MAT 520: Problem Set 1

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Problem 1

Prove that $\mathbb{C}^n$ with its Euclidean topology is a topological vector space, i.e., show that vector addition and scalar multiplication are continuous with respect to the Euclidean topology.

Solution

Proof. We first show continuity of vector addition. Let $z, w \in \mathbb{C}^n$ be two arbitrary vectors, and let $\epsilon > 0$. Then, if we let $\delta := \frac{\epsilon}{2} > 0$, we have for every $(\tilde{z}, \tilde{w}) \in B_\delta(z) \times B_\delta(w)$ that

$$|(\tilde{z} + \tilde{w}) - (z + w)| \leq |\tilde{z} - z| + |\tilde{w} - w| < \delta + \delta = \epsilon,$$

where the first inequality is the triangle inequality. Since such a δ exists for every ϵ, we see that the addition map is continuous at $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. Since this holds for all $z, w \in \mathbb{C}^n$, vector addition is continuous.

Similarly, let $z \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$ be arbitrary. Let $\epsilon > 0$ be arbitrary, and set $\delta := \min\left\{1, \frac{\epsilon}{1+|\alpha|+|z|}\right\} > 0$. Then, for every $\tilde{z} \in B_\delta(z)$ and every $\beta \in B_\delta(\alpha)$, we have that

$$\begin{aligned} |\beta\tilde{z} - \alpha z| &= |\beta\tilde{z} - \beta z + \beta z - \alpha z| \leq |\beta\tilde{z} - \beta z| + |\beta z - \alpha z| \\ &= |\beta||\tilde{z} - z| + |z||\beta - \alpha| \leq |\beta|\delta + |z|\delta \end{aligned}$$

Next, since $|\beta - \alpha| < \delta$, the reverse triangle inequality grants that $||\beta| - |\alpha|| < \delta \implies |\beta| < \delta + |\alpha|$. So,

$$|\beta\tilde{z} - \alpha z| \leq (\delta + |\alpha|)\delta + |z|\delta = \delta^2 + (|\alpha| + |z|)\delta \leq (1 + |\alpha| + |z|)\delta \leq \epsilon,$$

where the second to last inequality comes from the fact that $\delta \leq 1$, and the last inequality follows from $\delta \leq \frac{\epsilon}{1+|\alpha|+|z|}$. So, since we may find such a δ for every $\epsilon > 0$, scalar multiplication is continuous at (z, α). Since this holds for every $z \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$, we have shown that scalar multiplication is continuous. ∎

Problem 2

Prove that $\mathbb{C}$ with the French metro metric is not homeomorphic (=topologically isomorphic) to $\mathbb{C}$ with the Euclidean metric. Conclude (why?) that $\mathbb{C}$ with the French metro metric is not a TVS.

Solution

Proof. Recall the French metro metric

$$d(z,w) = \begin{cases} |z-w| & \exists \alpha \in \mathbb{R} \text{ s.t. } z = \alpha w \\ |z| + |w| & \text{else} \end{cases}$$

Pick a $z \in \mathbb{C}$ that is nonzero, and let $\delta \in (0, |z|)$ be arbitrary. In the French metro metric, we have

$$B_\delta(z) = \{w \in \mathbb{C} : \quad w = rz \text{ for some } r \in (1-\delta, 1+\delta)\}$$

Since metric spaces are T_1 and so singletons are closed, we see that $B := B_\delta(z) \setminus \{z\}$ is an open set with two connected components. Suppose by way of contradiction that there were a homeomorphism f going from $\mathbb{C}$ with the French metro metric to $\mathbb{C}$ with the Euclidean metric. Then, we should have that $f(B)$ is a set in $\mathbb{C}$ that is open in the Euclidean topology and has two connected components by the properties of homeomorphisms. However, by bijectivity of f we have

$$f(B) = f(B_\delta(z)) \setminus \{f(z)\}$$

So, $f(B_\delta(z))$ is a connected open set in $\mathbb{C}_{\text{usual}}$ which, upon removal of a single point, becomes two disjoint connected sets. This is impossible in $\mathbb{C}_{\text{usual}}$ since removing a point from open disks in $\mathbb{C}$ keeps the disk connected. So, we arrive at a contradiction, and so there can be no homeomorphism between these spaces.

Suppose by way of contradiction that $\mathbb{C}$ with the French metro metric were a TVS. Then, since the identity map is a vector space isomorphism from $\mathbb{C}_{\text{usual}}$ to $\mathbb{C}$ with the French metro metric (which is finite-dimensional), Theorem 1.21(a) from Rudin would guarantee that the identity map is also a homeomorphism. However, we just ruled out that possibility, and so we see that $\mathbb{C}$ with the French metro metric cannot be a TVS. ∎

Problem 3

Prove that if X is a TVS and $A, B \subseteq X$, then $\overline{A} + \overline{B} \subseteq \overline{A + B}$

Solution

Proof. Let $x \in \overline{A} + \overline{B}$ be arbitrary. Then, $x = x_a + x_b$ for some $x_a \in \overline{A}$ and $x_b \in \overline{B}$. Let $U \in \text{Nbhd}(0_X)$ be arbitrary. By Lemma 2.6, there is some $W \in \text{Nbhd}(0_X)$ for which $W + W \subseteq U$. By Theorem 1.13(a) in Rudin, we know that

$$\overline{A} = \bigcap_{U \in \text{Nbhd}(0_X)} (A + U)$$

and similarly for B. In particular, we know that $x_a \in A + W$ and $x_b \in B + W$. So, there exist $a \in A$, $b \in B$, and $w_a, w_b \in W$ such that

$$x = x_a + x_b = a + b + w_a + w_b = (a + b) + (w_a + w_b)$$

So, since $a + b \in A + B$ and $w_a + w_b \in W + W \subseteq U$, x can be written as a sum of an element of $A + B$ with an element of U, and so $x \in (A + B) + U$. Since this holds for all $U \in \text{Nbhd}(0_X)$, we have

$$x \in \bigcap_{U \in \text{Nbhd}(0_X)} ((A + B) + U) = \overline{A + B},$$

where the last equality is again by Theorem 1.13(a) in Rudin. Since this holds for all $x \in \overline{A} + \overline{B}$, the result is proven. ■

Problem 4

Prove that if X is a TVS and $A \subseteq X$ is a vector subspace, then so is $\overline{A}$.

Solution

Proof. Let $x, y \in \overline{A}$ and $\alpha \in \mathbb{C}$. Certainly, since $0_X \in A$ we have that $0_X \in \overline{A}$. We wish to show that $x + y \in \overline{A}$ and $\alpha x \in \overline{A}$, since then $\overline{A}$ will be closed under the vector operations. We know by Theorem 1.13(a) in Rudin that

$$\overline{A} = \bigcap_{U \in \text{Nbhd}(0_X)} (A + U)$$

So, for every $W \in \text{Nbhd}(0_X)$, it holds that $x, y \in A + W$. Let $U \in \text{Nbhd}(0_X)$ be an arbitrary neighborhood of the origin. By Lemma 2.6, there is a $W \in \text{Nbhd}(0_X)$ such that $W + W \subseteq U$. So, since $x, y \in A + W$, we know that $x = a_x + w_x$ and $y = a_y + w_y$ for some $a_x, a_y \in A$ and $w_x, w_y \in W$. Therefore,

$$x + y = (a_x + a_y) + (w_x + w_y)$$

Since A is a vector subspace, we know that $a_x + a_y \in A$ as well. Also, we know that $w_x + w_y \in W + W \subseteq U$. So, $x + y$ can be written as the sum of an element of A and an element of U, and so $x + y \in A + U$. Since this holds for every $U \in \text{Nbhd}(0_X)$, we find that $x + y \in \overline{A}$ as desired.

If $\alpha = 0$ then clearly $\alpha x \in \overline{A}$ (since A contains 0_X by definition of vector subspace and $A \subseteq \overline{A}$), and so suppose without loss of generality that $\alpha \neq 0$. Let $U \in \text{Nbhd}(0_X)$ be arbitrary. Define $W := \frac{1}{\alpha}U$; since scaling by $\frac{1}{\alpha}$ is a homeomorphism in a TVS and maps 0_X to 0_X, it must be that $W \in \text{Nbhd}(0_X)$. We note that

$$\begin{aligned} \alpha x \in A + U &\iff \alpha x = a + u \text{ for some } a \in A \text{ and } u \in U \\ \iff x = \frac{1}{\alpha}a + \frac{1}{\alpha}u &\text{ for some } a \in A \text{ and } u \in U \iff x \in \frac{1}{\alpha}A + \frac{1}{\alpha}U \end{aligned}$$

Since A is a vector subspace, we know that $A = \frac{1}{\alpha}A$, and so

$$\alpha x \in A + U \iff x \in A + W$$

Since $x \in \overline{A}$ and W is a neighborhood of the origin, we know by Theorem 1.13(a) in Rudin that $x \in A + W$, and so $\alpha x \in A + U$. Since this holds for all $U \in \text{Nbhd}(0_X)$, then

$$\alpha x \in \bigcap_{U \in \text{Nbhd}(0_X)} (A + U) = \overline{A},$$

completing the proof. ■

Problem 5

Prove that if X is a TVS and $A \subseteq X$, then $2A \subseteq A + A$.

Solution

Proof. Let $x \in 2A$; then, $x = 2a = a + a$ for some $a \in A$. So, x can be written as the sum of two elements of A (namely, a and a), and so $x \in A + A$. Since this holds for all $x \in 2A$, we find

$$2A \subseteq A + A$$

■

Problem 6

Prove that any union and any intersection of balanced sets is balanced.

Solution

Proof. Let $\{A_\gamma\}_{\gamma \in I}$ be any collection of balanced sets (I need not be countable). Let $\alpha \in \mathbb{C}$ be such that $|\alpha| \leq 1$. For the first part, let $x \in \alpha \left(\bigcup_{\gamma \in I} A_\gamma\right)$. Then, $x = \alpha y$ for some $y \in \bigcup_{\gamma \in I} A_\gamma$; since y is in the union, there is some A_γ such that $y \in A_\gamma$. So, $x = \alpha y \in \alpha A_\gamma \subseteq A_\gamma$ by the fact that A_γ is balanced. Thus, $x \in \bigcup_{\gamma \in I} A_\gamma$. Therefore,

$$\alpha \left(\bigcup_{\gamma \in I} A_\gamma \right) \subseteq \bigcup_{\gamma \in I} A_\gamma$$

for all $|\alpha| \leq 1$, as desired.

Now, let $x \in \alpha \left(\bigcap_{\gamma \in I} A_\gamma\right)$. Then, $x = \alpha y$ for some $y \in \bigcap_{\gamma \in I} A_\gamma \implies y \in A_\gamma \quad \forall \gamma \in I$. Since every A_γ is balanced, we know that $x = \alpha y \in \alpha A_\gamma \subseteq A_\gamma$ for every γ. So, $x \in \bigcap_{\gamma \in I} A_\gamma$. Therefore,

$$\alpha \left(\bigcap_{\gamma \in I} A_\gamma \right) \subseteq \bigcap_{\gamma \in I} A_\gamma$$

for all $|\alpha| \leq 1$, and the intersection is balanced. ■

Problem 7

Prove that if A and B are balanced, then so is $A + B$.

Solution

Proof. Let $\alpha \in \mathbb{C}$ be such that $|\alpha| \leq 1$, and let $x \in \alpha(A + B)$. Then, $x = \alpha y$ for some $y \in A + B$, meaning that $x = \alpha(a + b) = \alpha a + \alpha b$ for some $a \in A$ and $b \in B$. We know that $\alpha a \in \alpha A \subseteq A$ by the fact that A is balanced, and similarly we know that $\alpha b \in B$. Thus, x is the sum of an element of A and an element of B, meaning $x \in A + B$. Since this holds for all $x \in \alpha(A + B)$, we find

$$\alpha(A + B) \subseteq A + B$$

Since this holds for all $|\alpha| \leq 1$, then $A + B$ is balanced. ■

Problem 8

Let X be a TVS. Prove that if $A, B \subseteq X$ are bounded (resp. compact) then $A + B$ is bounded (resp. compact).

Solution

Proof. Suppose first that A, B are bounded. Let $U \in \text{Nbhd}(0_X)$ be arbitrary. By Lemma 2.6, there is some $W \in \text{Nbhd}(0_X)$ such that $W + W \subseteq U$. By boundedness, there are some $t_A, t_B > 0$ such that

$$s > t_A \implies A \subseteq sW \qquad \text{and} \qquad s > t_B \implies B \subseteq sW$$

Now, let $t_U := \max\{t_A, t_B\}$, and let $s > t$ be arbitrary. We wish to show that $A + B \subseteq sU$. So, suppose that $x \in A + B$, and so $x = a + b$ for some $a \in A \subseteq sW$ and $b \in B \subseteq sW$. Then, $a = sw_a$ and $b = sw_b$ for some $w_a, w_b \in W$, and so

$$x = a + b = sw_a + sw_b = s(w_a + w_b) \in s(W + W)$$

Lastly, since $W + W \subseteq U$, it must be that $s(W + W) \subseteq sU$, and so $x \in sU$. Since this holds for all $x \in A + B$, we have $A + B \subseteq sU$. Since this holds for all $U \in \text{Nbhd}(0_X)$ and all $s > t_U$, this means that $A + B$ is bounded.

Now, suppose that A and B are both compact. We wish to show that $A + B$ is also compact. To this end, let $\bigcup_{\alpha \in I} U_\alpha$ be an open cover of $A + B$. For each U_α, let $F_\alpha \subseteq X \times X$ denote the preimage of U_α under the addition map (i.e. $F_\alpha = \{(x, y) \in X \times X : \quad x + y \in U_\alpha\}$. Since addition is continuous in a TVS, each F_α is open. Furthermore, we note that

$$A \times B \subseteq \bigcup_{\alpha \in I} F_\alpha,$$

since each tuple $(a, b) \in A \times B$ maps under addition to an element $a + b \in U_\alpha$ for some α, and so $(a, b) \in F_\alpha$ for this α. So, $\{F_\alpha\}_\alpha$ is an open cover of $A \times B$, which is compact, and so there is a finite subcover $A \times B \subseteq \bigcup_{i=1}^n F_i$. We claim that

$$A + B \subseteq \bigcup_{i=1}^n U_i$$

To see this, let $a + b \in A + B$ with $a \in A$ and $b \in B$. Then, $(a, b) \in F_i$ for some $i \leq n$, and so taking the image under the addition map we see that $a + b \in U_i$ for that i. Thus, $a + b \in \bigcup_{i=1}^n U_i$. We have just constructed a finite subcover of an arbitrary open cover of $A + B$, therefore proving compactness of $A + B$.
■

Problem 9

Find two closed sets A, B for which $A + B$ is not closed.

Solution

Proof. Let $X = \mathbb{R}$ be the line as a TVS, and define

$$A := \{-n : \quad n \in \mathbb{N}\}$$

and

$$B := \left\{n - \frac{1}{n} : \quad n \in \mathbb{N}\right\}$$

Since

$$A^C = \bigcup_{n \in \mathbb{N}} (-n-1, -n) \cup (-1, \infty)$$

is an open set, we see that A is closed. Furthermore, B is closed since

$$B^C = (-\infty, 0) \cup \bigcup_{n \in \mathbb{N}} \left(n - \frac{1}{n},\ n + 1 - \frac{1}{n+1}\right)$$

is an open set. However, we claim that $A + B$ is not closed. To see this, note that for every $n \in \mathbb{N}$ we have that $-\frac{1}{n} \in A + B$ since $-n \in A$ and $n - \frac{1}{n} \in B$. Also, $-\frac{1}{n} \to 0$. However, $0 \notin A + B$. To see this, suppose by way of contradiction that $0 = a + b$ for some $-n_1 \in A$ and $n_2 - \frac{1}{n_2} \in B$. So,

$$0 = n_2 - \frac{1}{n_2} - n_1 \implies n_1 = n_2 - \frac{1}{n_2}$$

In order for $\frac{1}{n_2}$ to equal $n_2 - n_1$ and be an integer, it must be that $n_2 = 1$. Therefore, $n_1 = 0$, which is a contradiction since $0 \notin A$. Therefore, there is a sequence $\{-\frac{1}{n}\}_n \subseteq A + B$ whose limit point is not in $A + B$. Therefore, $A + B$ cannot be closed. ∎

Problem 10

If X, Y are TVS with $\dim(Y) < \infty$, and $\Lambda : X \to Y$ is linear with $\Lambda(X) = Y$, show that Λ is an open mapping. Show further that if $\ker(\Lambda)$ is closed, then Λ is continuous.

Solution

Proof. Suppose that Λ is linear and surjective, with $\dim(Y) = n < \infty$. Let $y_1, ..., y_n$ denote a basis of Y. Define the linear map $f : \mathbb{C}^n \to Y$ via

$$f(z) := \sum_{i=1}^{n} z_i y_i$$

Since $\{y_i\}_{i=1}^n$ is a basis of Y, the map f is bijective, and so it is a vector space isomorphism. By Theorem 1.21(a) in Rudin, it is therefore also a homeomorphism. Now, since Λ is surjective, we may define a map $g : \mathbb{C}^n \to X$ given by

$$g(z) = \sum_{i=1}^{n} z_i \Lambda^{-1}(y_i),$$

where $\Lambda^{-1} : Y \to X$ is any right inverse for Λ (i.e. for each y_i the preimage $\Lambda^{-1}(\{y\})$ is nonempty; use the Axiom of Choice to pick an element of this set and call it $\Lambda^{-1}(y_i)$). The map g is certainly linear, which by Lemma 1.20 in Rudin means that g is continuous. Define the map $\varphi : Y \to X$ given by $\varphi := g \circ f^{-1}$; since g is continuous and f is homeomorphic we find that φ is continuous. Now, for any $y \in Y$ we have that $y = \sum_{i=1}^n a_i y_i$, and so $f^{-1}(y) = (a_1, ..., a_n)$ uniquely by definition of a basis. Therefore, $\varphi(y) = g((a_1, ..., a_n)) = \sum_{i=1}^n a_i \Lambda^{-1}(y_i)$. By linearity of Λ, we get that

$$(\Lambda \circ \varphi)(y) = \Lambda\left(\sum_{i=1}^{n} a_i \Lambda^{-1}(y_i)\right) = \sum_{i=1}^{n} a_i \Lambda(\Lambda^{-1}(y_i)) = \sum_{i=1}^{n} a_i y_i = y$$

So, $\Lambda \circ \varphi$ is the identity over all of Y. We have therefore constructed a right inverse φ for Λ that is continuous. So, for any open set $U \in \mathrm{Open}(X)$, we see that $\varphi^{-1}(U)$ is open in Y by continuity of φ, where φ^{-1} denotes the preimage. However, $\varphi^{-1}(U) = \Lambda(U)$, since

$$y \in \varphi^{-1}(U) \iff \varphi(y) \in U \iff \Lambda(\varphi(y)) \in \Lambda(U) \iff y \in \Lambda(U),$$

where the first equivalence comes from the definition of the preimage of φ, the second equivalence comes from the definition of the image of Λ, and the last equivalence comes from the fact that $\Lambda(\varphi(y)) = y$ for all y. So, we find that $\Lambda(U)$ is open in Y. Since this holds for all $U \in \mathrm{Open}(X)$, Λ is an open map.

Suppose further that $\ker(\Lambda)$ is closed. We know that $\ker(\Lambda)$ is a vector subspace since $x, y \in \ker(\Lambda)$ and $\alpha \in \mathbb{C}$ implies $\Lambda(x + y) = \Lambda(x) + \Lambda(y) = 0_Y + 0_Y = 0_Y$ and $\Lambda(\alpha x) = \alpha\Lambda(x) = \alpha 0_Y = 0_Y$, and so $x + y, \quad \alpha x \in \ker(\Lambda)$. By Theorem 1.41 in Rudin, this means that the canonical quotient map $\pi : X \to X/\ker(\Lambda)$ sending $x \to x + \ker(\Lambda)$ is continuous, where we use $x + \ker(\Lambda)$ to denote cosets of $\ker(\Lambda)$ by elements x. Furthermore, note that the map $h : X/\ker(\Lambda) \to Y$ mapping $x + \ker(\Lambda) \mapsto \Lambda(x)$ is a vector space isomorphism by the First Isomorphism Theorem from abstract algebra. (Precisely, viewing Λ as a surjective group homomorphism and X, Y as additive groups, the theorem guarantees that h is a bijective homomorphism; we would also need to show that $h(\alpha x) = \alpha h(x) \quad \forall \alpha \in \mathbb{C}$, which follows directly from linearity of Λ and the fact that $\ker(\Lambda)$ is a vector subspace). So, as h is a vector space isomorphism to a finite-dimensional TVS, Theorem 1.21 in Rudin guarantees that h is also a homeomorphism, and so it certainly is continuous. Thus, $h \circ \pi$ is continuous; we claim that $\Lambda = h \circ \pi$ over X. Indeed,

$$h(\pi(x)) = h(x + \ker(\Lambda)) = \Lambda(x)$$

Thus, Λ is continuous, and we are done. ■

Problem 11

Let $C := \{f : [0,1] \to \mathbb{C} : \quad f \text{ is continuous}\}$ and define

$$d(f,g) := \int_{[0,1]} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx$$

Show that d is a metric on C, show that C is a vector space (with pointwise addition and scalar multiplication), and show that the topology which d induces on C makes it into a TVS. Show that that TVS has a countable local base.

Solution

Proof. We start by showing that d is a metric. Firstly, for all $f \in C$ we have

$$d(f,f) = \int_{[0,1]} \frac{|f(x) - f(x)|}{1 + |f(x) - f(x)|} dx = \int_{[0,1]} \frac{0}{1} dx = 0$$

The property that $d(f,g) = d(g,f)$ is clear from the symmetry of the definition. Next, suppose that $f, g \in C$ are such that $f \neq g$. Define the function $h : [0,1] \to \mathbb{R}$ by $h(x) := |f(x) - g(x)|$; then, h is nonnegative and continuous since $f - g$ and $|\cdot|$ are both continuous. Define the set

$$E := \{x \in [0,1] : \quad h(x) > 0\}$$

Clearly, E is nonempty since $f \neq g$ somewhere. Suppose by way of contradiction that $m(E) = 0$, where $m(\cdot)$ denotes the Lebesgue measure. Let $x \in E$, and let $0 < \epsilon < h(x)$. By continuity of h at x, there exists a $\delta > 0$ such that for all $y \in (x - \delta, x + \delta) \subseteq [0,1]$, we have

$$|h(x) - h(y)| < \epsilon \implies h(x) - h(y) < \epsilon < h(x) \implies h(y) > 0 \implies y \in E$$

So, $(x - \delta, x + \delta) \subseteq E$, and so by monotonicity of measure we have that $2\delta = m((x - \delta, x + \delta)) \leq m(E) = 0$. This is a contradiction, and so $m(E) > 0$. We may write

$$\begin{aligned} d(f,g) &= \int_E \frac{h(x)}{1 + h(x)} dx + \int_{[0,1] \setminus E} \frac{h(x)}{1 + h(x)} dx \\ &= \int_E \frac{h(x)}{1 + h(x)} dx + \int_E \frac{0}{1} dx \\ &\geq \int_E h(x) dx, \end{aligned}$$

where the inequality comes from the fact that h is nonnegative. Now, we know by inner regularity of the Lebesgue measure that if we set $\delta := m(E)$, we may select a closed set $F \subseteq E$ such that

$$m(E \setminus F) < \delta = m(E) \implies m(E) = m(F) + m(E \setminus F) < m(F) + m(E) \implies m(F) > 0$$

Define $a := \inf_{x \in F} \{h(x)\}$. Since $F \subseteq [0,1]$ is closed and bounded in $\mathbb{R}$, it is compact, and so the continuous function h attains its infimum a at some point; this necessarily means that $a > 0$ (if a were 0 then $h(t_a) = 0$ for some $t_a \in F \subseteq E$, contradicting our definition of E). As such, we may write

$$d(f,g) \geq \int_E h(x) dx \geq \int_F h(x) dx \geq a m(F) > 0$$

The last thing that remains to be shown is the triangle inequality. To this end, we prove the following lemma.

Lemma 1. *For any $a, b \in (0, \infty)$, we have that*

$$\frac{1}{1+a} + \frac{1}{1+b} \leq 1 + \frac{1}{1+a+b}$$

and

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + b$$

Proof of Lemma 1. For the first part, note that

$$LHS := \frac{1}{1+a} + \frac{1}{1+b} = \frac{1+a+1+b}{1+a+b+ab} = \frac{2+a+b}{1+a+b+ab}$$

Since $ab > 0$, we get that

$$LHS \leq \frac{2+a+b}{1+a+b} = 1 + \frac{1}{1+a+b}$$

To see the second part, note that

$$\frac{a+b}{1+a+b} \leq \frac{a+b+ab}{1+a+b} \leq \frac{a+b+ab}{1+a} = \frac{a}{1+a} + b,$$

proving the lemma. ■

From here, we note that for any $f, g, h \in C$, the above lemma grants

$$\begin{aligned}
d(f,g) = 1 - \int_{[0,1]} \frac{1}{1+|f(x)-g(x)|} dx &\leq 1 - \int_{[0,1]} \frac{1}{1+|f(x)-h(x)|+|h(x)-g(x)|} dx \\
&\leq 1 + 1 - \int_{[0,1]} \left(\frac{1}{1+|f(x)-h(x)|} + \frac{1}{1+|h(x)-g(x)|} \right) dx \\
&= \left(1 - \int_{[0,1]} \frac{1}{1+|f(x)-h(x)|} dx \right) + \left(1 - \int_{[0,1]} \frac{1}{1+|h(x)-g(x)|} dx \right) \\
&= d(f,h) + d(h,g),
\end{aligned}$$

and the triangle inequality is proven. So, d is a metric. In fact, we have that d is translation invariant, since for all $f, g, h \in C$ we know that $d(f+h, g+h) = \int_{[0,1]} \frac{|f+h-(g+h)|}{1+|f+h-(g+h)|} = \int_{[0,1]} \frac{|f-g|}{1+|f-g|} = d(f,g)$.

Next, we know that C is closed under the pointwise addition and pointwise scalar multiplication operations since continuity is preserved under such operations. So, C is a vector space. Since our topology is induced by a metric, we know that it is automatically T_1. We wish to show that addition and scalar multiplication are continuous. So, let $f, g \in C$ be arbitrary and let $\epsilon > 0$. Then, setting $\delta := \frac{\epsilon}{2}$, we have that for any $\tilde{f} \in B_\delta(f)$ and $\tilde{g} \in B_\delta(g)$ (where $B_r(\cdot)$ denotes an open ball of radius r),

$$\begin{aligned}
d(\tilde{f}+\tilde{g}, f+g) &\leq d(\tilde{f}+\tilde{g}, \tilde{f}+g) + d(\tilde{f}+g, f+g) \\
&= d(\tilde{g}, g) + d(\tilde{f}, f) \\
&< \delta + \delta = \epsilon,
\end{aligned}$$

where the first line is the triangle inequality, the second line uses the translation invariance of d, and the third line comes from our selection of δ. So, since we can find such balls $B_\delta(f)$ and $B_\delta(g)$ for any ϵ, the addition map is continuous at $(f, g) \mapsto f + g$. Since this holds for all $f, g \in C$, vector addition is continuous. To see that scalar addition is continuous, let $f \in C$ and $\alpha \in \mathbb{C}$ be arbitrary, and let $\epsilon > 0$. Let $s := \sup_{x \in [0,1]}\{f(x)\}$ (and so s is finite since f is continuous on a compact domain). Define

$$\delta := \min\left\{1, \frac{\epsilon}{1+|\alpha|+s}\right\}$$

Then, for any $\tilde{f} \in B_\delta(f)$ and any $\beta \in \mathbb{C}$ with $|\beta - \alpha| < \delta$, we have

$$\begin{aligned}
d(\beta\tilde{f}, \alpha f) = 1 - \int_{[0,1]} \frac{1}{1 + |\beta\tilde{f}(x) - \alpha f(x)|}dx &= 1 - \int_{[0,1]} \frac{1}{1 + |\beta\tilde{f}(x) - \beta f(x) + \beta f(x) - \alpha f(x)|}dx \\
&\leq 1 - \int_{[0,1]} \frac{1}{1 + |\beta\tilde{f}(x) - \beta f(x)| + |\beta f(x) - \alpha f(x)|}dx \\
&\leq 1 - \int_{[0,1]} \frac{1}{1 + |\beta||\tilde{f}(x) - f(x)| + |\beta - \alpha||f(x)|}dx
\end{aligned}$$

By the reverse triangle inequality, $||\beta| - |\alpha|| \leq |\beta - \alpha| < \delta \implies |\beta| < \delta + |\alpha|$, and so

$$\begin{aligned}
d(\beta\tilde{f}, \alpha f) &< 1 - \int_{[0,1]} \frac{1}{1 + (\delta + |\alpha|)|\tilde{f}(x) - f(x)| + \delta|f(x)|}dx \\
&\leq 1 - \int_{[0,1]} \frac{1}{1 + (\delta + |\alpha|)|\tilde{f}(x) - f(x)| + \delta s}dx
\end{aligned}$$

There are two cases: either $\delta + |\alpha| \leq 1$ or $\delta + |\alpha| > 1$. If $\delta + |\alpha| \leq 1$, then from the above bound we may get

$$d(\beta\tilde{f}, \alpha f) \leq 1 - \int_{[0,1]} \frac{1}{1 + |\tilde{f}(x) - f(x)| + \delta s}dx = \int_{[0,1]} \frac{\delta s + |\tilde{f}(x) - f(x)|}{1 + |\tilde{f}(x) - f(x)| + \delta s}dx$$

Now, we may apply the second result from Lemma 1 with $a = |\tilde{f}(x) - f(x)|$ and $b = \delta s$ to see that

$$d(\beta\tilde{f}, \alpha f) \leq \int_{[0,1]} \left(\frac{|\tilde{f}(x) - f(x)|}{1 + |\tilde{f}(x) - f(x)|} + \delta s \right) dx = d(\tilde{f}, f) + \delta s \leq \delta + \delta s = \delta(1 + s)$$

In the case when $\delta + |\alpha| > 1$, we instead continue with

$$\begin{aligned}
d(\beta\tilde{f}, \alpha f) &\leq 1 - \int_{[0,1]} \frac{1}{1 + (\delta + |\alpha|)|\tilde{f}(x) - f(x)| + \delta s}dx \\
&\leq 2 - \left(\int_{[0,1]} \frac{1}{1 + \delta s}dx + \int_{[0,1]} \frac{1}{1 + (\delta + |\alpha|)|\tilde{f}(x) - f(x)|}dx \right) \\
&= 1 - \frac{1}{1 + \delta s} + (\delta + |\alpha|) \int_{[0,1]} \frac{|\tilde{f}(x) - f(x)|}{1 + (\delta + |\alpha|)|\tilde{f}(x) - f(x)|}dx \\
&\leq 1 - \frac{1}{1 + \delta s} + (\delta + |\alpha|) \int_{[0,1]} \frac{|\tilde{f}(x) - f(x)|}{1 + |\tilde{f}(x) - f(x)|}dx \\
&= \frac{\delta s}{1 + \delta s} + (\delta + |\alpha|)d(\tilde{f}, f) < \delta s + (\delta + |\alpha|)\delta \\
&= \delta^2 + \delta(|\alpha| + s),
\end{aligned}$$

where the second line applies Lemma 1. Since $\delta \leq 1$, we see that in this second case

$$d(\beta\tilde{f}, \alpha f) \leq \delta + \delta(|\alpha| + s) = \delta(1 + |\alpha| + s)$$

Since $|\alpha| \geq 0$, this means that in both cases we always have that

$$d(\beta\tilde{f}, \alpha f) \leq \delta(1 + |\alpha| + s)$$

So, since $\delta \leq \frac{\epsilon}{1+|\alpha|+s}$, we have completed the proof that scalar multiplication is continuous. So, since the topology induced by d is T_1 and yields that the vector addition and scalar multiplication are continuous, we find that this is a TVS.

To finish up, note that the collection $\{B_r(g)\}_{(r,g)\in\mathbb{R}_+\times C}$ forms a basis for the topology induced by d, where

$$B_r(g) := \{f \in C : \quad d(f,g) < r\}$$

Furthermore, we have that $B_r(0_C) \subseteq B_q(0_C)$ for any $r < q$ by definition. We claim that the collection

$$\mathcal{B} := \{B_{\frac{1}{n}}(0_C) : \quad n \in \mathbb{N}\}$$

forms a countable local basis at 0_C for the topology induced by d. To this end, let $U \in \text{Nbhd}(0_C)$ be any open neighborhood containing the origin. Then, $U = \bigcup_{(r,g)\in I} B_r(g)$ for some $I \subseteq \mathbb{R}_+ \times C$. So, $0_C \in B_r(g)$ for some $r > 0$ and $g \in C$. For any $f \in B_\delta(0_C)$, we have that

$$d(f,g) = 1 - \int_{[0,1]} \frac{1}{1+|f-g|}$$

FINISH ∎

Problem 12

Let V be a neighborhood of zero in a TVS X. Prove that $\exists f : X \to \mathbb{R}$ continuous such that $f(0) = 0$ and $f(x) = 1$ for all $x \in X \setminus V$.

Solution

Proof. We proceed as hinted in Rudin exercise 1.21. Suppose without loss of generality that V is balanced, since we may find a balanced $W \in \text{Nbhd}(0_X)$ with $W \subseteq V$ by Theorem 1.14 in Rudin (if f is 1 outside W, then it is 1 outside V). By Rudin Theorem 1.14 and Lemma 2.6, we may select a balanced $V_1 \in \text{Nbhd}(0_X)$ such that $V_1 + V_1 \subseteq V$. Similarly, for any $n \in \mathbb{N}$, if $V_n \in \text{Nbhd}(0_X)$ we may select a $V_{n+1} \in \text{Nbhd}(0_X)$ balanced such that $V_{n+1} + V_{n+1} \subseteq V_n$. So, for each $n \in \mathbb{N}$, we have that V_n is open, balanced, and satisfies

$$V_n + V_n \subseteq V_{n-1} \quad \forall n > 1$$

Now, let B be the set of all rational numbers in $[0, 1]$ with a finite binary representation. By this, we mean let

$$B := \left\{ q \in (0,1) \cap \mathbb{Q} : q = \sum_{n \in N_q} 2^{-n} \text{ for some finite } N_q \subseteq \mathbb{N} \right\} \cup \{1\}$$

For each $q \in B \setminus \{1\}$ define the set

$$A(q) := \sum_{n \in N_q} V_n,$$

where the above sum is a finite sum of sets. Firstly, each $A(q)$ is balanced by Problem 7, since it is a finite sum of balanced sets. We claim that $A(q) \subseteq V$ for all $q \in B \setminus \{1\}$. To see this, $q \in B \setminus \{1\}$ and let $m_1, ..., m_k$ be the set N_q written in increasing order (which can be done by finiteness of N_q). We note that $V_{m_j} \subseteq V_i$ for all $i < m_j$ by construction. So, $V_{m_j} + V_{m_j} \subseteq V_{m_{j-1}}$ by monotonicity of the sequence $\{m_k\}_k$ and selection of the V_n's. Using these two facts, we see

$$\begin{aligned} A(q) = V_{m_k} + \ldots + V_{m_1} &\subseteq V_{m_k} + V_{m_k} + V_{m_{k-1}} + \ldots + V_{m_1} \\ &\subseteq V_{m_{k-1}} + V_{m_{k-1}} + V_{m_{k-2}} + \ldots + V_{m_1} \\ &\subseteq V_{m_{k-2}} + V_{m_{k-2}} + \ldots + V_{m_1} \\ &\subseteq \ldots \subseteq V_{m_1} + V_{m_1} \subseteq V_1 + V_1 \subseteq V \end{aligned}$$

In the above, we used the fact that that $V_{m_k} \subseteq V_{m_k} + V_{m_k}$ in the first line (it contains the origin), the fact that $V_{m_k} + V_{m_k} \subseteq V_{m_{k-1}}$ to go to the second line, the fact that $V_{m_{k-1}} + V_{m_{k-1}} \subseteq V_{m_{k-2}}$ to go to the third line, and the fact that $V_{m_1} \subseteq V_1$ in the last line.

Next, define $A(1) = X$. Thus, for every $x \in V$ there is some $q \in B$ for which $x \in A(q)$. Now, define a function $f : X \to [0, 1]$ via

$$f(x) := \inf_{q \in B} \{q : x \in A(q)\}$$

We note that $f(0_X) = 0$, since $0_X \in V_n \ \ \forall n \in \mathbb{N} \implies 0_X \in A(q) \ \ \forall q \in B$, and so $f(0_X) = \inf_{q \in B}\{q\} = 0$ (to see this last statement, note that B contains the sequence $\{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^k}, ...\}$ whose infimum is 0). Furthermore, for every $x \notin V$ we know that $x \notin A(q)$ if $q \in B \setminus \{1\}$, and so $f(x) = 1$ necessarily. Thus, all that remains to be proven is that this map f is continuous. We start with the following lemmas.

Lemma 2. *The set B is dense in* $[0, 1]$.

Proof of Lemma 2. The statement that B is dense in $[0, 1]$ holds if and only if $B \cap (r - \delta, r + \delta) \neq \emptyset$ for every $r \in (0, 1)$ and any $\delta > 0$ (since these intervals $(r - \delta, r + \delta)$ generate the usual subspace topology

Problem 12 continued on next page. . .

on $[0,1]$). So, let $r \in (0,1)$ and $\delta > 0$ be arbitrary such that $(r-\delta, r+\delta) \subseteq [0,1]$. Write r in terms of its infinite binary expansion

$$r = \sum_{n \in N_r} 2^{-n}$$

for some not-necessarily-finite $N_r \subseteq \mathbb{N}$. Let $k > -\log_2(\delta)$ be an integer, and define

$$\tilde{r} := \sum_{\substack{n \in N_r \\ n \leq k}} 2^{-n}$$

to be the k-truncated binary representation of r. Then, certainly $\tilde{r} \in B$ since $\tilde{r} \in (0,r) \subseteq [0,1]$ and it has a finite binary representation. Also,

$$|r - \tilde{r}| = \sum_{\substack{n \in N_r \\ n > k}} 2^{-n} \leq \sum_{n > k} 2^{-n} = 2^{-k} < 2^{\log_2(\delta)} = \delta,$$

where the above holds since we are summing nonnegative terms that form a geometric series. So $\tilde{r} \in (r-\delta, r+\delta)$. Thus, B is dense in $[0,1]$. ■

Lemma 3. *Each $A(q)$ is open for $q \in B$, and for any $p, q \in B$ we have*

$$A(q) + A(p) \subseteq A(q+p)$$

Thus, for any $p, q \in B$ with $p < q$ we have

$$A(p) \subseteq A(q)$$

Proof of Lemma 3. First, we prove openness. If $q = 1$, then $A(q) = X$ which is open. So, suppose $q \in B \setminus \{1\}$. Let N_q be the set of nonzero coefficient indices in the binary expansion of q. Since $q \in B$, we know N_q is finite; thus, $A(q)$ is a finite sum of open sets and is therefore open (to see this final statement, we observe that if A, B open, then $A + B = \bigcup_{a \in A}(a + B)$, and so $A + B$ is a union of open sets). So, $A(q)$ is an open set.

Now, let $p, q \in B$. If $p + q \geq 1$ then the result holds trivially. So, suppose that $p + q < 1$, and so $p, q, p+q \in B \setminus \{1\}$ (we know that $p + q \in B$ since the sum of two finite binary expansions is itself a finite binary expansion). Let N_p, N_q,$N_{p+q} \subseteq \mathbb{N}$ be the sets of indices of the nonzero terms in the binary expansions of p, q, and $p+q$, respectively; we know that they will all disagree with each other somewhere. We restate the proposition about binary addition that was used in the proof of Theorem 1.24 in Rudin: namely, that

$$m := \min\{(N_p \setminus N_q) \cup (N_q \setminus N_p)\} \in N_{p+q} \cap N_p^C \cap N_q^C$$

In words, this states that at the first coefficient where p and q disagree, $p+q$ will have a value of 1 while both p and q will have a value of 0; this can be seen as an immediate consequence of carrying over while performing the addition $p+q$ in the binary expansion. Regardless, we see that for this m, we have

$$A(p) = \sum_{n \in N_p \setminus \{m\}} V_n$$

and

$$A(q) = \sum_{n \in N_q \setminus \{m\}} V_n$$

So,

$$A(p) + A(q) \subseteq \sum_{N_{p+q}} V_n = A(p+q),$$

proving the second part. To see the last part, we note that since $0_X \in A(t)$ for every $t \in B$, for any $p < q$ with $p, q \in B$ we have

$$A(p) \subseteq A(p) + A(q-p) \subseteq A(q),$$

where we know that $q - p$ is in B since the difference of finite binary representations is again a finite binary representation. ■

Lemma 4. *For all $x, y \in X$, we have that*

$$|f(x) - f(y)| \leq f(x-y)$$

Proof of Lemma 4. Let $x, y \in X$. We claim that $f(x+y) \leq f(x) + f(y)$. To this end, let $\epsilon > 0$. By density of B in $[0,1]$, we may find a $p, q \in B$ such that $f(x) < p < f(x) + \frac{\epsilon}{2}$ and $f(y) < q < f(y) + \frac{\epsilon}{2}$. Since $f(x) < p$ we know by definition of an infimum that $x \in A(r)$ for some $r < p$, and so $x \in A(p)$ by the last statement of Lemma 3. Similarly, $y \in A(q)$. Thus, $x + y \in A(p) + A(q) \subseteq A(p+q)$ by Lemma 3, and so it must be that $f(x+y) \leq p + q$. Therefore,

$$f(x+y) < f(x) + f(y) + \epsilon,$$

and taking $\epsilon \to 0$ yields that $f(x+y) \leq f(x) + f(y)$.

Therefore, for any $x, y \in X$ we know that

$$f(y) = f(x + (y-x)) \leq f(x) + f(y-x) \implies f(y) - f(x) \leq f(y-x)$$

Similarly, $f(x) - f(y) \leq f(x-y)$. Lastly, for each $r \in B$ since $A(r)$ is balanced and therefore symmetric, we know that $y - x \in A(q) \iff x - y \in A(q)$, and so $f(x-y) = f(y-x)$. The lemma follows immediately. ■

Now, let $x \in X$ be arbitrary; we wish to show f is continuous at x. Let $\epsilon > 0$ be arbitrary. We seek an open set $U \in \mathrm{Nbhd}(x)$ for which $y \in U \implies |f(y) - f(x)| < \epsilon$. Let $q \in B \cap (0, \epsilon)$, which is nonempty by Lemma 2. Then, $A(q) \in \mathrm{Nbhd}(0_X)$ by Lemma 3, and so $U := x + A(q) \in \mathrm{Nbhd}(x)$. Furthermore, for any $y \in U$ we have that $y - x \in A(q)$, and so by Lemma 4 we know

$$|f(y) - f(x)| \leq f(y-x) \leq q < \epsilon,$$

where the second inequality above comes from the fact that $y - x \in A(q)$, and the last comes from our selection of $q \in B \cap (0, \epsilon)$. So, f is continuous on X. ■

Problem 13

Let X be the VS of all continuous functions $f : (0,1) \to \mathbb{C}$. For any $f \in X$ and $r > 0$, set

$$V(f,r) := \{g \in X : \quad |g(x) - f(x)| < r \ \forall x \in (0,1)\}$$

and set $Open(X)$ as the topology generated by $\{V(f,r)\}_{f \in X, r>0}$ (is this collection a basis or a sub-basis for a topology?). Show that w.r.t. $Open(X)$, vector addition is continuous but scalar multiplication is not.

Solution

Proof. To see that vector addition is continuous, let $g, h \in X$ be arbitrary and let $U \in \text{Nbhd}(g+h)$ be an arbitrary neighborhood of their sum. Let r be small enough that $V(g+h, r) \subseteq U$. We seek neighborhoods $V_g \in \text{Nbhd}(g)$ and $V_h \in \text{Nbhd}(h)$ for which $V_g + V_h \subseteq V(g+h, r)$. So, let $V_g := V(g, \frac{r}{2})$ and $V_h := V(h, \frac{r}{2})$. Now, for any $\tilde{g} \in V_g$ and $\tilde{h} \in V_h$, we have that for all $x \in (0,1)$,

$$|\tilde{g}(x) + \tilde{h}(x) - (g(x) + h(x))| \leq |\tilde{g}(x) - g(x)| + |\tilde{h}(x) - h(x)| < \frac{r}{2} + \frac{r}{2} = r,$$

and so $\tilde{g} + \tilde{h} \in V(g+h, r) \subseteq U$. Since we were able to find such neighborhoods V_g and V_h for all $U \in \text{Nbhd}(g+h)$, we see that vector addition is continuous.

i have no idea how to show scalar multiplication isnt continuous ■