MAT 520: Problem Set 10

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Problem 1

Provide an example for a non-normal operator $A \in \mathcal{B}(\mathcal{H})$ and a point in the resolvent set $z \in \rho(A)$ where

$$\|(A - z\mathbb{1})^{-1}\| \leq \frac{1}{\operatorname{dist}(z, \sigma(A))}$$

does *not* hold.

Solution

Proof. Let $\mathcal{H} = \mathbb{R}^2$ and $A \in \mathcal{B}(\mathcal{H})$ be the operator given by the matrix

$$A := \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Then, we know that $\sigma(A) = \{2\}$. So, for $z = 1 \in \rho(A)$, we see that

$$(A - \mathbb{1})^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

We may compute the operator norm using the fact that for an $n \times n$ matrix B, we have $\|B\|_{\text{op}} = \sqrt{\lambda_{\max}(B^* B)}$ is the square root of the largest eigenvalue of $B^* B$. So, letting $B := (A - \mathbb{1})^{-1}$, we seek eigenvalues of the matrix

$$B^* B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

We see that λ is an eigenvalue iff $(1 - \lambda)(2 - \lambda) - 1 = 0 \iff \lambda^2 - 3\lambda + 1 = 0 \iff \lambda = \frac{3 \pm \sqrt{5}}{2}$. Therefore, $\lambda_{\max}(B^* B) = \frac{3+\sqrt{5}}{2}$, and so

$$\|(A - \mathbb{1})^{-1}\| = \sqrt{\frac{3 + \sqrt{5}}{2}} \approx 1.618 > 1$$

However, we have that

$$\frac{1}{\operatorname{dist}(z, \sigma(A))} = \frac{1}{|1 - 2|} = 1$$

Therefore, for this choice of $\mathcal{H}$, A, and z, we have that

$$\|(A - z\mathbb{1})^{-1}\| > \frac{1}{\operatorname{dist}(z, \sigma(A))}$$

■

Problem 2

Let $A \in \mathcal{B}(\mathcal{H})$ be an operator with polar decomposition $A = U|A|$. Define functions $f_n : [0, \infty) \to \mathbb{R}$ via

$$f_n(x) := \begin{cases} \frac{1}{x} & x \geq \frac{1}{n} \\ n & x \leq \frac{1}{n} \end{cases}$$

Prove that

$$U = \operatorname*{s-lim}_{n \to \infty} A f_n(|A|)$$

Solution

Proof. We wish to show that

$$\mathbb{1} = \operatorname*{s-lim}_{n \to \infty} |A| f_n(|A|),$$

since the main result will follow by applying U to both sides. Since $|A|$ is self-adjoint, we may apply the functional calculus. Define $g_n : [0, \infty) \to \mathbb{R}$ via

$$g_n(x) := x f_n(x) = \begin{cases} 1 & x \geq \frac{1}{n} \\ nx & x \leq \frac{1}{n} \end{cases}$$

Letting $I : [0, \infty) \to [0, \infty)$ be the identity map $x \mapsto x$, we have that $g_n(x) = I(x) f_n(x)$. Then, by the homomorphism property of the functional calculus and the fact that $I(|A|) = |A|$,

$$g_n(|A|) = I(|A|) f_n(|A|) = |A| f_n(|A|)$$

So, we want to show that $\text{s-lim}_{n \to \infty} g_n(|A|) = \mathbb{1}$, and the result will follow. To see this, we simply note that $g_n(x)$ converges to 1 for every $x \in (0, \infty)$, and so $g_n \to 1$ pointwise a.e. on $[0, \infty)$. Then, over $\sigma(|A|)$ we see that $g_n \to 1$ pointwise and $\|g_n\|_\infty \leq 1 < \infty$. So, the measurable functional calculus (Theorem VII.2(d) in Reed & Simon) gives that $g_n(|A|) \to \mathbb{1}$ strongly. The main result follows. ■

Problem 3

Prove that if $A \in \mathcal{B}(\mathcal{H})$ is normal, then

$$\|A\| = r(A)$$

where $r(\cdot)$ is the spectral radius.

Solution

Proof. By the spectral mapping theorem and the continuous functional calculus on the function $z \mapsto |z|^2$ (which we may apply since A is normal, see Theorem 8.40 in the lecture notes), we have that

$$\sigma(|A|^2) = \{|z|^2 : \ z \in \sigma(A)\}$$

Thus,

$$r(|A|^2) = \sup_{z \in \sigma(A)} \{|z|^2\},$$

and so

$$r(A) = \sup_{z \in \sigma(A)} \{|z|\} = \sqrt{r(|A|^2)}$$

We know that $\||A|^2\| = r(|A|^2)$ by Theorem 8.6 in the lecture notes since $|A|^2$ is self-adjoint. So, by this and the C^* identity,

$$r(A) = \sqrt{\||A|^2\|} = \sqrt{\|A\|^2} = \|A\|$$

■

Problem 4

Let $A \in \mathcal{B}(\mathcal{H})$ be normal. Show there exists some finite measure space (M, μ) and a unitary $U : \mathcal{H} \to L^2(M, \mu)$ such that there exists a bounded Borel function $f : M \to \mathbb{C}$ such that

$$(UAU^*\psi)(m) = f(m)\psi(m) \quad (m \in M,\ \psi \in L^2(M, \mu))$$

Solution

Proof. We may express A as a linear combination of two self-adjoint operators via

$$A = \mathbb{R}\mathrm{e}\,\{A\} + i\,\mathbb{I}\mathrm{m}\,\{A\}$$

Furthermore, since A is normal we see that these two operators commute. Applying the result of Problem 5 with $A_1 := \mathbb{R}\mathrm{e}\,\{A\}$ and $A_2 := \mathbb{I}\mathrm{m}\,\{A\}$ yields the finite measure space and unitary such that $\mathbb{R}\mathrm{e}\,\{A\}$ is represented as multiplication by f and $\mathbb{I}\mathrm{m}\,\{A\}$ is represented as multiplication by g. A is therefore represented as multiplication by $f + ig : M \to \mathbb{C}$ as desired. ■

Problem 5

Show that if $A, B \in \mathcal{B}(\mathcal{H})$ are two self-adjoint operators such that $[A, B] = 0$, then there exists a finite measure space (M, μ) and a unitary $U : \mathcal{H} \to L^2(M, \mu)$ such that there are two bounded Borel functions $f, g : M \to \mathbb{R}$ which obey

$$(UAU^*\psi)(m) = f(m)\psi(m)$$
$$(UBU^*\psi)(m) = g(m)\psi(m)$$

for all $m \in M$ and $\psi \in L^2(M, \mu)$.

Solution

Proof. Suppose that A and B commute. We claim that for all $\Omega \subseteq \sigma(A)$ and $\Sigma \subseteq \sigma(B)$ Borel it holds that $[\chi_\Omega(A), \chi_\Sigma(B)] = 0$, or in other words that the spectral projections commute. Note that for each continuous function $f : \sigma(A) \to \mathbb{C}$, we have that

$$f(A)B = Bf(A)$$

since we may uniformly approximate f by polynomials (Stone-Weierstrass) and the above trivially holds for polynomial functions. For any $\psi \in \mathcal{H}$ we may approximate $\chi_\Omega(\cdot)$ in the $\mu_{A,\psi}$ measure by continuous functions on $\sigma(A)$, which reveals that

$$\chi_\Omega(A)B\psi = B\chi_\Omega(A)\psi$$

since the region of error from the approximation can be made to have approximately small $\mu_{A,\psi}$ measure. Since this holds for all $\psi \in \mathcal{H}$, we see that

$$\chi_\Omega(A)B = B\chi_\Omega(A)$$

which gives that

$$\chi_\Omega(A)\chi_\Sigma(B) = \chi_\Sigma(B)\chi_\Omega(A)$$

for all Borel $\Omega \subseteq \sigma(A)$ and $\Sigma \subseteq \sigma(B)$. Define the set of rectangles

$$\mathcal{R} := \{E \subseteq \mathbb{R}^2 : \ E = \Omega \times \Sigma \text{ for } \Omega \subseteq \sigma(A) \text{ and } \Sigma \subseteq \sigma(B) \text{ Borel}\}$$

We will define a functional calculus starting with simple functions that can be written as linear combinations of characteristic functions of disjoint rectangles from $\mathcal{R}$. Write

$$\mathcal{S} := \left\{ f : \ f = \sum_{i=1}^n \alpha_i \chi_{R_i} \text{ with } \{R_i\}_i \subseteq \mathcal{R} \text{ pairwise disjoint} \right\}$$

to be the set of simple functions. For such $f \in \mathcal{S}$, we define

$$f(A, B) := \sum_{i=1}^n \alpha_i \chi_{\Omega_i}(A)\chi_{\Sigma_i}(B)$$

where $R_i = \Omega_i \times \Sigma_i$. We stop to note that for all such f, $\|f(A, B)\| \leq \|f\|_\infty$. To see this, observe that $\|f\|_\infty = \max_i\{|\alpha_i|\}$. Since the Ω_i's are disjoint from each other and similarly for the Σ_i's, we see that the projection operators $\{\chi_{\Omega_i}(A)\chi_{\Sigma_i}(B)\}_i$ are all pairwise orthogonal. So, for any $\psi \in \mathcal{H}$ we that

$$\|f(A, B)\psi\|^2 = \sum_{i=1}^n |\alpha_i|^2 \|\chi_{\Omega_i}(A)\chi_{\Sigma_i}(B)\psi\|^2 \leq \|f\|_\infty^2 \sum_{i=1}^n \|\chi_{\Omega_i}(A)\chi_{\Sigma_i}(B)\psi\|^2 \leq \|f\|_\infty^2 \|\psi\|^2,$$

where the last inequality comes from the Pythagorean theorem. So, this defines a functional calculus

$$\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$$

which is clearly a linear transformation and satisfies $\|\phi(f)\| \leq \|f\|_\infty$. Thus, since the set $\mathcal{S}$ is dense in $C(\sigma(A)\times\sigma(B))$ equipped with the uniform norm (we may apply Stone-Weierstrass since $\mathcal{S}$ separates points), we may use the BLT theorem to construct a continuous functional calculus $\phi : C(\sigma(A)\times\sigma(B)) \to \mathcal{B}(\mathcal{H})$.

We now construct a Borelian functional calculus. For any $\psi \in \mathcal{H}$ we see that the map

$$C(\sigma(A)\times\sigma(B)) \ni f \mapsto \langle\psi, \phi(f)\psi\rangle = \langle\psi, f(A,B)\psi\rangle$$

is a real-valued, continuous linear functional on $C(\sigma(A)\times\sigma(B))$. Therefore, by the Riesz-Markov theorem, there is a unique Borel measure μ_ψ on $\mathbb{R}^2$ with $\mu_\psi(\mathbb{R}^2) = \|\psi\|^2$ and

$$\langle\psi, f(A,B)\psi\rangle = \int_{\mathbb{R}^2} f(z)d\mu_\psi(z)$$

These measures μ_ψ are spectral measures, and we may use the polarization identity to uniquely define $\langle\psi, f(A,B)\varphi\rangle$ for any bounded, Borel-measurable function $f : \sigma(A)\times\sigma(B) \to \mathbb{R}$. Thus upgrades us to a Borelian functional calculus.

We proceed to a spectral theorem. Call a vector $\psi \in \mathcal{H}$ *cyclic* for (A,B) if $\operatorname{span}\{f(A,B)\psi : f \in C(\sigma(A)\times\sigma(B))\}$ is dense in $\mathcal{H}$. Then, by basically the same proof as Lemma 10.22 in the lecture notes, we see that if ψ is cyclic for (A,B) then there is a unitary operator $U : \mathcal{H} \to L^2(\sigma(A)\times\sigma(B), \mu_\psi)$ for which

$$\begin{aligned}(UAU^*f)(x_1,x_2) &= x_1 f(x_1,x_2) \quad ((x_1,x_2) \in \sigma(A)\times\sigma(B))\\ (UBU^*f)(x_1,x_2) &= x_2 f(x_1,x_2) \quad ((x_1,x_2) \in \sigma(A)\times\sigma(B))\end{aligned}$$

Now, all we must do is decompose $\mathcal{H}$ into a direct sum of spaces which each admit a cyclic vector, which can be done cleanly via Zorn's lemma. So, we see that

$$\mathcal{H} = \bigoplus_{n=1}^N \mathcal{H}_n$$

where $N \in \mathbb{N}\cup\{\infty\}$ and each $\mathcal{H}_n$ admits a cyclic vector ψ_n. We see that $\mathcal{H}$ is unitarily equivalent to $\bigoplus_{n=1}^N L^2(\sigma(A)\times\sigma(B), \mu_{\psi_n})$ and for all $n = 1,\dots,N$ and $f_n \in L^2(\sigma(A)\times\sigma(B), \mu_{\psi_n})$,

$$\begin{aligned}(UAU^*f)_n(x_1,x_2) &= x_1 f_n(x_1,x_2) \quad ((x_1,x_2) \in \sigma(A)\times\sigma(B))\\ (UBU^*f)_n(x_1,x_2) &= x_2 f_n(x_1,x_2) \quad ((x_1,x_2) \in \sigma(A)\times\sigma(B))\end{aligned}$$

Set $M := \bigoplus_{n=1}^N(\sigma(A)\times\sigma(B))$ and $\mu := \bigotimes_{n=1}^N \mu_{\psi_n}$, The above statment now reads that, for every $f \in L^2(M,\mu)$, we have

$$\begin{aligned}(UAU^*f)(m) &= F_A(m)f(m) \quad (m \in M)\\ (UBU^*f)(m) &= F_B(m)f(m) \quad (m \in M)\end{aligned}$$

where F_A acts on the $\sigma(A)$ part of each constituent $\sigma(A)\times\sigma(B)$ in M and F_B acts on the $\sigma(B)$ parts. We note that $\mu(M) = \sum_{n=1}^N \|\psi_n\|^2$; if we select ψ_n's such that $\|\psi_n\| < 2^{-n}$ (which we may do since scaling does not change cyclicity in $\mathcal{H}_n$), we see that $\mu(M) \leq 1$. ∎

Problem 6

Prove that for $A \in \mathcal{B}(\mathcal{H})$ self-adjoint and $\chi_{\cdot}(A)$ the projection-valued measure of A, we have

$$\lambda \in \sigma(A) \iff \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) \neq 0 \quad (\varepsilon > 0)$$

Solution

Proof. We will show the contrapositives for both directions. Namely, we prove the following:

$$\lambda \in \rho(A) \iff \exists \varepsilon > 0 \text{ s.t. } \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) = 0$$

($\Longleftarrow$) Suppose that $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) = 0$ for some $\varepsilon > 0$; we want to show that this implies $\lambda \in \rho(A)$. Note that for all $\varphi, \psi \in \mathcal{H}$,

$$0 = \left\langle \varphi, \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)\psi \right\rangle = \mu_{A,\varphi,\psi}((\lambda-\varepsilon,\lambda+\varepsilon))$$

Therefore, for all $\varphi, \psi \in \mathcal{H}$ we see that for all $z \in \operatorname{spt}(\mu_{A,\varphi,\psi})$ we have $|z-\lambda| \geq \varepsilon$, and so the map sending $z \mapsto \frac{1}{z-\lambda}$ is bounded on $\operatorname{spt}(\mu_{A,\varphi,\psi})$. In other words, we have that $\left|\frac{1}{z-\lambda}\right| \leq \frac{1}{\varepsilon}$ for $\mu_{A,\varphi,\psi}$-a.e. $z \in \mathbb{R}$. By the bounded measurable functional calculus, we therefore see that

$$\left\langle \varphi, (A-\lambda\mathbb{1})^{-1}\psi \right\rangle = \int_{\mathbb{R}} \frac{1}{z-\lambda} d\mu_{A,\varphi,\psi}(z)$$

converges for all $\varphi, \psi \in \mathcal{H}$. Since all the matrix elements are defined, the resolvent $(A-\lambda\mathbb{1})^{-1}$ exists. In particular, $\lambda \in \rho(A)$.

($\Longrightarrow$) Suppose that $\lambda \in \rho(A)$. Since $\rho(A)$ is open, there is an $\varepsilon > 0$ such that $(\lambda-\varepsilon,\lambda+\varepsilon) \subseteq \rho(A)$. Since $\operatorname{spt}(\mu_{A,\psi}) \subseteq \sigma(A)$ for all $\psi \in \mathcal{H}$, this reveals that for all $\psi \in \mathcal{H}$,

$$(\lambda-\varepsilon,\lambda+\varepsilon) \subseteq \operatorname{spt}(\mu_{A,\psi})^C \implies \int_{\mathbb{R}} \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(z) d\mu_{A,\psi}(z) = 0$$

By construction of the measurable functional calculus, for all $\psi \in \mathcal{H}$ we have

$$\left\langle \psi, \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)\psi \right\rangle = \int_{\mathbb{R}} \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(z) d\mu_{A,\psi}(z) = 0$$

So, by Theorem 7.11 in the lecture notes, since all the diagonal elements of $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)$ are 0 we know that $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) = 0$. So, there exists an $\varepsilon > 0$ such that $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) = 0$. ■

Problem 7

Prove that the **only** norm-closed $*$-ideals in $\mathcal{B}(\mathcal{H})$ are $\{0\}$, $\mathcal{K}(\mathcal{H})$, and $\mathcal{B}(\mathcal{H})$.

Solution

Proof. Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a norm-closed $*$-ideal. Clearly, $0 \in \mathcal{A}$. Also, $\mathcal{A}$ is a vector subspace. We claim the following:

(1) If $\mathcal{A}$ contains any nonzero operator, then $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{A}$

(2) If $\mathcal{A}$ contains any noncompact operator, then $\mathcal{A} = \mathcal{B}(\mathcal{H})$

The result follows from these facts. To see this, we will exhaust all the possible cases. Suppose first that $\mathcal{A}$ contains no nonzero compact operator and no nonzero noncompact operator; then, $\mathcal{A} = \{0\}$. Next, suppose that $\mathcal{A}$ contains some nonzero compact operator (and so $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{A}$ by (1)), but no nonzero noncompact operator. Then, $\mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \subseteq \mathcal{A}^C \implies \mathcal{A} \subseteq \mathcal{K}(\mathcal{H})$, and so $\mathcal{A} = \mathcal{K}(\mathcal{H})$. Lastly, suppose that $\mathcal{A}$ contains some noncompact operator; (2) then implies that $\mathcal{A} = \mathcal{B}(\mathcal{H})$. So, to complete the proof it suffices to show (1) and (2).

(1) We will show that $\mathcal{A}$ contains every rank-one operator of the form $\varphi \otimes \psi^*$, $\varphi, \psi \in \mathcal{H}$, since then $\mathcal{A}$ will contain every finite-rank operator by properties of a vector space. From this, we will see that $\mathcal{A}$ contains $\mathcal{K}(\mathcal{H})$ by the fact that $\mathcal{K}(\mathcal{H})$ is the norm-closure of the set of finite-rank operators and $\mathcal{A}$ is norm-closed. So, let $\varphi, \psi \in \mathcal{H}$ be arbitrary. Let $B \in \mathcal{A}$ be nonzero. Then, there is some $\phi \in \mathcal{H}$ such that $B(\phi)$ is nonzero. By the Hahn-Banach theorem (linear functionals separate points), there is some $\eta \in \mathcal{H}$ such that $\langle \eta, B(\phi) \rangle = 1$. We claim that

$$\varphi \otimes \psi^* = (\varphi \otimes \eta^*) B (\phi \otimes \psi^*),$$

which by the two-sided-ideal property would mean that $\varphi \otimes \psi^* \in \mathcal{A}$. To see this, note that for any $\xi \in \mathcal{H}$,

$$(\varphi \otimes \eta^*) B (\phi \otimes \psi^*)(\xi) = (\varphi \otimes \eta^*) B(\langle \psi, \xi \rangle \phi) = \langle \psi, \xi \rangle \langle \eta, B(\phi) \rangle \varphi = \langle \psi, \xi \rangle \varphi = (\varphi \otimes \psi^*)(\xi)$$

So, $\varphi \otimes \psi^* \in \mathcal{A}$ for all $\varphi, \psi \in \mathcal{H}$. Thus, $\mathcal{A}$ contains all the finite-rank operators, and by norm-closure we have that $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{A}$.

(2) Suppose now that $\mathcal{A}$ contains a noncompact operator $S \in \mathcal{A}$. By noncompactness, $\text{im}(S)$ must have a closed infinite-dimensional subspace, call it M. Define the closed vector subspace $N := S^{-1}(M) \cap \ker(S)^\perp$. Then, consider the restricted bounded linear operator $S|_N : N \to M$. By construction, $S|_N$ is both injective (since $\ker(S) \cap N = \{0\}$) and surjective (since $\text{im}(S|_N) = M$). Since N and M are both closed subspaces of $\mathcal{H}$ they are Banach spaces, and so $S|_N^{-1} : M \to N$ is continuous by the inverse mapping theorem. Let P_M be the orthogonal projection onto M, and define the map $T \in \mathcal{B}(\mathcal{H})$ via

$$T = S|_N^{-1} P_M$$

Then, for all $\varphi \in M$ we see that $ST\varphi = \varphi$ whereas for all $\varphi \in M^\perp$ we see that $ST\varphi = 0$. So, $ST = P_M$. Now, let $R \in \mathcal{B}(\mathcal{H})$ be arbitrary; we will show that $R \in \mathcal{A}$. Let $\{\varphi_n\}_{n \in \mathbb{N}} \subseteq M$ be an orthonormal basis of M and let $\{\psi_n\}_{n=1}^N \subseteq \overline{\text{im}(R)}$ be an orthonormal basis of $\overline{\text{im}(R)}$ (N may be finite or countably infinite, which is fine since $\{\varphi_n\}_n$ is infinite). Let $U \in \mathcal{B}(\mathcal{H})$ be the map sending $\psi_n \mapsto \varphi_n$ and extended linearly, such that $U \equiv 0$ over $\overline{\text{im}(R)}^\perp$. Then, U is a partial isometry and $\text{im}(U) \subseteq M$. Note that for all $\xi \in \mathcal{H}$, we have that

$$R\xi \in \ker(U)^\perp \implies R\xi = |U|^2 R\xi = U^* U R\xi$$

However, we know that $UR\xi \in M$ and so $STUR\xi = UR\xi$. Thus, $R\xi = U^* STUR\xi$. Since this holds for all $\xi \in \mathcal{H}$, we see that $R = U^* STUR$, which means that $R \in \mathcal{A}$ by the two-sided-ideal property. Thus, $\mathcal{A} = \mathcal{B}(\mathcal{H})$. ∎

Problem 8

Let $\mathcal{H} := \ell^2(\mathbb{Z})$ and on it define the discrete Laplacian

$$-\Delta := 2\mathbb{1} - R - R^*$$

where R is the bilateral right shift operator.

(a) For $x \in \mathbb{Z}$, is δ_x a cyclic vector for $-\Delta$?

(b) Define $f : \mathbb{C}^+ \to \mathbb{C}$ via

$$f(z) = \left\langle \delta_0, (-\Delta - z\mathbb{1})^{-1}\delta_0 \right\rangle$$

Find an explicit expression for f using the Fourier series.

(c) Calculate

$$\lim_{\varepsilon \to 0^+} \mathbb{Im}\left\{f(E + i\varepsilon)\right\}$$

for the two cases $E \in (0, 4)$ and $E \in \mathbb{R} \setminus (0, 4)$.

(d) Calculate the spectral measure of $(-\Delta, \delta_0)$ and determine its type (with respect to the Lebesgue decomposition theorem where the reference measure is the Lebesgue measure, i.e., ac, sc, or pp).

Solution

Proof. (a) Pick an $x \in \mathbb{Z}$. Then, we see that applying $-\Delta$ symmetrically introduces positions to the left and right of x. Put differently, we have that for all $n, k \in \mathbb{N}$,

$$\langle \delta_{x+k}, (-\Delta)^n \delta_x \rangle = \langle \delta_{x-k}, (-\Delta)^n \delta_x \rangle$$

By linearity and continuity of the inner product, we see that for all $\psi \in \overline{\text{span}\{(-\Delta)^n \delta_x : n \in \mathbb{N}\}}$ and all $k \in \mathbb{N}$ we have

$$\langle \delta_{x+k}, \psi \rangle = \langle \delta_{x-k}, \psi \rangle$$

Therefore we clearly cannot have that $\delta_{x+k} \in \overline{\text{span}\{(-\Delta)^n \delta_x : n \in \mathbb{N}\}}$ for any $k \in \mathbb{N}$, which means that $\overline{\text{span}\{(-\Delta)^n \delta_x : n \in \mathbb{N}\}}$ is not dense. Thus, δ_x isn't cyclic.

(b) Let $\mathcal{F} : \mathcal{H} \to L^2(\mathbb{S}^1)$ be the Fourier transform from Problem Set 8. Then, since it is unitary we have that for all $z \in \mathbb{C}^+$,

$$f(z) = \left\langle \delta_0, (-\Delta - z\mathbb{1})^{-1}\delta_0 \right\rangle = \left\langle \mathcal{F}\delta_0, \mathcal{F}(-\Delta - z\mathbb{1})^{-1}\mathcal{F}^*\mathcal{F}\delta_0 \right\rangle_{L^2(\mathbb{S}^1)}$$

We know that $(\mathcal{F}\delta_0)(x) = e^{-i0x} = 1$, the constant 1 function. Also, we know from Problem Set 8 and linearity that $\mathcal{F}(-\Delta)\mathcal{F}^*$ is the multiplication operator by the function $\theta \mapsto 2 - 2\cos(\theta)$. So, by the functional calculus we know that $\mathcal{F}(-\Delta - z\mathbb{1})^{-1}\mathcal{F}^*$ is the multiplication operator by the function $\theta \mapsto \frac{1}{2-2\cos(\theta)-z}$. So, we may compute this inner product in $L^2(\mathbb{S}^1)$ to see that for $z \in \mathbb{C}^+$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_0^{2\pi} \overline{(\mathcal{F}\delta_0)(\theta)} \frac{1}{2 - 2\cos(\theta) - z} (\mathcal{F}\delta_0)(\theta) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(2-z) - 2\cos(\theta)} d\theta \end{aligned}$$

It is a well-known fact from residue integration that $\int_0^{2\pi} \frac{1}{a + b\cos(\theta)} d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}$, where the result may be complex for general $a, b \in \mathbb{C}$. Therefore, we have that

$$f(z) = \frac{1}{\sqrt{(2-z)^2 - 4}} = \frac{1}{\sqrt{z^2 - 4z}} \quad (z \in \mathbb{C}^+)$$

(c) Let $E \in \mathbb{R}$ and $\varepsilon > 0$. Noting that $\mathbb{Im}\{\cdot\}$, $\sqrt{\cdot}$, and $\frac{1}{\cdot}$ are all continuous away from 0, we see that when $E \notin \{0,4\}$ we may directly compute the limit as

$$\lim_{\varepsilon\to 0^+} \mathbb{Im}\{f(E+i\varepsilon)\} = \mathbb{Im}\left\{\frac{1}{\sqrt{\lim_{\varepsilon\to 0^+}(E+i\varepsilon)^2 - 4(E+i\varepsilon)}}\right\} = \mathbb{Im}\left\{\frac{1}{\sqrt{E^2-4E}}\right\}$$

In particular, when $E > 4$ or $E < 0$ the imaginary part is 0. When $E \in (0,4)$ we have that

$$\mathbb{Im}\left\{\frac{1}{\sqrt{E^2-4E}}\right\} = \mathbb{Im}\left\{\frac{1}{i\sqrt{4E-E^2}}\right\} = \frac{1}{\sqrt{4E-E^2}} > 0$$

When E is 0 or 4 we see that $(E+i\varepsilon)^2 - 4(E+i\varepsilon) = E^2 - \varepsilon^2 + 2Ei\varepsilon - 4E - 4i\varepsilon = -\varepsilon^2 \pm 4i\varepsilon$, and so

$$\mathbb{Im}\{f(E+i\varepsilon)\} = \mathbb{Im}\left\{\frac{1}{\sqrt{-\varepsilon^2 \pm 4i\varepsilon}}\right\} = \mathbb{Im}\left\{\sqrt{\frac{1}{-\varepsilon^2 \pm 4i\varepsilon}}\right\}$$

Clearly, as $\varepsilon \to 0^+$ we see that this approaches infinity. To summarize,

$$\lim_{\varepsilon\to 0^+} \mathbb{Im}\{f(E+i\varepsilon)\} = \begin{cases} 0 & E \in \mathbb{R}\setminus[0,4] \\ \frac{1}{\sqrt{4E-E^2}} & E \in (0,4) \\ \infty & E \in \{0,4\} \end{cases}$$

We take the time here to also note that for $E \in \{0,4\}$,

$$\lim_{\varepsilon\to 0^+} \varepsilon\,\mathbb{Im}\{f(E+i\varepsilon)\} = \lim_{\varepsilon\to 0^+} \mathbb{Im}\left\{\sqrt{\frac{1}{-1 \pm \frac{4i}{\varepsilon}}}\right\} = 0$$

(d) Let $\mu_{-\Delta,\delta_0}$ be the spectral measure of $(-\Delta,\delta_0)$. Write the Radon-Nikodym decomposition of $\mu_{-\Delta,\delta_0}$ w.r.t. the Lebesgue measure as

$$\mu_{-\Delta,\delta_0} = f\lambda + \mu_{\text{sing}}$$

By Lemma 10.10 in the lecture notes and part (c), we see that

$$f(\lambda) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\mathbb{Im}\{f(\lambda+i\varepsilon)\} = \begin{cases} 0 & \lambda \in \mathbb{R}\setminus[0,4] \\ \frac{1}{\pi\sqrt{4\lambda-\lambda^2}} & \lambda \in (0,4) \\ \infty & \lambda \in \{0,4\} \end{cases}$$

By Lemma 10.11 in the lecture notes, $\text{spt}(\mu_{\text{sing}}) = \{0,4\}$, but at these values $\mu_{-\Delta,\delta_0}$ has no point masses. Thus, the spectral measure $\mu_{-\Delta,\delta_0}$ is absolutely continuous w.r.t. Lebesgue with support on $[0,4]$ and a density given by f above. ■

Problem 9

Let $\mathcal{H} := \ell^2(\mathbb{Z})$ and on it define the multiplication operator $V(X)$ via

$$(V(X)\psi)(x) := V(x)\psi(x) \quad (x \in \mathbb{Z}, \psi \in \mathcal{H}),$$

where $V : \mathbb{Z} \to \mathbb{R}$ is some bounded sequence.

(a) For $x \in \mathbb{Z}$, is δ_x a cyclic vector for $V(X)$?

(b) For any $x \in \mathbb{Z}$, define $f_x : \mathbb{C}^+ \to \mathbb{C}$ via

$$f_x(z) = \langle \delta_x, (V(X) - z\mathbb{1})^{-1}\delta_x \rangle$$

Find an explicit expression for f_x.

(c) Calculate both

$$\lim_{\varepsilon \to 0^+} \mathbb{Im}\{f_x(E + i\varepsilon)\}$$

and

$$\lim_{\varepsilon \to 0^+} \varepsilon\, \mathbb{Im}\{f_x(E + i\varepsilon)\}$$

for all $E \in \mathbb{R}$ (separate into cases).

(d) Calculate the spectral measure of $(V(X), \delta_0)$ and determine its type (with respect to the Lebesgue decomposition theorem where the reference measure is the Lebesgue measure, i.e., ac, sc, or pp).

Solution

Proof. (a) For any $x \in \mathbb{Z}$, note that $V(X)\delta_x = V(x)\delta_x$. In particular, we have that for all $x \in \mathbb{Z}$,

$$\{V(X)^n \delta_x : \ n \in \mathbb{N}\} \subseteq \operatorname{span}\{\delta_x\}$$

Clearly, this set cannot be dense in $\mathcal{H}$, and so no δ_x is cyclic for $V(X)$.

(b) Note that for each $x \in \mathbb{Z}$ and each $z \in \mathbb{C}^+$ we have

$$(V(X) - z\mathbb{1})\left(\frac{\delta_x}{V(x) - z}\right) = \delta_x \implies (V(X) - z\mathbb{1})^{-1}\delta_x = \left(\frac{1}{V(x) - z}\right)\delta_x$$

So, we see that

$$f_x(z) = \left\langle \delta_x, \left(\frac{1}{V(x) - z}\right)\delta_x \right\rangle = \frac{1}{V(x) - z}$$

(c) From the above, we get that for all $E \in \mathbb{R}$ and all $\varepsilon > 0$,

$$\mathbb{Im}\{f_x(E + i\varepsilon)\} = \mathbb{Im}\left\{\frac{1}{(V(x) - E) - i\varepsilon}\right\}$$

Letting $y := V(x) - E$ for notation, we can rationalize

$$\frac{1}{y - i\varepsilon} = \frac{y + i\varepsilon}{y^2 + \varepsilon^2} = \frac{y}{y^2 + \varepsilon^2} + i\frac{\varepsilon}{y^2 + \varepsilon^2}$$

From this, it is obvious that

$$\mathbb{Im}\{f_x(E + i\varepsilon)\} = \frac{\varepsilon}{(V(x) - E)^2 + \varepsilon^2}$$

If $E \neq V(x)$, then we find that

$$\lim_{\varepsilon \to 0^+} \mathbb{I}\mathrm{m}\left\{f_x(E+i\varepsilon)\right\} = \lim_{\varepsilon \to 0^+} \varepsilon\, \mathbb{I}\mathrm{m}\left\{f_x(E+i\varepsilon)\right\} = 0$$

However, if $E = V(x)$ then

$$\lim_{\varepsilon \to 0^+} \mathbb{I}\mathrm{m}\left\{f_x(E+i\varepsilon)\right\} = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} = \infty \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \varepsilon\, \mathbb{I}\mathrm{m}\left\{f_x(E+i\varepsilon)\right\} = 1$$

(d) Let $\mu_{V(X),\delta_0}$ be the spectral measure of $(V(X), \delta_0)$. Write the Radon-Nikodym decomposition of $\mu_{V(X),\delta_0}$ w.r.t. the Lebesgue measure as

$$\mu_{V(X),\delta_0} = f\lambda + \mu_{\text{sing}}$$

By Lemma 10.10 in the lecture notes and part (c), we see that

$$f(\lambda) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \mathbb{I}\mathrm{m}\left\{f_0(\lambda+i\varepsilon)\right\} = \begin{cases} 0 & \lambda \neq V(0) \\ \infty & \lambda = V(0) \end{cases}$$

So, $f = 0$ Lebesgue a.e., which means $f\lambda = 0$. Therefore, $\mu_{V(X),\delta_0}$ has no absolutely continuous part. By Lemma 10.11 in the lecture notes, $\text{spt}(\mu_{\text{sing}}) = \{V(0)\}$, and at this value $\mu_{V(X),\delta_0}$ has a point mass. From the above, we see that μ_{sing} is precisely equal to a point mass at $V(0)$. Thus, the spectral measure $\mu_{V(X),\delta_0}$ is pure point and equal to a point mass at $V(0)$. $\blacksquare$

Problem 10

On $\mathcal{H} := \ell^2(\mathbb{N})$, let R be the unilateral right shift operator. Calculate $\ker R$, $\ker R^*$ and $\mathrm{im}(R)$ and show that R is a Fredholm operator. Calculate its Fredholm index.

Solution

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be the standard orthonormal basis of $\mathcal{H}$. Then, R is the operator sending

$$Re_n := e_{n+1} \quad (n \in \mathbb{N})$$

and extended linearly. Then, we see that R preserves the norm of each e_n, which means that it is an isometry over all of $\mathcal{H}$ by the Pythagorean theorem and the fact that $\{e_n\}_n$ is an orthonormal basis. In particular, this means that R has trivial kernel and closed image. Let

$$M := \mathrm{span}\{e_1\} = \{\psi \in \mathcal{H} : \ \psi(j) = 0 \text{ for all } j > 1\}$$

Then, we see that

$$\mathrm{im}(R) = M^\perp = \{\psi \in \mathcal{H} : \ \psi(1) = 0\}$$

We have already seen that R^* is the unilateral left shift, i.e.

$$R^* e_n := \begin{cases} e_{n-1} & n > 1 \\ 0 & n = 1 \end{cases}$$

From this wee see immediately that $\ker(R^*) = M$.

Now, we see that $\dim\ker(R) = 0$. Also, we know that $\dim\mathrm{coker}(R) = \dim\ker(R^*) = \dim M = 1$. Therefore, R is Fredholm with a Fredholm index of $0 - 1 = -1$. ∎

Problem 11

Show that on $\mathcal{H} := \ell^2(\mathbb{N})$, $\frac{1}{X}$ where X is the position operator is *not* a Fredholm operator by calculating $\operatorname{im}(\frac{1}{X})$ and showing that it is not closed.

Solution

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be the standard orthonormal basis of $\mathcal{H}$. The operator $\frac{1}{X}$ can be expressed as a multiplication operator on this basis via

$$\frac{1}{X}e_n := \frac{1}{n}e_n$$

Clearly, $\frac{1}{X}$ is bounded and linear, and so $\frac{1}{X} \in \mathcal{B}(\mathcal{H})$. For any $\psi \in \mathcal{H}$ we may write

$$\psi = \sum_{n\in\mathbb{N}} \psi(n)e_n$$

where this convergence is in the norm on $\mathcal{H}$. So, we have that

$$\frac{1}{X}\psi = \sum_{n\in\mathbb{N}} \frac{\psi(n)}{n}e_n$$

Therefore, we may write

$$\operatorname{im}\left(\frac{1}{X}\right) = \left\{\varphi \in \mathcal{H} : \sum_{n\in\mathbb{N}} n^2|\varphi(n)|^2 < \infty\right\}$$

since any such φ can be expressed as $\frac{1}{X}\psi$ for a $\psi \in \mathcal{H}$, whereas any $\varphi \in \mathcal{H}$ that does not have this summability condition cannot. We claim that this set is not closed. To see this, for each $N \in \mathbb{N}$ define

$$\psi_N := \sum_{n=1}^{N} \frac{1}{n}e_n$$

Clearly, each $\psi_N \in \operatorname{im}(\frac{1}{X})$ since it has finitely many nonzero terms. Furthermore, we see that $\{\psi_N\}_N$ has a limit ψ, since

$$\|\psi\|_{\mathcal{H}}^2 = \left\|\lim_{N\to\infty} \psi_N\right\|_{\mathcal{H}}^2 = \left\|\sum_{n=1}^{\infty} \frac{1}{n}e_n\right\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

However, this limit is certainly not in $\operatorname{im}(\frac{1}{X})$ since ψ does not satisfy the harder summability condition

$$\sum_{n\in\mathbb{N}} n^2|\psi(n)|^2 = \sum_{n\in\mathbb{N}} n^2\frac{1}{n^2} = \sum_{n\in\mathbb{N}} 1 = \infty$$

So, there is a sequence of elements in $\operatorname{im}(\frac{1}{X})$ converging to a vector that is not in $\operatorname{im}(\frac{1}{X})$, and so $\operatorname{im}(\frac{1}{X})$ is not closed. By Proposition 9.45 in the lecture notes, this means that $\frac{1}{X}$ is not Fredholm. since $\operatorname{coker}(\frac{1}{X})$ is not finite-dimensional. ■