} f(x) & x > 0 \\ 0 & x = 0 \end{cases}$$

Note that each $|f_n(x)| \leq |f(x)|$ for all x, and so it is in $\mathcal{H}$ because f is continuous and therefore bounded since we are on a compact domain. Furthermore, each f_n is has $f_n(0) = 0$ and is certainly absolutely continuous with bounded (and so square-integrable) derivative via $f_n(x) = f(x) \int_0^x e^{-1/(nt)}/(nt^2) dt$ for all x; so, $f_n \in \mathcal{D}(A_2)$. Let M be such that $|f| \leq M$. For any $\varepsilon > 0$, we see that for $n > N$,

$$\|f_n - f\|^2 = \int_0^1 |f_n(x) - f(x)|^2 \mathrm{d}x \leq M^2 \int_0^1 |e^{-1/(nx)} - 1|^2 \mathrm{d}x$$

This integrand is bounded above by $(|1| + |1|)^2 = 4$, and so by dominated convergence and the fact that $e^{-1/(nx)}$ converges to 1 pointwise a.e. as $n \to \infty$, we see that $\|f_n - f\| \to 0$, which means that $\mathcal{D}(A_2)$ is dense in $\mathcal{H}$.

To see closedness of A_1, let $\{(\varphi_n, A_1\varphi_n)\}_n \subseteq \Gamma(A_1)$ converge to some $(\varphi, \psi) \in \mathcal{H} \times \mathcal{H}$; we claim that $(\varphi, \psi) \in \Gamma(A_1)$ as this will imply that $\Gamma(A_1)$ is closed and thus A_1 is closed. We know that $\varphi_n \to \varphi$ and $A_1\varphi_n \to \psi \implies \varphi_n' \to i\psi$. We will show that the convergence $\varphi_n \to \varphi$ is uniform by showing it is uniformly Cauchy. Let $\varepsilon > 0$ and $x \in [0,1]$ be arbitrary. Then, for all $n, m \in \mathbb{N}$

$$\begin{aligned} |\varphi_n(x) - \varphi_m(x)| &= \left| \varphi_n(0) + \int_0^x \varphi_n'(t) dt - \varphi_m(0) - \int_0^x \varphi_m'(t) dt \right| \\ &\leq |\varphi_n(0) - \varphi_m(0)| + \int_0^x |\varphi_n'(t) - \varphi_m'(t)| dt \\ &\leq |\varphi_n(0) - \varphi_m(0)| + \int_0^1 |\varphi_n'(t) - \varphi_m'(t)| dt \\ &\leq |\varphi_n(0) - \varphi_m(0)| + \|\varphi_n' - \varphi_m'\|, \end{aligned}$$

where for the last line we used the Holder inequality $\|\cdot\|_{L^1} \leq \|\cdot\|_{L^2}$. Since $\varphi_k(0) = \varphi_k(x) - \int_0^x \varphi_k'(t) dt$, we see

$$|\varphi_n(0) - \varphi_m(0)|^2 = \int_0^1 |\varphi_n(0) - \varphi_m(0)|^2 dt = \int_0^1 \left| \varphi_n(t) - \varphi_m(t) + \int_0^t (\varphi_m'(s) - \varphi_n'(s)) ds \right|^2 dt$$

So, letting ξ_k be the map sending $t \mapsto \int_0^t \varphi_k'(s)ds$,

$$|\varphi_n(0) - \varphi_m(0)| = \|\varphi_n - \varphi_m + \xi_n - \xi_m\| \leq \|\varphi_n - \varphi_m\| + \|\xi_n - \xi_m\|$$

We have already seen that $|\xi_n(x) - \xi_m(x)| \leq \|\varphi_n' - \varphi_m'\|$ by the Holder estimate, and so $\|\xi_n - \xi_m\| \leq \|\varphi_n' - \varphi_m'\|$ as well. Combining everything, we see that

$$|\varphi_n(x) - \varphi_m(x)| \leq \|\varphi_n - \varphi_m\| + 2\|\varphi_n' - \varphi_m'\|$$

Since both of these sequences converge and therefore are Cauchy, certainly the convergence $\varphi_n \to \varphi$ is uniform. We now claim that $\varphi(x) = \varphi(0) + \int_0^x (i\psi(t))dt$ for a.e. x. Let $\varepsilon > 0$; then, there is some $n \in \mathbb{N}$ independent of x for which $|\varphi(x) - \varphi_n(x)|, |\varphi(0) - \varphi_n(0)|$, and $\|\varphi_n' - i\psi\|$ are all $\leq \frac{\varepsilon}{3}$ for all $x \in [0,1]$. Then, for all x,

$$\begin{aligned}\left|\varphi(x) - \varphi(0) - \int_0^x (i\psi(t))dt\right| &= \left|\varphi(x) - \varphi(0) - \int_0^x (i\psi(t))dt + \varphi_n(x) - \varphi_n(x)\right| \\ &\leq |\varphi(x) - \varphi_n(x)| + \left|\varphi_n(x) - \varphi(0) - \int_0^x (i\psi(t))dt\right| \\ &\leq \frac{\varepsilon}{3} + |\varphi_n(0) - \varphi(0)| + \left|\int_0^x (\varphi_n'(t) - i\psi(t))dt\right| \\ &\leq \frac{2\varepsilon}{3} + \int_0^1 |\varphi_n'(t) - i\psi(t)|dt \\ &\leq \frac{2\varepsilon}{3} + \|\varphi_n' - i\psi\| \leq \varepsilon,\end{aligned}$$

where to get the last line we again used our favorite Holder estimate. Since this holds for all $\varepsilon > 0$, we see that $\left|\varphi(x) - \varphi_0 - \int_0^x (i\psi(t))dt\right|$ for all x, and so $\varphi(x) = \varphi(0) + \int_0^x (i\psi(t))dt$ for all x. Therefore, φ is absolutely continuous and $\varphi' = i\psi$. So, $\varphi \in \mathcal{D}(A_1)$ and $(\varphi, \psi) \in \Gamma(A_1)$, from which closedness follows.

The proof that A_2 is closed goes similarly. Let $\{(\varphi_n, A_2\varphi_n)\}_n \subseteq \Gamma(A_2)$ converge to some $(\varphi, \psi) \in \mathcal{H} \times \mathcal{H}$; we claim that $(\varphi, \psi) \in \Gamma(A_2)$. We know that $\varphi_n \to \varphi$ and $A_2\varphi_n \to \psi \implies \varphi_n' \to i\psi$. The exact same proof as above shows that $\varphi_n \to \varphi$ uniformly and therefore that $\varphi(x) = \varphi(0) + \int_0^x (i\psi(t))dt$, which means that φ is absolutely continuous and $-i\varphi' = \psi$. The last thing we must show is that $\varphi(0) = 0$, which follows clearly from the fact that $\varphi_n \to \varphi$ uniformly and so $\varphi(0) = \lim_n \varphi_n(0) = 0$. So, $\varphi \in \mathcal{D}(A_2)$ and so $(\varphi, \psi) \in \Gamma(A_2)$.

We conclude by showing the spectral results. For A_1, note that for any $\lambda \in \mathbb{C}$ we have that the map ψ_λ sending $x \mapsto e^{i\lambda x}$ is absolutely continuous and so $\psi_\lambda \in \mathcal{D}(A_1)$ (since $e^{i\lambda x} = 1 + i\lambda \int_0^x e^{i\lambda t}dt$) and we have that

$$(A_1 - \lambda\mathbb{1})\psi_\lambda = -i(i\lambda)\psi_\lambda - \lambda\psi_\lambda = 0 \implies \psi_\lambda \in \ker(A_1 - \lambda\mathbb{1}) \implies \lambda \in \sigma(A_1)$$

So, $\sigma(A_1) = \mathbb{C}$. For A_2, we claim that $\rho(A_2) = \mathbb{C}$. To see this, we will simply show that the resolvent $(A_2 - \lambda\mathbb{1})^{-1}$ exists for every $\lambda \in \mathbb{C}$. Equivalent, we show that for each $\lambda \in \mathbb{C}$ and each $\psi \in \mathcal{H}$, the equation

$$(A_2 - \lambda\mathbb{1})\varphi = \psi$$

has a unique solution $\varphi \in \mathcal{D}(A_2)$. Let $M_\lambda : \mathcal{D}(A_2) \to \mathcal{D}(A_2)$ be the multiplication operator sending $\xi(x) \mapsto e^{i\lambda x}\xi(x)$; this is clearly a bijection. So, we want to show that the equation

$$(A_2 - \lambda\mathbb{1})M_\lambda\xi = \psi$$

has a unique solution $\xi \in \mathcal{D}(A_2)$. Since $(M_\lambda\xi)'(x) = e^{i\lambda x}\xi'(x) + i\lambda e^{i\lambda x}\xi(x)$, the above differential equation reads

$$-ie^{i\lambda x}\xi'(x) - i^2\lambda e^{i\lambda x}\xi(x) - \lambda e^{i\lambda x}\xi(x) = \psi(x) \quad (\xi(0) = 0)$$

$$\iff \xi'(x) = ie^{-i\lambda x}\psi(x) \quad (\xi(0) = 0)$$
$$\iff \xi(x) = \int_0^x ie^{-i\lambda t}\psi(t)dt$$

This $\xi \in \mathcal{D}(A_2)$ defined above is the unique solution, which means that $(A_2 - \lambda\mathbb{1})$ is invertible and so $\lambda \in \rho(A_2)$. Thus, $\sigma(A_2) = 0$. ■

Problem 3

Show that if A is a symmetric operator on a Hilbert space $\mathcal{H}$ then the following are equivalent:

(a) A is essentially self-adjoint.

(b) $\ker(A^* \pm i\mathbb{1}) = \{0\}$.

(c) $\overline{\operatorname{im}(A \pm i\mathbb{1})} = \mathcal{H}$.

Solution

Proof. (a $\implies$ b) Suppose first that A is essentially self-adjoint. Then, $\overline{A}$ is self-adjoint. So,

$$\ker((\overline{A})^* \pm i\mathbb{1}) = \{0\}$$

by applying Theorem 11.26 in the lecture notes to $\overline{A}$. By Theorem 11.17 in the lecture notes, $(\overline{A})^* = A^*$, which means that

$$\ker(A^* \pm i\mathbb{1}) = \{0\}$$

as desired.

(b $\implies$ c) Suppose now that $\ker(A^* \pm i\mathbb{1}) = \{0\} \implies \ker((\overline{A})^* \pm i\mathbb{1}) = \{0\}$, where the equality follows from Theorem 11.17. Then, again applying Theorem 11.26 to $\overline{A}$, we see that

$$\operatorname{im}(\overline{A} \pm i\mathbb{1}) = \mathcal{H}$$

Note that we certainly have that $A \pm i\mathbb{1}$ is closable and with the same domain as A, and so

$$\overline{A \pm i\mathbb{1}} = (A \pm i\mathbb{1})^{**} = A^{**} \pm i\mathbb{1} = \overline{A} \pm i\mathbb{1}$$

Thus,

$$\operatorname{im}(\overline{A \pm i\mathbb{1}}) = \mathcal{H}$$

Let $T := A \pm i\mathbb{1}$ for notation. Write $\Gamma(T) = \{(\varphi, T\varphi) : \ \varphi \in \mathcal{D}(T)\}$ for the graph of T. Then, by Claim 11.11 in the lecture notes we know that $\overline{\Gamma(T)} = \Gamma(\overline{T})$. Since $\operatorname{im}(\overline{T}) = \mathcal{H}$, for every $\psi \in \mathcal{H}$ there is an element $\varphi \in \mathcal{D}(\overline{T})$ and a sequence $\{(\varphi_n, T\varphi_n)\}_{n\in\mathbb{N}} \subseteq \Gamma(T)$ such that $\overline{T}\varphi = \psi$ and

$$(\varphi_n, T\varphi_n) \to (\varphi, \psi)$$

In particular, we see that $T\varphi_n \to \psi$, which means that $\psi \in \overline{\operatorname{im}(T)}$. Since this holds for all $\psi \in \mathcal{H}$, we see that $\overline{\operatorname{im}(A \pm i\mathbb{1})} = \mathcal{H}$.

(c $\implies$ a) Again let $T := A \pm i\mathbb{1}$ for notation. Suppose that $\overline{\operatorname{im}(T)} = \mathcal{H}$. Let $\psi \in \ker(T^*) = \ker(A^* \mp i\mathbb{1})$ be arbitrary. Then,

$$T^*\psi = 0 \implies \langle T^*\psi, \varphi\rangle = 0 \ \ \forall \varphi \in \mathcal{D}(T) \implies \langle \psi, T\varphi\rangle = 0 \ \ \forall \varphi \in \mathcal{D}(T) \implies \psi \in \operatorname{im}(T)^\perp$$

By Claim 7.8 in the lecture notes, $\operatorname{im}(T)^\perp = (\overline{\operatorname{im}(T)})^\perp = \mathcal{H}^\perp = \{0\}$. So, $\psi = 0$, and therefore

$$\ker(A^* \pm i\mathbb{1}) = \{0\}$$

Since $A^* = (\overline{A})^*$, we find that $\overline{A}$ is closed and $\ker((\overline{A})^* \pm i\mathbb{1}) = \{0\}$. So, by Theorem 11.26 in the lecture notes, $\overline{A}$ is self-adjoint. Thus, A is essentially self-adjoint. ∎

Problem 4

Let $A := -i\partial$ on

$$\mathcal{D}(A) := \{\psi \in \mathcal{A} : \ \psi(0) = \psi(1) = 0\}$$

with $\mathcal{A}$ as above.

(a) Show that A is symmetric as an operator $A : \mathcal{D}(A) \to L^2([0,1])$.

(b) Calculate A^* (along with $\mathcal{D}(A^*)$) and conclude A is closed, symmetric but *not* self-adjoint.

(c) For any $\alpha \in \mathbb{C}$, $|\alpha| = 1$, define $A_\alpha := -i\partial$ on the domain

$$\mathcal{D}(A_\alpha) := \{\psi \in \mathcal{A} : \ \psi(0) = \alpha\psi(1)\}$$

Show that A_α is self-adjoint, and that it is an extension of A, and is extended by A^*:

$$A \subseteq A_\alpha \subseteq A^*$$

Conclude that A has uncountably many self-adjoint extensions.

Solution

Proof. (a) To see that it is symmetric, let $\psi, \varphi \in \mathcal{D}(A)$. We explicitly compute via integration by parts that

$$\begin{aligned}
\langle \psi, A\varphi \rangle &= \int_0^1 \overline{\psi(x)}(-i\varphi'(x))\mathrm{d}x \\
&= [\overline{\psi(x)}(-i\varphi(x))]_{x=0}^1 - \int_0^1 \overline{\psi'(x)}(-i\varphi(x))\mathrm{d}x \\
&= \int_0^1 \overline{-i\psi'(x)}\varphi(x)\mathrm{d}x \\
&= \langle A\psi, \varphi \rangle ,
\end{aligned}$$

where we used that $\psi(0) = \psi(1) = \varphi(0) = \varphi(1) = 0$ to eliminate the boundary term and observed that $\overline{\psi}' = \overline{\psi'}$ (which can be seen by separately differentiating the real and imaginary parts). Therefore, A is symmetric.

(b) We will show that $\mathcal{D}(A^*) = \mathcal{A}$. To see this, let $\{K_\delta\}_\delta \subseteq L^2([0,1])$ be an approximation to the identity as defined in Chapter 3.2 of Stein III (in particular, each K_δ may be compactly-supported, real-valued, and infinitely-differentiable). Letting $J_\delta \in \mathcal{B}(L^2([0,1])$ be the operator sending $\varphi \to \varphi * K_\delta$, we see that J_δ is indeed linear. To see boundedness, note that by definition of an approximation to the identity there is a constant C for which $\int_0^1 |K_\delta| < C$ for all δ. By Young's inequality, we see that

$$\|J_\delta\varphi\| = \|\varphi * K_\delta\|_{L^2} \leq \|\varphi\|_{L^2}\|K_\delta\|_{L^1} \leq C\|\varphi\|$$

and so J_δ is bounded. By Theorem 2.1 of Chapter 3 in Stein III, we know that for Lebesgue a.e. $x \in [0,1]$ and continuous $f : [0,1] \to \mathbb{C}$,

$$(f * K_\delta)(x) \to f(x) \text{ as } \delta \to 0$$

If f is continuous on the compact domain $[0,1]$ it is also bounded and so clearly in $L^2([0,1])$. Again by Young's inequality, $\|f * K_\delta\|_{L^\infty} \leq \|f\|_{L^2}\|K_\delta\|_{L^2} = \|f\|_{L^2}$. Therefore, the sequence $\{f * K_\delta\}_\delta$ is a uniformly bounded sequence converging to f pointwise a.e., which means that $f * K_\delta \to f$ in $L^2([0,1])$ as well by dominated convergence. It is known that on a compact domain, the continuous functions are dense in L^2. So, by a standard $\frac{\varepsilon}{3}$ argument we see that $J_\delta\varphi \to \varphi$ in $\|\cdot\|_{L^2}$ as $\delta \to 0$ for all $\varphi \in L^2([0,1])$. Thus, $J_\delta \to \mathbb{1}$ strongly.

Problem 4 continued on next page. . .

Now, note that for all $0 < \alpha < \beta < 1$, if we define

$$g_\delta^{(\alpha,\beta)}(x) := \int_0^x K_\delta(t-\beta) - K_\delta(t-\alpha)dt,$$

then for small enough δ we definitely have that $g_\delta^{(\alpha,\beta)} \in \mathcal{D}(A)$. Since $|g_\delta^{(\alpha,\beta)}|$ is uniformly bounded w.r.t. δ by $2C$ and $g_\delta^{(\alpha,\beta)}(x) \to \chi_{(\alpha,\beta)}(x)$ pointwise, we see that $g_\delta^{(\alpha,\beta)} \to \chi_{(\alpha,\beta)}$ in L^2 as $\delta \to 0$ by the dominated convergence theorem. Now, let $\psi \in \mathcal{D}(A^*)$ be arbitrary. We have that by definition of the adjoint and continuity of the inner product,

$$\left\langle Ag_\delta^{(\alpha,\beta)}, \psi \right\rangle = \left\langle g_\delta^{(\alpha,\beta)}, A^*\psi \right\rangle \to \left\langle \chi_{(\alpha,\beta)}, A^*\psi \right\rangle = \int_\alpha^\beta (A^*\psi)(x)\mathrm{d}x$$

We may compute the left hand side via

$$Ag_\delta^{(\alpha,\beta)}(x) = -iK_\delta(x-\beta) + iK_\delta(x-\alpha)$$

and so

$$\begin{aligned}\left\langle Ag_\delta^{(\alpha,\beta)}, \psi \right\rangle &= -i\int_0^1 K_\delta(x-\beta)\psi(x)\mathrm{d}x + i\int_0^1 K_\delta(x-\alpha)\psi(x)\mathrm{d}x \\ &= -i(K_\delta * \psi)(\beta) + i(K_\delta * \psi)(\alpha) \\ &= -i(J_\delta\psi)(\beta) + i(J_\delta\psi)(\alpha)\end{aligned}$$

By strong convergence of J_δ, we see that as $\delta \to 0$,

$$\left\langle Ag_\delta^{(\alpha,\beta)}, \psi \right\rangle \to -i\psi(\beta) + i\psi(\alpha)$$

So, by uniqueness of limits, we see that

$$-i(\psi(\beta) - \psi(\alpha)) = \int_\alpha^\beta (A^*\psi)(x)\mathrm{d}x$$

So, ψ is absolutely continuous and therefore $\mathcal{D}(A^*) \subseteq \mathcal{A}$. Furthermore, the above shows that $(-i\partial)\psi = A^*\psi$. We already saw in the proof of Problem 2 that $\mathcal{A} \subseteq \mathcal{D}(A^*)$ since we may apply integration by parts to any absolutely continuous function. So, $A^* = -i\partial$ with domain $\mathcal{D}(A^*) = \mathcal{A}$, which means A is not self-adjoint.

To show that A is closed, we will show that $A = A^{**}$ (though replicating the proofs of closedness from Problem 2's A_1 and A_2 would work too). We know that $A^{**} \subseteq A^*$ since A is symmetric, and so $A^{**} = -i\partial$. To determine its domain, let $\psi \in \mathcal{D}(A^{**})$. Then, we have that for all $\varphi \in \mathcal{D}(A^*) = \mathcal{A}$,

$$\begin{aligned}\langle A^*\varphi, \psi\rangle = \langle \varphi, A^{**}\psi\rangle = \langle \varphi, -i\psi'\rangle &= \int_0^1 \overline{\varphi(x)}(-i\psi'(x))\mathrm{d}x \\ &= -i[\overline{\varphi(x)}\psi(x)]_{x=0}^1 + i\int_0^1 \overline{\varphi'(x)}\psi(x)\mathrm{d}x \\ &= -i[\overline{\varphi(x)}\psi(x)]_{x=0}^1 + \langle -i\varphi', \psi\rangle \\ &= -i[\overline{\varphi(x)}\psi(x)]_{x=0}^1 + \langle A^*\varphi, \psi\rangle\end{aligned}$$

So, we must have that $-i[\overline{\varphi(x)}\psi(x)]_{x=0}^1 = 0$ for every $\varphi \in \mathcal{A}$, which is only possible if $\psi(0) = \psi(1) = 0$. Therefore, $\psi \in \mathcal{D}(A)$, which means that $\mathcal{D}(A^{**}) \subseteq \mathcal{D}(A)$ and so $A^{**} \subseteq A$. Since $A \subseteq A^{**}$ always, we have shown that $A = A^* * = \overline{A}$, and so A is closed.

Problem 4 continued on next page. . .

(c) We know that $A_\alpha^* = -i\partial$ by similar logic to what we had in part (b) (note that all the J_δ and $g_\delta^{(\alpha,\beta)}$ nonsense did not depend on any boundary information of ψ), and so all we must do is find its domain. Let $\psi \in \mathcal{D}(A_\alpha^*)$ be arbitrary. Then, for all $\varphi \in \mathcal{D}(A_\alpha)$, we have by yet another integration by parts that

$$\langle A_\alpha^* \psi, \varphi \rangle = \langle \psi, A_\alpha \varphi \rangle = \int_0^1 \overline{\psi(x)}(-i\varphi'(x)) = -i[\overline{\psi(x)}\varphi(x)]_{x=0}^1 + \langle -i\psi', \varphi \rangle = i[\overline{\psi(x)}\varphi(x)]_{x=0}^1 + \langle A_\alpha^* \psi, \varphi \rangle$$

So, for all $\varphi \in \mathcal{D}(A_\alpha)$ we must have

$$i[\overline{\psi(x)}\varphi(x)]_{x=0}^1 = 0 \implies \frac{1}{\alpha}\overline{\psi(1)}\varphi(0) - \overline{\psi(0)}\varphi(0) = 0$$

The only way for this to hold for all φ is if

$$\overline{\psi(1)} = \alpha\overline{\psi(0)} \implies \alpha\psi(1) = \psi(0),$$

where the implication follows since $\overline{\alpha} = \frac{1}{\alpha}$ for $|\alpha| = 1$. Therefore, $\psi \in \mathcal{D}(A_\alpha)$, and so $\mathcal{D}(A_\alpha^*) = \mathcal{D}(A_\alpha)$. In particular, A_α is self-adjoint. Since all of A, A_α, and A^* act as $-i\partial$ and we have that

$$\mathcal{D}(A) \subseteq \mathcal{D}(A_\alpha) \subseteq \mathcal{A} = \mathcal{D}(A^*),$$

we see

$$A \subseteq A_\alpha \subseteq A^*$$

So, we conclude that A has uncountably many self-adjoint extensions. ∎

Problem 5

Show that A is closable iff $\overline{\Gamma(A)} = \Gamma(B)$ for some operator B. Show that this operator B is the closure $\overline{A}$ of A.

Solution

Proof. ($\implies$) Suppose that A is closable. Let B be any closed extension of A with domain $\mathcal{D}(B) \supseteq \mathcal{D}(A)$. By definition of an extension, we know that $\Gamma(A) \subseteq \Gamma(B)$. Since B is closed, so too is $\Gamma(B)$, which means that

$$\overline{\Gamma(A)} \subseteq \Gamma(B)$$

Define the operator $R : \mathcal{D}(R) \to \mathcal{H}$ via

$$\mathcal{D}(R) := \{\psi \in \mathcal{H} : \ (\psi, \varphi) \in \overline{\Gamma(A)} \text{ for some } \varphi \in \mathcal{H}\}$$

and

$$R\psi = \varphi \text{ for the } \varphi \in \mathcal{H} \text{ such that } (\psi, \varphi) \in \overline{\Gamma(A)}$$

We know that this definition is unique since, if there were two $(\psi, \varphi_1), (\psi, \varphi_2) \in \overline{\Gamma(A)}$, we would have that $(\psi, \varphi_1), (\psi, \varphi_2) \in \Gamma(B)$ and so $\varphi_1 = B\psi = \varphi_2$. Clearly, this construction means that $\Gamma(R) = \overline{\Gamma(A)}$. Furthermore, this means that $\Gamma(R) \subseteq \Gamma(B) \implies R \subseteq B$. Since this holds for all closed extensions B of A, we see that R is the minimal closed extension of A. Therefore, $R = \overline{A}$.

($\impliedby$) Suppose that $\overline{\Gamma(A)} = \Gamma(B)$ for some operator B. We know that B is a closed operator since its graph is closed. Furthermore,

$$\Gamma(A) \subseteq \Gamma(B) \implies A \subseteq B$$

So, B is a closed extension of A. In particular, A is closable. ∎

Problem 6

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and $\psi \in \mathcal{H}$ any vector which is not a finite linear combination of $\{\varphi_n\}_n$. Let $\mathcal{D}$ be the set of vectors which are finite linear combinations of $\{\varphi_n\}_n$ and of ψ. Define $A : \mathcal{D} \to \mathcal{H}$ via

$$A\left(b\psi + \sum_{i=1}^{N} a_i\varphi_i\right) := b\psi.$$

Calculate $\Gamma(A)$ and show that $\overline{\Gamma(A)}$ is not the graph of a linear operator.

Solution

Proof. Note that for any $\eta \in \text{span}\{\varphi_n\}_n$ we have that $A\eta = 0$. Also, $A\psi = \psi$. Let $\xi \in \mathcal{D}$ be arbitrary; then, $\xi = b\psi + \eta$ for some $\eta \in \text{span}\{\varphi_n\}_n$, and so

$$(\xi, A\xi) = (b\psi, b\psi) + (\eta, 0)$$

Since this holds for all $\xi \in \mathcal{D}$, we see that

$$\Gamma(A) = \{(\xi, A\xi) \in \mathcal{H}^2 : \ \xi \in \mathcal{D}\} = \text{span}\{(\psi, \psi)\} \oplus \text{span}\{(\varphi_n, 0)\}_{n\in\mathbb{N}}$$

So,

$$\begin{aligned} \overline{\Gamma(A)} &= \text{span}\{(\psi, \psi)\} \oplus \overline{\text{span}\{(\varphi_n, 0)\}_{n\in\mathbb{N}}} \\ &= \text{span}\{(\psi, \psi)\} \oplus \{(\xi, 0) : \ \xi \in \mathcal{H}\} \\ &= \{(\varphi, b\psi) : \ \varphi \in \mathcal{H} \text{ and } b \in \mathbb{C}\}, \end{aligned}$$

where for the second equality we used that the closure of the span of an orthonormal basis is the whole space, and for the third line we noted that for any b and φ, it holds that $(\varphi - b\psi, 0) \in \{(\xi, 0) : \ \xi \in \mathcal{H}\}$ and so $(b\psi, b\psi) + (\varphi - b\psi, 0) = (\varphi, b\psi) \in \overline{\Gamma(A)}$. Suppose by way of contradiction that $\overline{\Gamma(A)} = \Gamma(B)$ for some linear operator B. Then, $(0, \psi) \in \Gamma(B)$, which means that $B(0) = \psi$ and so $\psi = 0$. This contradicts the fact that $\psi \notin \text{span}\{\varphi_n\}_n$, and so A is not a closable operator. ■

Problem 7

Let $A : \mathcal{D}(A) \to H$ be injective.

(a) Show that if A is closed and has a closed range then $\exists C \in (0, \infty)$ such that

$$\|A\psi\| \geq C\|\psi\| \quad (\psi \in \mathcal{D}(A)) \tag{1}$$

(b) Show that if A has dense closed range and obeys (1) then A is closed.

(c) Show that if A is closed and obeys (1) then it has a closed range.

Solution

Proof. (a) Suppose that A is closed with closed range. Define the map $\widetilde{A} : \mathcal{D}(A) \to \operatorname{im}(A)$, which is then a bijection since A is injective. Furthermore, we see that $\Gamma(\widetilde{A}) = \Gamma(A)$ and so $\Gamma(\widetilde{A})$ is closed in $\mathcal{H} \times \operatorname{im}(A)$ since $\Gamma(A)$ is closed. Since $\operatorname{im}(A)$ is closed and is therefore a Banach space, we may apply the closed graph theorem to see that $\widetilde{A}^{-1} : \operatorname{im}(A) \to \mathcal{D}(A)$ is bounded. Let $M \in (0, \infty)$ be such that

$$\|\widetilde{A}^{-1}\varphi\| \leq M\|\varphi\| \quad (\varphi \in \operatorname{im}(A))$$

Then, for any $\psi \in \mathcal{D}(A)$ we may apply the above with $\varphi := A\psi$ to see that

$$\|\widetilde{A}^{-1}A\psi\| \leq M\|A\psi\| \implies \|\psi\| \leq M\|A\psi\| \implies \|A\psi\| \geq \frac{1}{M}\|\psi\|$$

The above holds for all $\psi \in \mathcal{D}(A)$.

(b) Suppose now that A has dense closed range and obeys (1). A dense closed set is the whole space, and so $\operatorname{im}(A) = \mathcal{H}$. Therefore, A is both injective and surjective, meaning that it is invertible. We claim that $\Gamma(A)$ is closed. To this end, suppose that $\{(\varphi_n, A\varphi_n)\}_{n\in\mathbb{N}} \subseteq \Gamma(A)$ is a sequence converging to some $(\varphi, \psi) \in \mathcal{H} \times \mathcal{H}$; we want to show that $\varphi \in \mathcal{D}(A)$ and $\psi = A\varphi$ since this would imply that $(\varphi, \psi) \in \Gamma(A)$ and therefore that $\Gamma(A)$ is closed. Since A is bijective, we see that for all $\xi \in \mathcal{H}$, $A^{-1}\xi \in \mathcal{D}(A)$ and so (1) reads

$$\|AA^{-1}\xi\| \geq C\|A^{-1}\xi\| \implies \|A^{-1}\xi\| \leq \frac{1}{C}\|\xi\|$$

Thus, the map $A^{-1} : \mathcal{H} \to \mathcal{D}(A)$ is a bounded linear bijection. Note that we separately have that $\varphi_n \to \varphi$ and $A\varphi_n \to \psi$. We may say that

$$\|A^{-1}\psi - \varphi_n\| = \|A^{-1}(\psi - A\varphi_n)\| \leq \frac{1}{C}\|\psi - A\varphi_n\|$$

Since $\|\psi - A\varphi_n\| \to 0$, we then find that $\|A^{-1}\psi\varphi_n\| \to 0$ and so $\varphi_n \to A^{-1}\psi$. By uniqueness of limits, this means that $\varphi = A^{-1}\psi$, which automatically yields both that $\varphi \in \mathcal{D}(A)$ and also that $A\varphi = \psi$. So, $\Gamma(A)$ is closed.

(c) Suppose now that A is closed (and so $\Gamma(A)$ is closed) and obeys (1). Let $\{\psi_n\}_{n\in\mathbb{N}} \subseteq \operatorname{im}(A)$ be a sequence converging to some $\psi \in \mathcal{H}$; we want to show that $\psi \in \operatorname{im}(A)$ as this will prove that A has closed range. Note that for each n there is some $\varphi_n \in \mathcal{D}(A)$ such that $\psi_n = A\varphi_n$. We note that by (1), for $n \neq m$ we have

$$\|\varphi_n - \varphi_m\| \leq \frac{1}{C}\|A\varphi_n - A\varphi_m\| = \frac{1}{C}\|\psi_n - \psi_m\|$$

Since the sequence $\{\psi_n\}_n$ converges it is Cauchy, which means that $\{\varphi_n\}_n$ is also Cauchy and so converges to some $\varphi \in \mathcal{H}$ by completeness. So, since $\varphi_n \to \varphi$ and $A\varphi_n \to \psi$, we see that

$$(\varphi_n, A\varphi_n) \to (\varphi, \psi) \text{ as } n \to \infty$$

Since each $(\varphi_n, A\varphi_n) \in \Gamma(A)$ and $\Gamma(A)$ is closed, this means that $(\varphi, \psi) \in \Gamma(A)$ as well. Therefore, $\psi = A\varphi$, and $\psi \in \operatorname{im}(A)$. So, A has closed image. ■

Problem 8

Calculate the adjoint of $-\partial^2 : C_0^\infty(\mathbb{R}) \to L^2(\mathbb{R})$. Determine if $-\partial^2$ is essentially self-adjoint. Here $C_0^\infty(\mathbb{R})$ is the set of functions $f : \mathbb{R} \to \mathbb{C}$ smooth of compact support.

Solution

Proof. We start by observing that $-\partial^2$ is indeed densely defined since the smooth functions of compact support are dense in $L^2(\mathbb{R})$, which is a good sign. Now, we show that $-\partial^2$ is symmetric. For every $\psi, \varphi \in \mathcal{D}(-\partial^2)$, we have that

$$\left\langle \psi, (-\partial^2)\varphi \right\rangle = \langle \psi, -\varphi'' \rangle$$

By integration by parts, we see that

$$\langle \psi, -\varphi'' \rangle = -\int_{\mathbb{R}} \overline{\psi}\varphi'' = -[\overline{\psi}\varphi']_{-\infty}^{\infty} + \int_{\mathbb{R}} \overline{\psi'}\varphi'$$

Since φ has compact support, certainly φ' also does, and so the boundary term vanishes. Applying integration by parts again,

$$\langle \psi, -\varphi'' \rangle = [\overline{\psi'}\varphi]_{-\infty}^{\infty} - \int_{\mathbb{R}} \overline{\psi''}\varphi = \langle -\psi'', \varphi \rangle = \left\langle (-\partial^2)\psi, \varphi \right\rangle,$$

where the boundary term again vanishes because of compact support. Since this holds for all $\psi, \varphi \in \mathcal{D}(-\partial^2)$, we find that $-\partial^2$ is symmetric. We will now compute its adjoint.

As we did in Problem 4, let $\{K_\delta\}_{\delta>0}$ be a real-valued, compactly-supported, infinitely-differentiable approximation to the identity, and let $J_\delta \in \mathcal{B}(L^2(\mathbb{R}))$ be the bounded linear operator sending

$$L^2(\mathbb{R}) \ni \psi \mapsto \psi * K_\delta \in L^2(\mathbb{R})$$

Note that even though the measure space is now infinite, we may still use Young's inequality for boundedness of J_δ since $\int_{\mathbb{R}} |K_\delta| < C$. We would like to show that J_δ converges strongly to $\mathbb{1}$. To see this, it suffices to show that $\|J_\delta\psi - \psi\| \to 0$ for all $\psi \in C_0^\infty(\mathbb{R})$, and then density of C_0^∞ will imply that $J_\delta \to \mathbb{1}$ strongly. So, let $\psi \in C_0^\infty(\mathbb{R})$, and we compute

$$\|J_\delta\psi - \psi\|^2 = \int_{\mathbb{R}} |(\psi * K_\delta)(x) - \psi(x)|^2 \mathrm{d}x$$

Note that if we let M denote an upper bound on ψ and E be the compact support of ψ, then

$$|(\psi * K_\delta)(x) - \psi(x)| \leq |(\psi * K_\delta)(x)| + M\chi_E(x)$$

$$\implies |(\psi * K_\delta)(x) - \psi(x)|^2 \leq |(\psi * K_\delta)(x)|^2 + 2M|(\psi * K_\delta)(x)| + M^2\chi_E(x)$$

Since ψ, K_δ are both compactly-supported, their convolution must be as well. It is also bounded since $|(\psi * K_\delta)(x)| \leq M \int_{\mathbb{R}} |K_\delta| \leq MC$. This shows that the function $|\psi * K_\delta|^2 + 2M|\psi * K_\delta| + M^2\chi_E$ is integrable, which means that it is a valid dominating function for the above integral. So, by the dominated convergence theorem, we see that $\|J_\delta\psi \to \psi\| \to 0$. So, $J_\delta \to \mathbb{1}$ on $C_0^\infty(\mathbb{R})$; by density and boundedness of J_δ, this means that $J_\delta \to \mathbb{1}$ strongly on all of $L^2(\mathbb{R})$.

Proceeding similarly to what we did in Problem 4, for any $0 < \alpha < \beta$ we define

$$g_\delta^{(\alpha,\beta)}(x) := c(x) \int_0^x \int_0^t K_\delta(s - \beta) - K_\delta(s - \alpha) ds dt$$

where $c(x) = \begin{cases} \frac{1}{x-\alpha} & x \in (\alpha, \beta) \\ 0 & \text{else} \end{cases}$. Then, each $g_\delta^{(\alpha,\beta)}$ is smooth and has compact support since K_δ does (for large enough s we expect $K_\delta(s-\beta) - K_\delta(s-\alpha) = 0$ since K_δ is compactly supported). Therefore, each $g_\delta^{(\alpha,\beta)} \in \mathcal{D}(-\partial^2)$. Furthermore, since $|g_\delta^{(\alpha,\beta)} - \chi_{(\alpha,\beta)}|$ is dominated by 0 for $x \leq \alpha$ and $2C$ otherwise and compactly supported we may apply dominated convergence. We know that $\int_0^t K_\delta(s-\beta) - K_\delta(s-\alpha)ds \to \chi_{(\alpha,\beta)}(t)$ pointwise, and so

$$g_\delta^{(\alpha,\beta)}(x) \to c(x) \int_0^x \chi_{(\alpha,\beta)}(t)dt$$

If $x \leq \alpha$ this equals 0, and if $x \geq \beta$ this also equals 0. However, for $x \in (\alpha, \beta)$, we have that this equals $\frac{1}{x-\alpha} \int_\alpha^x dt = 1$. So, $g_\delta^{(\alpha,\beta)} \to \chi_{(\alpha,\beta)}$ pointwise. By dominated convergence, $g_\delta^{(\alpha,\beta)} \to \chi_{(\alpha,\beta)}$ in L^2 as well.

Now, let $\psi \in \mathcal{D}((-\partial^2)^*)$ be arbitrary. We have that by definition of the adjoint and continuity of the inner product,

$$\left\langle -\partial^2 g_\delta^{(\alpha,\beta)}, \psi \right\rangle = \left\langle g_\delta^{(\alpha,\beta)}, (-\partial^2)^* \psi \right\rangle \to \left\langle \chi_{(\alpha,\beta)}, (-\partial^2)^* \psi \right\rangle = \int_\alpha^\beta ((-\partial^2)^*\psi)(x)\mathrm{d}x$$

We may compute the left hand side via

$$-\partial^2 g_\delta^{(\alpha,\beta)} = -K_\delta(x-\beta) + K_\delta(x-\alpha)$$

and so

$$\left\langle -\partial^2 g_\delta^{(\alpha,\beta)}, \psi \right\rangle = -\int_0^\infty K_\delta'(x-\beta)\psi(x)\mathrm{d}x + \int_0^\infty K_\delta'(x-\alpha)\psi(x)\mathrm{d}x$$

By integration by parts, for $\gamma \in (0, \infty)$ we have that

$$\int_0^\infty K_\delta'(x-\gamma)\psi(x)\mathrm{d}x = []$$

i'm pretty sure I messed up, something is not right. I hope I was on the right track :) ■

Problem 9

Let $-i\partial : C_0^\infty([0,\infty)) \to L^2([0,\infty))$ where the domain is the set of smooth functions with compact support away from the origin. Is it essentially self-adjoint?

Solution

Proof. Let $A := -i\partial$ and $\mathcal{H} := L^2([0,\infty))$ for notation. The usual integration by parts trick reveals that A is symmetric since a function having "compact support away from the origin" means that it is 0 at 0 and $[N,\infty)$ for some large enough N, and so the boundary term from integration by parts will vanish. We claim that it is not essentially self-adjoint.

Define $\psi \in \mathcal{H}$ via $\psi(x) = e^{-x}$. We have that

$$\mathcal{D}(A^*) = \{\varphi \in \mathcal{H} : \ \exists \xi \in \mathcal{H} \text{ s.t. } \forall \eta \in \mathcal{D}(A),\ \langle \varphi, A\eta \rangle = \langle \xi, \eta \rangle\}$$

We claim that $\psi \in \mathcal{D}(A^*)$. To see this, let $\eta \in \mathcal{D}(A)$ be arbitrary. Then,

$$\langle \psi, A\eta \rangle = \int_0^\infty -ie^{-x}\eta'(x)\mathrm{d}x = i\left[e^{-x}\eta(x)\right]_{x=0}^\infty - (-i)\int_0^\infty (-e^{-x})\eta(x)\mathrm{d}x$$

Since η is compactly supported away from 0,

$$\langle \psi, A\eta \rangle = \int_0^\infty \overline{ie^{-x}}\eta(x)\mathrm{d}x = \langle -i\psi', \eta \rangle$$

Note that $-i\psi'(x) = -i(-e^{-x}) = ie^{-x}$ and so $-i\psi' = i\psi \in \mathcal{H}$. We find that $A^*\psi = i\psi$ and therefore that $\psi \in \mathcal{D}(A^*)$. Note that

$$(A^* - i\mathbb{1})\psi = i\psi - i\psi = 0 \implies \psi \in \ker(A^* - i\mathbb{1})$$

Since $\psi \neq 0$, Problem 3 above tells us that A is not essentially self-adjoint. ■