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We will proceed with

① Rudin functional analysis (topo. vector spaces, convexity, open mappings)

② Banach spaces/alg. → spectral theories

③ Hilbert/operators

④ Fredholm theory, C-* alg., Schatten classes

* All vector spaces will be over $\mathbb{C}$

Consider two different topologies on $\mathbb{C}$:

$\tau_1 = \tau_{usual}$ and τ_2 the topo induced by the metric $d(z,w) = \begin{cases} |z-w| & z = \alpha w \text{ for } \alpha \in \mathbb{R} \\ |z| + |w| & \text{else} \end{cases}$

Clearly, $(\mathbb{C}, \tau_1)$ and $(\mathbb{C}, \tau_2)$ are <u>not homeomorphic</u>, even though the vector spaces are the same. We will now define a notion to unite the two notions.

<u>Defn:</u> (Topological Vector Space)

X is a topological vector space if it is a ($\mathbb{C}$-)vector space and also a topological space with open sets $\mathrm{Open}(X)$ in a compatible way:

- $+ : X \times X \to X$ is continuous w.r.t. $\mathrm{Open}(X)$
- $\cdot : \mathbb{C} \times X \to X$ is continuous w.r.t. product topo. $\mathbb{C} \times X$

Furthermore, we assume X is $T_1 \iff$ singletons are closed.

We claim that $(\mathbb{C}, \tau_1)$ is a TVS but $(\mathbb{C}, \tau_2)$ is NOT.

To check whether $+$ is continuous, we may note that the basic open sets of a metrizable topo is $B_\varepsilon(z)$. So, we may see $+^{-1}(B_\varepsilon(z)) = \bigcup_{w \in X} B_\varepsilon(w) \times \{v \in X : \|w+v-z\| < \varepsilon\}$

Example: ℓ^p spaces for $p \in (0,\infty)$

$$\ell^p(\mathbb{N}) = \ell^p(\mathbb{N}\to\mathbb{C}) = \left\{ a:\mathbb{N}\to\mathbb{C} \,\middle|\, \sum_{n\in\mathbb{N}} |a(n)|^p < \infty \right\}$$

It is a $\mathbb{C}$-vector space. If $p \geq 1$, $\exists$ norm $\Rightarrow$ metric $\Rightarrow$ topo.
If $p<1$, $\exists$ metric $\Rightarrow$ topo.

We claim $\ell^p(\mathbb{N})$ is a TVS of infinite dimension. Furthermore, $\ell^p(\mathbb{N}) \not\cong \ell^q(\mathbb{N})$ if $p \neq q$ as TVS, even though they are the same vector space. The theory of TVS will allow us to discern between these.

TVS

Examples:

not normable! { - For $U \subseteq \mathbb{R}^n$ open, $C(U\to\mathbb{C})$ (continuous) is a $\mathbb{C}$-vector space. We may give it a topology.
- For $U \subseteq \mathbb{R}^n$ open, $H(U\to\mathbb{C})$ (holomorphic) . . .

Defn: (Bounded & Balanced sets)

Let X be a TVS. We say $S \subseteq X$ is **bounded** if for any neighborhood N of a point $o \in S$, $S \subseteq tN$ for large enough t.

S is **balanced** (star shaped) if $\alpha S \subseteq S$ $\forall \alpha \in \mathbb{C}$ with $|\alpha| \leq 1$.

S is **absorbing** if $\forall x \in X$, $\exists t>0$ s.t. $x \in tA$

⚠ warning: TVS boundedness does not always agree with metric boundedness (though it does if the metric is induced by a norm.

Remark:

Recall a local basis at $p \in X$ is a collection $\mathcal{B} \subseteq \mathrm{Nbhd}(X)$ st. $\forall N \in \mathrm{Nbhd}(p)$, $\exists B \in \mathcal{B}$ st. $B \subseteq N$.

Furthermore, by hypothesis we have two homeomorphisms

$T_\varphi : X \to X$, $\varphi \mapsto \varphi + \psi$ with inverse $T_{-\varphi}$ and $M_\lambda : X \to X$, $\varphi \mapsto \lambda\varphi$ with inverse $M_{\frac{1}{\lambda}}$, $\lambda \neq 0$

So, a local basis at $p \in X$ is sent to a local basis at q by T_{p-q}, and so it is sufficient to specify a local basis to define a topology on X.

local basis $\to$ basis $\to$ topo.

Special types of TVS

① X is **locally convex** if $\exists$ local basis of 0 (origin) consisting of convex sets.

② X is **locally bounded** if . . . bounded sets

③ X is **locally compact** if $\exists N \in \mathrm{Nbhd}(0)$ s.t. $\overline{N}$ is compact

④ X is an **F-space** if τ is metrizable from a complete, translation-invariant metric.

⑤ X has the **Heine-Borel** property if closed + bounded $\Rightarrow$ compact

Lemma:

$\forall W \in \mathrm{Nbhd}(0)$, $\exists U \in \mathrm{Nbhd}(0)$ s.t. $U = -U$ and $U + U \subseteq W$.

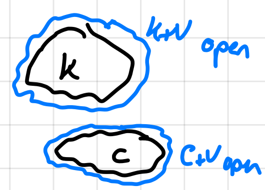
Proof: $+ : X \times X \to X$ is continuous at 0, and so $\exists V_1, V_2 \in \mathrm{Nbhd}(0)$ s.t. $V_1 + V_2 \subseteq W$.
Let $U := V_1 \cap V_2 \cap (-V_1) \cap (-V_2) \Rightarrow 0 \in U$, U is open, and $U = -U$

$\square$

Lemma (separation)

If X is a TVS with $C \subseteq X$ closed and $K \subseteq X$ compact with $C \cap K = \emptyset$, then $\exists V \in \mathrm{Nbhd}(0)$ s.t.

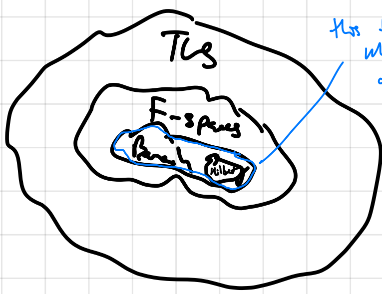
$$(C+V) \cap (K+V) = \emptyset$$



Furthermore, since $C+V$ is open, $K+V \subseteq (C+V)^c \Rightarrow \overline{K+V} \subseteq (C+V)^c \Rightarrow (C+V) \cap \overline{(K+V)} = \emptyset$.

Moreover, if we take $K = \{0\}$ and $C = U^c$ with $U \in \mathrm{Nbhd}(0)$, then
$\exists V \in \mathrm{Nbhd}(0)$ s.t. $\overline{V} \cap U^c = \emptyset \Rightarrow \overline{V} \subseteq U$. $\leftarrow$ for any neighborhood, we may find another neighborhood whose closure sits inside

9/7 - (read Rudin Ch. 1)



this tends to be where we are interested, but results are often more general!

Proof of separation lemma: Suppose WOLOG that $K \neq \emptyset$. Let $x \in K \Rightarrow x \notin C$. So, $C^c \in Nbhd(x) \Rightarrow C^c + \{-x\} \in Nbhd(0)$. Applying our symmetrization lemma, $\exists V_x \in Nbhd(0)$ s.t. $V_x = -V_x$ and $V_x + V_x + V_x \subseteq C^c + \{-x\} \Leftrightarrow (\{x\} + V_x + V_x + V_x) \cap C = \emptyset$

Since $V_x = -V_x$, $(\{x\} + V_x + V_x) \cap (C + V_x) = \emptyset$

Certainly, $\bigcup_{x \in K} (\{x\} + V_x)$ is an open cover, and so

$$K \subseteq \bigcup_{j=1}^{n} (\{x_j\} + V_{x_j}).$$ Define $V := \bigcap_{j=1}^{n} V_{x_j}$. Then,

$$K + V \subseteq \bigcup_{j=1}^{n} (\{x_j\} + V_{x_j} + V) \subseteq \bigcup_{j=1}^{n} (\{x_j\} + U_{x_j} + V_{x_j})$$

By construction, $(\{x_j\} + U_{x_j} + V_{x_j}) \cap (C + V_{x_j}) = \emptyset \quad \forall j$

$\Rightarrow (\{x_j\} + U_{x_j} + V_{x_j}) \cap (C + V) = \emptyset.$

So, $(K + V) \cap (C + V) = \emptyset.$

□

Lemma:

Let $A \subseteq X$. Then, $\bar{A} \subseteq \bigcap_{U \in Nbhd(0)} (A + U)$.

Proof: We know $x \in \bar{A} \Leftrightarrow U \cap A \neq \emptyset \ \forall U \in Nbhd(x) \Leftrightarrow (\{x\} + U) \cap A \neq \emptyset \ \forall U \in Nbhd(0)$
$\Leftrightarrow x \in A + (-U) \ \forall U \in Nbhd(0) \Leftrightarrow x \in A + U \ \forall U \in Nbhd(0).$

$U \in Nbhd(0) \Leftrightarrow -U \in Nbhd(0)$

□

Lemma:

If $E \subseteq X$ is bounded, then $\bar{E}$ is too.

Proof: Let $V \in Nbhd(0)$. Then, $\exists W \in Nbhd(0)$ s.t. $\bar{W} \subseteq V$. Since E is bounded, $E \subseteq tW$ for large $t \Rightarrow \bar{E} \subseteq t\bar{W} \subseteq tV$. □

shorthand for $Nbhd(0)$

Lemma:
1. $\forall U \in N(0)$, $\exists V \in N(0)$ balanced s.t. $V \subseteq U$.
2. $\forall U \in N(0)$ convex, $\exists V \in N(0)$ balanced & convex s.t. $V \subseteq U$.

Proof: 1. Let $U \in N(0)$. Scalar mult. is continuous and so $\exists \delta > 0$ s.t. $\forall \alpha \in (0, \alpha)$ & $W \in Nbhd(0)$, $\alpha W \subseteq U$.
Let $V_\delta := \bigcup_{\alpha \in (0,\delta)} \alpha W$. Then, V_δ is open. Furthermore V_δ is balanced as multiplying by $|\beta| \leq 1$ rearranges terms.

2. See Rudin.

□

Theorem:

For any $U \in \mathcal{N}(0)$,

① $X \subseteq \bigcup_{n=1}^{\infty} r_n U \quad \forall \{r_n\}_n \subseteq (0,\infty)$ with $r_n \to \infty$.

② Any compact K is bounded.

③ If U bounded, then $\forall \{\delta_n\}_n \subseteq (0,\infty)$ with $\delta_n > \delta_{n+1}$ and $\delta_n \to 0$ the family $\{\delta_n U\}_n \subseteq \mathcal{P}(X)$ is a local basis for X. ← note that we don't need to say "at 0" by TVS

Proof: ① For a fixed $x \in X$, the map $F_x : \mathbb{C} \to X$ sending $\alpha \mapsto \alpha x$ is continuous. So, $F_x^{-1}(U) \in \text{Open}(\mathbb{C})$ and contains 0. So, it contains $\frac{1}{r_n}$ for large n
$\Leftrightarrow \frac{1}{r_n} x \in U \Leftrightarrow x \in r_n U \Rightarrow X \subseteq \bigcup_n r_n U$.

② Let $K \subseteq X$ be compact, $U \in \mathcal{N}(0)$. By the above lemma, $\exists W \subseteq U$ s.t. $W \in \mathcal{N}(0)$ is balanced. By ①, $K \subseteq \bigcup_{n=1}^{\infty} nW \overset{\text{compact}}{\Longrightarrow} K \subseteq \bigcup_{j=1}^{m} n_j W$
Since W is balanced, $kW \subseteq lW$ for $|k| \leq |l| \Rightarrow K \subseteq n_m W \Rightarrow K \subseteq n_m U$.
So, K is bounded.

③ Do this!

$\square$

We now look at linear maps on TVS's.

Prop:
Let X, Y be TVS. Then, $\Lambda : X \to Y$ linear and continuous at 0
$\Rightarrow \Lambda$ is continuous everywhere.

Proof: Let $U \subseteq Y$ be open; we WTS $\Lambda^{-1}(Y)$ is open in X. Suppose WOLOG $U, \Lambda^{-1}U \neq \emptyset$. Then, $\exists x \in X$ s.t. $0_Y \in U + \{\Lambda x\}$. Continuity of Λ at 0_X and linearity (i.e. $\Lambda 0_X = 0_Y$) $\Rightarrow \Lambda^{-1}(U + \{\Lambda x\}) \in \mathcal{N}(0_X)$. Since $\Lambda^{-1}(U + \{\Lambda x\}) = \Lambda^{-1}U + \{x\}$ by linearity,
$\Lambda^{-1}U + \{x\} \in \text{Open}(X) \Longleftrightarrow \Lambda^{-1}U \in \text{Open}(X)$.

$\square$

Theorem:

If $\Lambda : X \to \mathbb{C}$ is linear and $\text{Ker}(\Lambda) \neq X$, then TFAE:

① Λ is continuous
② $\text{Ker}(\Lambda)$ is closed in X
③ $\text{Ker}(\Lambda)$ is NOT dense in X
④ $\exists U \in \mathcal{N}(0)$ s.t. $\Lambda|_U$ is a bounded map.

Proof: ($1 \Rightarrow 2$) Λ cont. $\Rightarrow \Lambda^{-1}(C)$ is closed in X for all closed $C \subseteq \mathbb{C}$.
Since $\{0\} \in \text{Closed}(\mathbb{C})$ and $\text{Ker}(\Lambda) = \Lambda^{-1}(\{0\})$, it is closed.

($2 \Rightarrow 3$) Since $\ker(\Lambda)$ is closed, $\overline{\ker(\Lambda)} = \ker(\Lambda) \neq X$ by assumption.

($3 \Rightarrow 4$) Suppose $\ker(\Lambda)$ isn't dense $\Leftrightarrow \text{int}(\ker(\Lambda)^c) \neq \emptyset$.
Let $x \in \text{int}(\ker(\Lambda)^c) \Rightarrow \exists U \in N(0)$ s.t. $\{x\} + U \subseteq \ker(\Lambda)^c$.
Suppose WOLOG that U is balanced. By linearity, ΛU is balanced as well. Suppose BWOC that $\Lambda U \subseteq \mathbb{C}$ is unbounded, yet balanced.
Then, $\Lambda U = \mathbb{C} \Rightarrow \exists y \in U$ s.t. $\Lambda y = -\Lambda x \Rightarrow x + y \in \ker(\Lambda) \cap (\{x\} + U)$. $\times$

($4 \Rightarrow 1$) Suppose $\exists U \in N(0)$ s.t. $\Lambda|_U$ is bounded. Then, $\exists M \in (0, \infty)$ s.t. $|\Lambda x| \leq M$ $\forall x \in U$. Let $\varepsilon > 0$ and define $W_\varepsilon := \frac{\varepsilon}{M} U$.
Then, $\forall x \in W_\varepsilon$ we know $|\Lambda x - \Lambda 0| = |\Lambda x| \leq \varepsilon$. So, Λ is continuous at $0 \Rightarrow \Lambda$ continuous.

$\square$

We now look at finite-dim TVS's (which it will turn out is always $\cong \mathbb{C}^n$).

Theorem:
Any linear $f: \mathbb{C}^n \to X$ is continuous.

Proof: Let $\{e_j\}_j$ be the standard basis for $\mathbb{C}^n$. Then, $f(z) = \sum_{j=1}^{n} z_j f(e_j)$ by linearity. Since each element of the sum is continuous, so is f.

$\square$

Theorem:
Let X be a TVS, and let $Y \subseteq X$ be a finite-dim subspace.
Let $\dim(Y) = n$. Then,

1. Y is closed in X.

2. Any vector space isomorphism $f: \mathbb{C}^n \to Y$ is a TVS isomorphism.

Proof:

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2. Let $f: \mathbb{C}^n \to Y$ be a vector space isomorphism $\Rightarrow f$ is bijective and linear.
Define $S := \{z \in \mathbb{C}^n \mid \sum_{j=1}^{n} |z_j|^2 = 1\} \cong \mathbb{S}^{2n-1}$. So, S is compact.

Since f is linear, it is continuous and so $f(S)$ is compact in Y.
Since $f 0_{\mathbb{C}^n} = 0_Y$, $0_Y \notin f(S)$. Thus, $\exists V \in N(0_Y)$ balanced s.t. $V \cap f(S) = \emptyset$.
Define $E := f^{-1}(V) = f^{-1}(V \cap Y) \subseteq \mathbb{C}^n$. Then, E is open and $E \cap S = \emptyset$.

We argue V balanced $\Rightarrow V$ path-connected, and so E is path-connected $\Rightarrow$ (because $\subseteq \mathbb{C}^n$) E connected.

So, E is a connected subset of $\mathbb{C}^n$ s.t. $E \cap S = \emptyset$, $0_{\mathbb{C}^n} \in E$.
Thus, $E \subseteq B_1(0_{\mathbb{C}^n})$. So, f^{-1} is a bounded map $Y \to \mathbb{C}^n$.
So, $(f^{-1})_i : Y \to \mathbb{C}$ is a bounded linear functional, and so its continuous by Lemma 2.14. By the defn of the product topology, f^{-1} is continuous.
So, f is a homeomorphism.

① We WTS $\overline{Y} \subseteq Y \iff Y^c \subseteq \overline{Y}^c$. Let $x \notin Y$ and consider $Z = \text{span}\{x, Y\}$. By ②, $Z \overset{\text{TVS isomorphism}}{\cong} \mathbb{C}^{n+1}$. So, x is not in the closure of Y in Z. By the defn of the subspace topology,

$$\text{Closure}_Z(Y) = \text{Closure}_X(Y) \cap Z \Rightarrow (\text{Closure}_Z(Y))^c = Z \setminus (\text{closure}_Z(Y) \cap Z)$$
$$= Z \setminus \text{Closure}_X(Y) \subseteq X \setminus \text{Closure}_X(Y)$$

So, $x \in Z \setminus \text{Closure}_Z(Y) \Rightarrow x \in X \setminus \text{Closure}_X(Y)$, and we are done.

$\square$

Theorem:

If X is a locally compact TVS, then $\dim X < \infty$.

Proof: Local compactness means $\exists V \in N(0_X)$ st. $\overline{V}$ is compact.
We can build a countable local basis at 0_X via $\{2^{-n}V\}_n$ via Thm. 2.12. Also, $\overline{V}$ compact $\Rightarrow$ $\overline{V}$ bounded $\Rightarrow$ V bounded.
We know that $\bigcup_{x \in X} \{x\} + \frac{1}{2}V$ is an open cover of $\overline{V}$, and so
$\exists \{x_1, \dots, x_m\}$ s.t. $\overline{V} \subseteq \bigcup_{j=1}^{m} \{x_j\} + \frac{1}{2}V$. Define $Y := \text{span}\{x_1, \dots, x_m\}$. Then, Y is closed in X by prev. theorem. Since Y is a vector subspace, $Y = \lambda Y$ $\forall \lambda \neq 0$.
Thus, $V \subseteq \overline{V} \subseteq Y + \frac{1}{2}V \Rightarrow V \subseteq Y + \frac{1}{2}(Y + \frac{1}{2}V) = Y + \frac{1}{4}V$

We may repeat this always to see that $V \subseteq Y + \frac{1}{2^n}V$ $\forall n \in \mathbb{N}$.
So,
$$V \subseteq \bigcap_{j \in \mathbb{N}} (Y + 2^{-j}V) \overset{\{2^{-j}V\} \text{ local basis}}{=} \overline{Y} \overset{Y \text{ closed}}{=} Y.$$ Thus, $kV \subseteq kY = Y$ $\forall k \in \mathbb{N}$
Since $X = \bigcup_{k \in \mathbb{N}} kV$, $X \subseteq Y \Rightarrow X = Y \Rightarrow X$ is finite-dim. $\square$

Theorem:

If X is a locally bounded TVS which obeys Heine-Borel property, then $\dim(X) < \infty$.

Proof: By local boundedness, $\exists V \in N(0_X)$ bounded. Thus, $\overline{V}$ is also bounded, which by Heine-Borel property means $\overline{V}$ compact. So, X is locally compact! Apply previous theorem.

$\square$

Remark: make everything finite-dim!!!

2. Banach Spaces

Defn:

Let V be a vector space. A norm on V is a map $\|\cdot\|: V \to [0,\infty)$ s.t.

① $\alpha\|x\| = \|\alpha x\|$ $\quad \forall \alpha \in \mathbb{C},\ x \in V$ (homogeneity)
② $\|x+y\| \leq \|x\| + \|y\|$ $\quad \forall x,y \in V$ (triangle ineq.)
③ $\|x\| = 0 \Rightarrow x = 0_V$ $\quad \forall x \in V$ (injectivity-ish)

Defn:

linear in 2nd slot, antilinear in 1st slot

A vector space V is an inner product space iff $\exists$ a sesquilinear map $\langle\cdot,\cdot\rangle: V^2 \to \mathbb{C}$ s.t.

① $\langle x,y\rangle = \overline{\langle y,x\rangle}$ ② Linear in 2nd argument ③ $\langle x,x\rangle > 0$ $\forall x \in V \setminus \{0\}$

It happens that inner products induce norms.
The converse is not always true.

Claim: If a normed vector space whose norm satisfies the $\square$-law
$$\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \quad \forall x,y \in V$$
then $\langle x,y\rangle := \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|ix-y\|^2 - i\|ix+y\|^2\right]$ is a valid inner product.

Furthermore, if $\square$-law is not obeyed, then there is no inner product which is compatible with that norm.

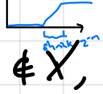
example: $\mathbb{C}^n$ with 1-norm $\|z\|_1 = \sum_i |z_i|$ doesn't satisfy $\square$-law, $n>1$.

Every norm induces a homogeneous metric $d: V^2 \to [0,\infty)$ via
$$d(x,y) = \|x-y\| \Rightarrow d(\alpha x, \alpha y) = |\alpha|\, d(x,y)$$

Def:

X is a Banach space iff its norm induces a complete metric.

Example $\mathbb{C}^n$ w/ 2-norm

Counterexample: $X := \{f: [0,1] \to \mathbb{C} \mid f \text{ continuous}\}$ with pointwise $+, \cdot$.
Define a norm $\|f\|_2 := \left(\int_{[0,1]} |f|^2\right)^{\frac{1}{2}}$. Then, $(X, \|\cdot\|_2)$ is a normed VS.
However, the sequence  ,  , ..., then it is Cauchy in L^2-norm.
Yet $f_n \to I_{[\frac{1}{2},1]} \notin X$, and so X isn't complete.

Defn:

In a Banach space X, a set $S \subseteq X$ is **dense** if $\forall x \in X$, $\forall \varepsilon > 0$, $\exists y \in S$: $d(x,y) < \varepsilon$ (equivalent to $\overline{S} = X$).

Defn:

A Banach space X is **separable** if $\exists$ a countable, dense subset.

Prop:

A Banach space X is a TVS.

Proof: Metric spaces are T_1, and so all we must show is that $+, \cdot$ is continuous.
Let $x, y \in X$ and $\varepsilon > 0$. We want $\delta_1, \delta_2 > 0$ s.t.

$$\tilde{x} \in B_{\delta_1}(x),\ \tilde{y} \in B_{\delta_2}(y) \Rightarrow (\tilde{x} + \tilde{y}) \in B_\varepsilon(x+y)$$

Pick $\delta_1 = \delta_2 = \varepsilon/2$, and then

$$\|(x+y) - (\tilde{x}+\tilde{y})\| \leq \|x - \tilde{x}\| + \|y - \tilde{y}\| \leq \delta_1 + \delta_2 = \varepsilon/2.$$

Same for $\cdot$. $\square$

Boundedness

Recall that TVS, boundedness of $S \Leftrightarrow \forall U \in N(0_X)$, $S \subseteq tN$ for sufficiently large t.

Also, in a normed VS, boundedness of $S \Leftrightarrow \sup_{x \in S} \{\|x\|\} < \infty$.

It turns out that these are equivalent in normed spaces. ← though not for metrics inhomogeneous

Let X, Y be Banach spaces. If $A: X \to Y$ is linear and continuous, then A is bounded, and so $S \subseteq X$ bdd $\Rightarrow AS \subseteq Y$ bdd in the TVS sense.
So,

$$\sup_{x \in B_1(0)} \|Ax\| < \infty.$$

Defn:

For X, Y Banach spaces and $A: X \to Y$ linear, define

$$\|A\|_{B(X \to Y)} := \sup_{x \in B_1(0_X)} \{\|Ax\|\}$$

Let $B(X \to Y) := \{A: X \to Y \mid \|A\|_{B(X\to Y)}$ is linear and continuous$\}$

In Banach spaces, we have continuous $\Leftrightarrow$ bounded.

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Claim:

If $A: X \to Y$ is a linear map between Banach spaces, then
$$\|A\|_{op} < \infty \iff A \text{ continuous}$$

Proof: ($\Leftarrow$) Already seen in Chapter 1.

($\Rightarrow$) We have $\|Ax - A\tilde{x}\|_Y = \|A(x-\tilde{x})\|_Y \leq \|A\|_{op} \|x-\tilde{x}\|_X$
So, A is $\|A\|_{op}$-Lipschitz, and thus continuous. □

Also, if $A, B : X \to X$ are linear operators on a Banach space, then
$$\|A \circ B\|_{op} \leq \|A\|_{op} \|B\|_{op} \quad \text{(submultiplicative)}$$
This ends up turning $\mathcal{B}(X)$ into a Banach algebra, as we will see later.

Claim: (Reed & Simon III.2)

$(\mathcal{B}(X \to Y), \|\cdot\|_{op})$ is a Banach space.

Proof: We know $\mathcal{B}(X \to Y)$ is a vector space and $\|\cdot\|_{op}$ is a norm. So, we must show completeness. Let $\{A_n\}_n$ be Cauchy w.r.t. $\|\cdot\|_{op}$. For any $x \in X$, $\{A_n x\}_n$ is Cauchy in Y by the earlier claim. So, $A_n x \to y$ for some $y \in Y$ by completeness. Define B sending $x \to \lim_{n\to\infty} A_n x$.

B is linear by linearity of the limit. Furthermore,
$$\|A_n - A_m\|_{op} \geq \big| \|A_n\|_{op} - \|A_m\|_{op} \big| \quad \text{by reverse } \Delta\text{-ineq.}$$
So, $\{\|A_n\|_{op}\}_n$ is Cauchy in $\mathbb{R}$, a complete space. So, $\exists \alpha \in \mathbb{R}$ s.t. $\lim_{n\to\infty} \|A_n\|_{op} = \alpha$. So, $\forall x \in X$,
$$\|Bx\|_Y = \lim_{n\to\infty} \|A_n x\|_Y \leq \lim_{n\to\infty} \|A_n\|_{op} \|x\|_X = \alpha \|x\|_X$$
So, B is bounded with $\|B\|_{op} \leq \alpha$. Thus, $B \in \mathcal{B}(X \to Y)$.
All that remains to show is that $\|B - A_m\|_{op} \to 0$.
For every $x \in X$, and any $m \in \mathbb{N}$, $\|(B - A_m)x\|_Y = \lim_{n\to\infty} \|(A_n - A_m)x\|_Y$
So, if $\|x\| \leq 1$, $\|(B-A_m)x\|_Y \leq \lim_{n\to\infty} \|A_n - A_m\|_{op} \Rightarrow \|B - A_m\|_{op} \leq \lim_{n\to\infty} \|A_n - A_m\|_{op}$ □

Defn:

$A \in \mathcal{B}(X \to Y)$ is an isometry iff $\|Ax\|_Y = \|x\|_X \quad \forall x \in X$.
So, X and Y are isometrically isomorphic iff $\exists A \in \mathcal{B}(X \to Y)$ linear and isomorphic.
This is the isomorphism in the category of Banach spaces.

Claim:

Any closed subspace of a Banach space is itself a Banach space.

2.1 Completeness

Defn:

If X is a topological space we say $S \subseteq X$ is nowhere dense iff
$$\operatorname{int}(\overline{S}) = \emptyset$$

Defn: (Baire)

Sets of the 1st Category: $S \subseteq X$ is meagre iff it is the countable union of nowhere dense sets.
"meagre"

Sets of the 2nd Category: sets that are NOT meagre

Ex/
- $\mathbb{Z} \subseteq \mathbb{R}_{usual}$ is nowhere dense
- $\mathbb{Q} \subseteq \mathbb{R}_{usual}$ is NOT nowhere dense
- $[0,1] \subseteq \mathbb{R}_{usual}$ is NOT "
- $\mathbb{R} \subseteq \mathbb{C}_{usual}$ is nowhere dense
- in a discrete space, $\emptyset$ is the only nowhere dense set.
- If X a TVS and $V \subseteq X$ is a vector subspace, V is either dense or nowhere dense.
- $C \subseteq [0,1]$ Cantor set is nowhere dense.

Claim:

In a topo. space X:
- $A \subseteq B$ and B is meagre, then so is A
- If $\{A_n\}_n$ are all meagre, then $\bigcup_{n}$ is too
- If E closed with $\operatorname{int}(E) = \emptyset$, then E is meagre
- If $h : X \to X$ is a homeomorphism, then $h(B)$ meagre $\Leftrightarrow$ B meagre

Theorem: (Baire Category Theorem)

If X is either a complete metric space or a locally compact Hausdorff space, then:

If $\{A_n\}_n \subseteq \mathrm{Open}(X)$ are dense then $\bigcap_n A_n$ is dense.

In particular, X is not meagre.

Proof: We prove BCT first for complete metric spaces. Let $\{V_j\}_j$ be open and dense. Let $W \in \mathrm{Open}(X)$ be arbitrary: we WTS $W \cap (\bigcap_j V_j) \neq \emptyset$. Since V_1 dense, $W \cap V_1 \neq \emptyset$. Then, $\exists x_1 \in W \cap V_1$, $r_1 \in (0, \frac{1}{2})$ s.t.

$$\overline{B_{r_1}(x_1)} \subseteq W \cap V_1$$

Proceeding inductively, we may always find $x_j \in B_{r_{j-1}}(x_{j-1}) \cap V_j$ and $r_j \in (0, \frac{1}{2^j})$ s.t. $\overline{B_{r_j}(x_j)} \subseteq B_{r_{j-1}}(x_{j-1}) \cap V_j$.

So, $\forall j$ we have
$$x_j \in B_{r_{j-1}}(x_{j-1}) \cap V_j \subseteq B_{r_{j-2}}(x_{j-2}) \cap V_{j-1} \cap V_j$$
$$\subseteq \dots \subseteq W \cap \left(\bigcap_{i=1}^{j} V_i\right)$$

We claim $\{x_j\}_j$ is Cauchy, since if $n, m \geq N$ we have
$$x_n, x_m \in B_{r_N}(x_N) \Rightarrow d(x_n, x_m) < 2 r_N$$

Since X is complete, $\exists x \in X$ s.t. $x_n \to x$, and so $x \in W \cap (\bigcap_{j \in \mathbb{N}} V_j)$.
Thus, $W \cap (\bigcap_j V_j) \neq \emptyset \Rightarrow \bigcap_j V_j$ is dense.

For the "in particular" part, let $\{E_j\}_j \subseteq X$ be a countable collection of nowhere dense sets. Then, $\mathrm{int}(\overline{E_j}) = \emptyset$ $\forall j$. So,

E_j nowhere dense $\Leftrightarrow [\mathrm{int}(\overline{E_j})]^c = X \Leftrightarrow \overline{(\overline{E_j})^c} = X \Leftrightarrow (\overline{E_j})^c$ is dense and open

BCT gives $\bigcap_j (\overline{E_j})^c \neq \emptyset \Rightarrow \bigcup_j \overline{E_j} \neq X \Rightarrow \bigcup_j E_j \neq X$

Since this holds $\forall \{E_j\}_j$ nowhere dense, we know X is not meagre.
$\square$

Corollary:

Complete metric spaces are uncountable.

Proof: $X = \bigcup_{x \in X} \{x\}$, and each $\{x\}$ is nowhere dense.
By BCT, X cannot be countable.

Theorem: (Banach–Steinhaus / Uniform Boundedness Principle)

Let X, Y be Banach spaces. Let $\mathcal{F} \subseteq \mathcal{B}(X \to Y)$.
If for all $x \in X$ we have $\sup_{A \in \mathcal{F}} \{ \|Ax\|_Y \} < \infty$,

then $\sup_{A \in \mathcal{F}} \{ \|A\|_{op} \} < \infty$.

Proof: Define $X_n := \left\{ x \in X : \sup_{A \in \mathcal{F}} \{ \|Ax\|_Y \} \le n \right\}$

Then, $X_n \in \text{Closed}(X)$ and $X = \bigcup_{n \in \mathbb{N}} X_n$. Since X is not meagre by BCT, $\exists n \in \mathbb{N}$ s.t. X_n is not nowhere dense. So,

$$\emptyset \neq \text{int}(\overline{X_n}) = \text{int}(X_n) \Rightarrow \exists x_0 \in X_n \subseteq X, \quad \varepsilon > 0 \quad \text{s.t.}$$

$\overline{B_\varepsilon(x_0)} \subseteq X_n$. So, $\forall u \in X$ with $\|u\|_X \le 1$,

$$\|Au\| = \frac{1}{\varepsilon} \|A(x_0 + \varepsilon u - x_0)\| \overset{\Delta}{\le} \frac{1}{\varepsilon} \underbrace{\|A(x_0 + \varepsilon u)\|_Y}_{\le n} + \frac{1}{\varepsilon} \underbrace{\|Ax_0\|_Y}_{\le n}$$

$$\le \frac{2n}{\varepsilon}$$

Since this bound doesn't depend on A or u, uniform boundedness follows.

$\square$

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We show a nice application of uniform boundedness below.

Prop:
Let X, Y, Z be Banach spaces and $B : X \times Y \to Z$ bilinear and continuous in each argument separately.
Then, B is jointly continuous.

Proof: $\forall x \in X$, $B(x, \cdot) : Y \to Z$ is a bounded (by continuity) linear mapping, and so $\exists C_x < \infty$ s.t. $\|B(x, y)\|_Z \le C_x \quad \forall y \in Y$ with $\|y\|_Y = 1$.

Define $\mathcal{F} := \{ B(\cdot, y) : X \to Z \ : \ \|y\|_Y = 1 \}$

By uniform boundedness, $\exists M < \infty$ s.t. $\forall \|y\|_Y = 1$, $\|B(\cdot, y)\|_{\mathcal{B}(X \to Z)} \le M$
By homogeneity,
$$\|B(x,y)\|_Z = \|y\|_Y \left\| B\left(x, \frac{y}{\|y\|_Y}\right) \right\|_Z$$
$$\le \|y\|_Y \left\| B\left(\cdot, \frac{y}{\|y\|_Y}\right) \right\|_{\mathcal{B}(X \to Z)} \|x\|_X$$
$$\le M \|x\|_X \|y\|_Y$$
Continuity follows immediately.

$\square$

Open Mapping Theorem

Defn:

A map $f: X \to Y$ between topological spaces is open if
$$U \in \mathrm{Open}(X) \Rightarrow f(U) \in \mathrm{Open}(Y)$$

Prop:

If X, Y are TVS, then a linear map $f: X \to Y$ is open iff $\forall U \in \mathcal{N}(0_x)$, $f(U)$ contains a neighborhood of 0_y.

Proof: ($\Rightarrow$) Let $U \in \mathcal{N}(0_x) \Rightarrow f(U) \in \mathrm{Open}(Y)$. By linearity, $0_x \in U \Rightarrow 0_y \in f(U)$.

($\Leftarrow$) Let $U \in \mathrm{Open}(X)$. Let $y \in f(U)$, meaning $\exists x \in U$ s.t. $f(x) = y$.
Also, $0_x \in U - \{x\} \Rightarrow U - \{x\} \in \mathcal{N}(0_x)$
Let $L \in \mathcal{N}(0_y)$ be s.t. $L \subseteq f(U - \{x\})$ as promised by the hypothesis.
Then,
$$L \subseteq f(U - \{x\}) = f(U) - f(\{x\}) = f(U) - \{y\} \Rightarrow \{y\} + L \subseteq f(U).$$
So, $\exists$ an open neighborhood of y contained in $f(U)$ $\forall y \in f(U)$.

$\square$

Theorem: (Open Mapping)

Let $A \in \mathcal{B}(X \to Y)$ be a bounded, linear map between Banach spaces.
Then,
$$A \text{ is surjective} \iff A \text{ is an open map}$$

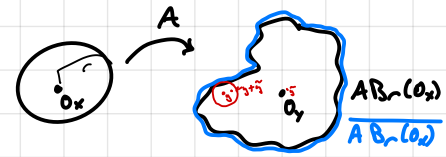
Proof: ($\Rightarrow$) We claim that for all $r > 0$.

(1) $\overline{AB_r(0_x)}$ has nonempty interior

(2) $\overline{AB_r(0_x)} \subseteq AB_{2r}(0_x)$

First, we show that (1)+(2) suffices to prove A is open.
Indeed, if (1) then $\exists y \in Y$ and $\varepsilon > 0$ s.t.
$$B_\varepsilon(y) \subseteq \overline{AB_r(0_x)}$$

So, there is some $x \in B_r(0_x)$ s.t. $Ax = y$
(if not, then every $B_\varepsilon(y) \cap \overline{AB_r(0_x)}^c \neq \emptyset$).



Furthermore, $\exists \tilde{y} \in B_\varepsilon(0_y)$: $y + \tilde{y} \in AB_r(0_x)$
$\Rightarrow \exists z \in B_r(0_x)$ s.t. $Az = y + \tilde{y} \Rightarrow A(z - x) = y + \tilde{y} - y = \tilde{y}$
and $\|z - x\|_X \le \|z\|_X + \|x\|_X \le 2r$. So, $\tilde{y} \in A\overline{B_{2r}(0_x)} \overset{A \text{ continuous}}{\subseteq} \overline{AB_{2r}(0_x)}$
$\Rightarrow B_\varepsilon(0_y) \subseteq \overline{AB_{2r}(0_x)}$

Applying (2), $B_\varepsilon(0_y) \subseteq AB_{4r}(0_x)$, and so $AB_{4r}(0_x)$ is open.

So, all we have to do is show (1) and (2).

(1) comes from Baire Category Theorem with $Y = \bigcup_{r\in\mathbb{N}} AB_r(0_X)$

(2): we wts $\overline{AB_r(0_X)} \subseteq AB_{2r}(0_X)$ $\forall r>0$.

Let $y \in \overline{AB_r(0_X)} \Rightarrow \forall \varepsilon>0$, $B_\varepsilon(y)\cap AB_r(0_X)\neq\emptyset$

So, $\forall\varepsilon>0$, $\exists x(\varepsilon)$ s.t. $Ax(\varepsilon)\in B_\varepsilon(0_Y)+y \Rightarrow Ax(\varepsilon)-y\in B_\varepsilon(0_Y)$

Pick ε s.t. $B_\varepsilon(0_Y)\subseteq \overline{AB_{r/2}(0_X)}$, which we can do by (1).

Then, $y-Ax_1\in B_\varepsilon(0_Y)\subseteq\overline{AB_{r/2}(0_X)}$. Repeat on $y-Ax_1$:

$\exists x_2\in B_{r/2}(0_X)$ s.t. $y-Ax_1-Ax_2\in B_{\varepsilon_2}(0_Y)\subseteq \overline{AB_{r/4}(0_X)}$

We thus have $x_n\in B_{2^{1-n}}(0_X)$ s.t. $y-\sum_{j=1}^{n} Ax_j \in B_{2^{1-n}}(0_Y)\subseteq\overline{AB_{2^{2-n}r}(0_X)}$

So, $\sum_{j=1}^{\infty} Ax_j = y$. Since $\|x_n\|<2^{1-n}$, $\sum_{j=1}^{\infty} x_j$ exists.

Then, $A\Big(\underbrace{\sum_{j=1}^{\infty} x_j}_{\in B_{2r}(0_X)}\Big)=y \Rightarrow y\in AB_{2r}(0_X)$.

($\Leftarrow$) Homework :)

$\square$

Theorem: (Inverse Mapping Theorem)

If $A \in \mathcal{B}(X \to Y)$ is a bijection, then $A^{-1} \in \mathcal{B}(X \to Y)$

Proof: A continuous & surjective $\Rightarrow$ A open $\xRightarrow{A \text{ injective}}$ A^{-1} continuous $\Rightarrow A^{-1}$ bounded. ∎

Prop:

If $A : X \to Y$ is a linear map between Banach spaces, then

$$A \text{ bounded} \iff A^{-1}(\overline{B_1(0_Y)}) \text{ has nonempty interior}$$

Proof: ($\Leftarrow$) Let x_0 be in the interior, and so $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x_0) \subseteq A^{-1}(\overline{B_1(0_X)})$
$\forall x \in X$ with $\|x\| < \varepsilon$, we have $x_0 + x \in B_\varepsilon(x_0)$, and so $\|A(x+x_0)\| \leq 1$.
So,
$$\|Ax\| \leq \|A(x+x_0)\| + \|Ax_0\| \leq 1 + \|Ax_0\|$$
If $\|\tilde{x}\| \leq 1$, then $\|\frac{\varepsilon}{2}\tilde{x}\| < \varepsilon \Rightarrow \|A\tilde{x}\| = \frac{2}{\varepsilon}\|A(\frac{\varepsilon}{2}\tilde{x})\| \leq \frac{2}{\varepsilon}(1 + \|Ax_0\|) < \infty$.

($\Rightarrow$) Homework :)

∎

Closed Graph Theorem

Defn:

The graph of a function $f : X \to Y$ is
$$\Gamma(f) := \{(x,y) \in X \times Y : y = f(x)\}$$

Theorem: (Closed graph)

Let $A : X \to Y$ be a linear map between Banach spaces. Then,
$$A \text{ bounded} \iff \Gamma(A) \in \text{Closed}(X \times Y)$$

Proof: ($\Rightarrow$) A bdd $\Rightarrow A$ continuous $\Rightarrow$ if $\{x_n\}_n \subseteq X$ s.t. $x_n \to x \in X$, then $Ax_n \to Ax$ in Y.
Let $\{(x_j, Ax_j)\}_j \subseteq \Gamma(A)$ be a sequence which converges to some $(x,y) \in X \times Y$. We WTS $y = Ax \Rightarrow (x,y) \in \Gamma(A)$, and so $\Gamma(A)$ would be closed

by first countability of $X \times Y$.

So, consider the two projection maps $\left.\begin{array}{l} p_1 : X \times Y \to X \\ p_2 : X \times Y \to Y \end{array}\right\}$ continuous by defn. of product topology

Then, $$x_j = p_1((x_j, Ax_j)) \to x \quad \text{by continuity of } p_1, p_2.$$
$$Ax_j = p_2((x_j, Ax_j)) \to y$$

So, since $Ax_j \to Ax$ by continuity of A, $Ax = y$.

($\Leftarrow$) Let $\Gamma(A) \in \text{Closed}(X \times Y)$. Then, $\Gamma(A)$ is itself a Banach space. Define $\tilde{A} : X \to \Gamma(A)$ s.t. $\tilde{A}x = (x, Ax)$. Then, $\tilde{A}$ is a bijection whose inverse is $p_1|_{\Gamma(A)}$, which is continuous. So, by inverse mapping theorem, $\tilde{A}$ is continuous, and so $A = p_2 \circ \tilde{A}$ is as well.

$\square$

A cool application!

Lemma: (Grothendieck)

If $p \in (1, \infty)$, then L^p embeds in L^∞.

Formally, let μ be a finite measure on Ω, and consider $S \in \text{Closed}(L^p(\Omega, \mu))$ as a closed subspace that is also contained in $L^\infty(\Omega, \mu)$.
Then, $\exists K < \infty$ s.t. $\forall f \in S$, $$\|f\|_\infty \leq K \|f\|_p$$

Proof: Let S have the subspace topology from $L^p(\Omega, \mu)$, and let $j : S \to L^\infty(\Omega, \mu)$ be the injection map.
Let $\{f_n\}_n$ be a sequence in S s.t. $f_n \to f$ in S, and $f_n \to g$ in L^∞.
Then, $$\|f - g\|_p \leq \|f - f_n\|_p + \|f_n - g\|_p \underset{\|\cdot\|_p \leq \|\cdot\|_\infty \text{ for finite measure spaces}}{\leq} \|f_n - f\|_p + \|f_n - g\|_\infty \to 0$$
So, $f = g$ μ-a.e., and so j has a closed graph.
By the closed graph theorem, $\exists K = \|j\|_{B(S \to L^\infty)} < \infty$ for which

$$\|jf\|_\infty \leq K \|f\|_p \Rightarrow \|f\|_\infty \leq K \|f\|_p.$$

$\square$

Remark: In fact, from the assumption on any p, we may show $\|f\|_\infty \leq M \|f\|_2$ over S.

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4. Convexity

Defn:

A partial order on a set X is a subset $R \subseteq X \times X$ s.t.

(1) reflexive: $(a,a) \in R \quad \forall a \in X$
(2) antisymmetric: $(a,b) \in R$ and $(b,a) \in R \Rightarrow a=b \quad \forall a,b \in X$
(3) transitive: $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R \quad \forall a,b,c \in X$

We say X is linearly ordered if $\forall a,b \in X$, either (a,b) or (b,a) in R.

We say $m \in Y$ is a maximal element of $Y \subseteq X$ if $\forall y \in Y$, $(m,y) \in R \Rightarrow y = m$.
We say u is an upper bound of $Y \subseteq X$ if $\forall y \in Y$, $(y,c) \in R$

Lemma (Zorn's Lemma)

Let X be a nonempty partially-ordered set s.t. any linearly-ordered subset has an upper bound. Then, any linearly-ordered subset of X has an upper bound that is a maximal element.

Theorem: ($\mathbb{R}$-Hahn-Banach)

Let X be an $\mathbb{R}$-vector space and $p: X \to \mathbb{R}$ s.t.
$$\underbrace{p(\alpha x + (1-\alpha)y) \le \alpha p(x) + (1-\alpha)p(y)}_{p \text{ convex}} \quad \forall x,y \in X \text{ and all } \alpha \in (0,1)$$

Suppose that $\lambda: Y \to \mathbb{R}$ is linear on a subspace $Y \subseteq X$ with $\lambda \le p$ over Y.
Then, $\exists \Lambda: X \to \mathbb{R}$ linear s.t. (1) $\Lambda|_Y = \lambda$ (extension)
(2) $\Lambda \le p$ on X (maintains bound)

Proof: Let $z \in X \setminus Y$. Define $\tilde{Y} := \operatorname{span}\{z, Y\} = (\mathbb{R}z) \oplus Y$.
We will define $\tilde{\lambda}: \tilde{Y} \to \mathbb{R}$ via $\tilde{\lambda}(az+y) = a\,\tilde{\lambda}(z) + \lambda(y)$ to preserve linearity. We wish to pick a value for $\tilde{\lambda}(z)$ to maintain the bound. To that end, let $y_1, y_2 \in Y$ and $\alpha, \beta > 0$. Then,

$$\begin{aligned}
\alpha\lambda(y_1) + \beta\lambda(y_2) &= \lambda(\alpha y_1 + \beta y_2) = (\alpha+\beta)\,\lambda\left(\tfrac{\alpha}{\alpha+\beta}y_1 + \tfrac{\beta}{\alpha+\beta}y_2\right) \\
&= (\alpha+\beta)\,\lambda\left(\tfrac{\alpha}{\alpha+\beta}(y_1 - \beta z) + \tfrac{\beta}{\alpha+\beta}(y_2 + \alpha z)\right) \\
&\underset{\lambda \le p}{\le} (\alpha+\beta)\,p\left(\tfrac{\alpha}{\alpha+\beta}(y_1 - \beta z) + \tfrac{\beta}{\alpha+\beta}(y_2 + \alpha z)\right)
\end{aligned}$$

$$\underset{p \text{ convex}}{\leq} \alpha p(y_1 - \beta z) + \beta p(y_2 + \alpha z)$$

$$\Rightarrow \frac{1}{\beta}\left[-p(y_1-\beta z) + \lambda(y_1)\right] \leq \frac{1}{\alpha}\left[p(y_2+\alpha z) - \lambda(y_2)\right] \qquad (*)$$

In particular, $\exists q \in \mathbb{R}$ s.t.

$$\sup_{\substack{\beta>0 \\ y_1\in Y}} \frac{1}{\beta}\left[-p(y_1-\beta z) + \lambda(y_1)\right] \leq q \leq \inf_{\substack{\alpha>0 \\ y_2\in Y}} \frac{1}{\alpha}\left[p(y_2+\alpha z) - \lambda(y_2)\right]$$

Define $\tilde{\lambda}(z) := q$. We WTS $\tilde{\lambda}(az+y) \leq p(az+y)$ $\forall a \in \mathbb{R}, y \in Y$.
Suppose WLOG that $a>0$. Apply $(*)$ with $\alpha = a$, $y_2 = y$ to see

$$\tilde{\lambda}(z) \leq \frac{1}{a}\left[p(y+az) - \lambda(y)\right] \Rightarrow \tilde{\lambda}(az+y) \leq p(az+y)$$

So, we can extend by 1 extra dimension without violating $\tilde{\lambda} \leq p$.

Next, let $\mathcal{E}$ be the collection of linear extensions of λ that are $\leq p$ on their subspaces of definition. Define a partial order $R \subseteq \mathcal{E}\times\mathcal{E}$ via $(e_1, e_2) \in R \iff X_1 \subseteq X_2$ and $e_2|_{X_1} = e_1$ (X_1, X_2 are the subspaces of definition)

Let $\{e_\alpha\}_{\alpha\in A} \subseteq \mathcal{E}$ be linearly ordered. We define an upper bound via
$e: \bigcup_{\alpha\in A} X_\alpha \to \mathbb{R}$ via $e(x) := e_\alpha(x)$ $\forall x \in X_\alpha$

By defn of $\mathcal{E}$, $e(x) \leq p(x)$, and so $e \in \mathcal{E}$. Clearly, $(e_\alpha, e) \in R$ $\forall \alpha \in A$.
So, every linearly-ordered subset of $\mathcal{E}$ has an upper bound.

By Zorn's Lemma, $\exists$ max element $e: X' \to \mathbb{R}$ st. $e \leq p$ on X'.
Suppose BWOC $X' \subsetneq X$; then, we could add another dimension to the extension and violate the maximal element property. So, $X' = X$. Thus, $e: X \to \mathbb{R}$ has $e \leq p$ and $e|_Y = \lambda$.

$\square$

Theorem: ($\mathbb{C}$-Hahn-Banach)

Let X be a $\mathbb{C}$-vector space and $p: X \to \mathbb{R}$ s.t.
$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y)$ $\forall x, y \in X$ and all $\alpha, \beta \in \mathbb{C}$ w/ $|\alpha| + |\beta| = 1$.

Suppose that $\lambda: Y \to \mathbb{C}$ is linear on a subspace $Y \subseteq X$ with $|\lambda| \leq p$ over Y.
Then, $\exists \Lambda: X \to \mathbb{C}$ linear s.t. (1) $\Lambda|_Y = \lambda$ (extension)
(2) $|\Lambda| \leq p$ on X (maintains bound)

Proof: Apply $\mathbb{R}$-Hahn-Banach on $\underset{\text{forgetful functor}}{\mathcal{F}_{\mathbb{R}}(X)}$ with the linear functional $\ell: \mathcal{F}_{\mathbb{R}}(X) \to \mathbb{R}$ via $\ell(y) := \mathrm{Re}(\lambda(y))$, $\ell \leq |\lambda| \leq p$ on Y. So, we get $L: \mathcal{F}_{\mathbb{R}}(x) \to \mathbb{R}$ s.t. $L|_Y = \ell$ and $L \leq p$. Define $\Lambda: X \to \mathbb{C}$ via $\Lambda(x) := L(x) - iL(ix)$

$\square$

Duality

Defn:

If X is a Banach space, we define its dual X^* to be the vector space $\mathcal{B}(X \to \mathbb{C})$ with the norm $\|\lambda\|_{op} = \sup\left\{ |\lambda(x)| : \begin{matrix} \|x\| \le 1 \\ x \in X \end{matrix} \right\}$

We have seen that the dual is a Banach space.

Examples

① Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q \in (1, \infty)$. We claim $(L^q(\mathbb{R}^n))^* = L^p(\mathbb{R}^n)$

Take $g \in L^p \mapsto G \in (L^q)^*$ via $G(f) := \int \bar{g} f$.
By Hölder's inequality, $\|G(f)\| \le \|g\|_p \|f\|_q \Rightarrow \|G\|_{op} \le \|g_p\| < \infty$
this is actually equality

In fact, this map $L^p \to (L^q)^*$ turns out to be an isometric isomorphism.
So, they are the <u>same Banach space.</u>

Theorem:

Let $(X^*)^*$ denote the double dual. Then, the map

$$J : X \to X^{**} \qquad x \mapsto (X^* \ni \lambda \mapsto \lambda(x))$$

is an <u>isometric</u> injection.

Proof: J will essentially be the evaluation map. We send a point x to the evaluation map that evaluates a functional at x. In math, J sends

$$x \mapsto (\underbrace{\lambda \mapsto \lambda(x)}_{\lambda \in B^*,\ \lambda(x) \in \mathbb{C}})$$

We want to show that $\|J(x)\|_{\mathcal{B}(X^* \to \mathbb{C})} = \|x\|_X$
For all $x \in X$, $\lambda \in X^*$,

$$|(J(x))(\lambda)| = |\lambda(x)| \le \|\lambda\|_{op} \|x\|_X$$

Taking a supremum over all $\lambda \in X^*$ with $\|\lambda\|_{op} \le 1$, $\|J(x)\|_{op} \le \|x\|_X$
To show the other direction, we seek a functional Λ s.t. $|J(x)\Lambda| \ge \|x\|_X$ for some $x \in X$.
Fix some $x_0 \in X$, and define a linear functional $\lambda : \mathbb{C}x_0 \to \mathbb{C}$ via $\alpha x_0 \mapsto \alpha \|x\|_X$
Clearly, we have an upper bound $p : X \to (0, \infty)$ via $p(y) = \|y\|_X$ (i.e. $\lambda \le p$ on $\mathbb{C}x_0$).
Applying Hahn-Banach, we get some $\Lambda : X \to \mathbb{C}$ s.t. $\Lambda(x_0) = \|x_0\|_X$ and

$$\|\Lambda\|_{op} = \sup\{|\lambda_0 y| : \|y\| \le 1\} \le 1.$$

Thus, $\|J(x_0)\|_{op} \ge |J(x_0)(\Lambda)| = \|x_0\|_X \Rightarrow \|J(x_0)\|_{op} = \|x_0\|_X \quad \forall x_0 \in X.$ □

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5. Duality, Weak Topologies, & Banach-Alaoglu

Defn:

We say that a Banach space X is reflexive if $X \cong X^{**}$, or equivalently if $J(X) = X^{**}$.

Lemma:

Let X be a Banach space and $Y \subseteq X$ a vector subspace. For any $\lambda \in Y^*$, there exists some $\Lambda \in X^*$ s.t. $\|\Lambda\|_{op} = \|\lambda\|_{op}$ and $\Lambda|_Y = \lambda$.

Proof: Define $p_\lambda : X \to [0,\infty)$ via $x \mapsto \|x\|_X \|\lambda\|_{op}$ $\Rightarrow$ $\lambda \leq p$ over Y.
By Hahn-Banach, there is some $\Lambda : X \to \mathbb{C}$ s.t.
$\forall x \in X$ $|\Lambda(x)| \leq p(x) = \|x\|_X \|\lambda\|_{op} \Rightarrow \|\Lambda\|_{op} \leq \|\lambda\|_{op}$
Next $\|\Lambda\|_{op} = \sup_{\substack{\|x\| \leq 1 \\ x \in X}} |\Lambda(x)| \geq \sup_{\substack{\|y\| \leq 1 \\ y \in Y}} |\lambda(y)| = \|\lambda\|_{op}$. $\square$

Lemma:

Let X be Banach. Then, for all $x \in X$
$$\|x\|_X = \sup\{|\lambda(x)| : \lambda \in X^* \text{ s.t. } \|\lambda\|_{op} \leq 1\}$$

Proof: $\|x\|_X = \|J(x)\|_{B(X^* \to \mathbb{C})} = \text{RHS}.$ $\square$

5.1 – Weak Topologies

Defn:

Let $(X, \|\cdot\|)$ be Banach. We define the weak topology on X as the "initial topology" generated by the collection of maps X^*.
Let's call it $\text{Open}_w(X)$. Then, $\text{Open}_w(X) \subseteq \text{Open}_{\|\cdot\|}(X)$
Then, $\text{Open}_w(X)$ is the smallest topology on X s.t. $\lambda : X \to \mathbb{C}$ is continuous for all $\lambda \in X^*$.
$\text{Open}_w(X)$ is generated by the sub-basis $\{\lambda^{-1}(U) : U \in \text{Open}(\mathbb{C}) \text{ and } \lambda \in X^*\}$
So,
$$U \in \text{Open}_w(X) \iff U = \bigcup_{\alpha \in I} \bigcap_{j=1}^{n_\alpha} \lambda_{\alpha j}^{-1}(E_{\alpha j}) \text{ for some } \lambda_{\alpha j} \in X^*, E_{\alpha j} \in \text{Open}(\mathbb{C}), n_\alpha \in \mathbb{N}$$

Lemma:

If X is an infinite-dimensional Banach space and $\mathcal{U} \in \mathrm{Open}_w(X)$, <u>then</u> $\mathcal{U}$ is <u>unbounded</u> in $\|\cdot\|_X$.
and open

Proof: Let $x_0 \in \mathcal{U}$. We find $\lambda_1, \ldots, \lambda_n \in X^*$ and $\varepsilon > 0$ s.t.

$$x_0 \in \underbrace{\bigcap_{j=1}^{n} \lambda_j^{-1}(B_\varepsilon(\lambda_j(x_0)))}_{\subseteq \mathcal{U}} = \bigcap_{j=1}^{n} \{x \in X : \underbrace{|\lambda_j(x) - \lambda_j(x_0)|}_{= \lambda_j(x - x_0)} < \varepsilon\}$$

$$= \{x_0\} + \bigcap_{j=1}^{n} \{x \in X : |\lambda_j(x)| < \varepsilon\}$$

$$= \{x_0\} + \bigcap_{j=1}^{n} \lambda_j^{-1}(B_\varepsilon(0_{\mathbb{C}}))$$

$$\supseteq \{x_0\} + \bigcap_{j=1}^{n} \ker(\lambda_j)$$

Thus, $\{x_0\} + \bigcap_{j=1}^{n} \ker(\lambda_j) \subseteq \mathcal{U}$. Furthermore, $x_0 \in \{x_0\} + \bigcap_{j=1}^{n} \ker(\lambda_j)$ clearly.

Define $\mathfrak{Z} : X \to \mathbb{C}^n$ via $x \mapsto (\lambda_1(x), \ldots, \lambda_n(x))$
$\Rightarrow \ker(\mathfrak{Z}) = \bigcap_{j=1}^{n} \ker(\lambda_j)$.

It cannot be that $\ker(\mathfrak{Z}) = \{0_X\}$, since we would then have an injection $X \hookrightarrow \mathbb{C}^n$, contradicting infinite dim. So,

$$\exists\, v \in \left(\bigcap_{j=1}^{n} \ker(\lambda_j)\right) \setminus \{0_X\}$$

By linearity, $x_0 + \alpha v \in \mathcal{U}$ $\forall \alpha \in \mathbb{C}$. Since $\|x_0 + \alpha v\|_X \geq \big|\, |\alpha| \|v\|_X - \|x_0\|_X \big|$, we may take $|\alpha|$ large enough that $\|x_0 + \alpha v\|_X \to \infty$.

$\square$

Corollary:

If X is an infinite-dim Banach space, then $\mathrm{Open}_w(X)$ is <u>not metric</u>.

Proof: Suppose BWOC it is. Then, $\exists d : X \times X \to [0, \infty)$ metric inducing $\mathrm{Open}_w(X)$.
Let $U_n := \{x \in X : d(0_X, x) < \frac{1}{n}\}$. By hypothesis, $U_n \in \mathrm{Open}_w(X)$.
By the previous result, $\forall n \in \mathbb{N}$ $\exists x_n \in U_n$ s.t. $\|x_n\|_X \geq n$.
Since $x_n \xrightarrow{d} 0_X$ by selection, then $\{\|x_n\|\}_n$ is bounded eventually.
$\forall\, U \in \mathrm{Nbhd}_w(0_X)$, a tail is contained in U. For this tail, . . .

$\square$

Lemma:

$(X, \mathrm{Open}_w(X))$ is a TVS.

Proof: Use separating seminorms $\lambda \mapsto p_\lambda(x,y) = |\lambda(x) - \lambda(y)|$.
Then, the collection $\{p_\lambda\}_{\lambda \in X^*}$ is separating: for two points, there will be disagreeing functionals. This leads to continuity of $+$ and $\cdot$, see Rudin 1.37.
To show T_1, we show $\{0_X\}$ is closed.

Let $x \in X \setminus \{0_X\}$. Then, $\exists \lambda \in X^*$ s.t. $\lambda(x) \neq 0$. Then, $\exists \varepsilon > 0$ s.t.
$x \notin \lambda^{-1}(B_\varepsilon(0_{\mathbb{C}})) \iff 0_X \notin \{x\} - \lambda^{-1}(B_\varepsilon(0_{\mathbb{C}})) \in \mathrm{Nbhd}_w(x)$.
So, $\{0_X\}$ is closed in the weak topology.

$\square$

Remark: - When $\dim X = \infty$, since this is a non-metric TVS, there are two inequivalent TVS structures.
- This contrasts the finite-dim case!

"never differentiate in public"

Lemma:

$$x_n \xrightarrow{w} x \iff \lambda(x_n) \xrightarrow{\mathbb{C}} \lambda(x) \quad \forall \lambda \in X^*$$

$$J(x_n)(\lambda) \to J(x)(\lambda)$$

In words, weak convergence $\iff$ pointwise convergence on λ's.

Proof: ($\Rightarrow$) Suppose $x_n \xrightarrow{w} x$. Then, $\forall V \in \mathrm{Nbhd}_w(x)$, $\exists M_V \in \mathbb{N}$ s.t. $n \geq M_V \Rightarrow x_n \in V$.
Let $\lambda \in X^*$, and let $U \in \mathrm{Nbhd}_{\mathbb{C}}(\lambda(x))$. Then, $\lambda^{-1}(U) \in \mathrm{Nbhd}_w(x)$. So, letting $V = \lambda^{-1}(U)$, we get $N_U = M_{\lambda^{-1}(U)}$ s.t. $\forall n \geq N_U$, $\lambda(x_n) \in U$.

($\Leftarrow$) Let $U \in \mathrm{Nbhd}_w(x)$. We may find $\lambda_1, \dots, \lambda_m \in X^*$ and $\varepsilon > 0$ s.t.
$$x \in \{x\} + \bigcap_{j=1}^{m} \lambda_j^{-1}(B_\varepsilon(0_{\mathbb{C}})) \subseteq U.$$
Pick n large enough that $\lambda_j(x_n) \in \{\lambda_j(x)\} + B_\varepsilon(0_{\mathbb{C}})$ $\forall j$. check this step

$\square$

Prop:

Every weakly-convergent sequence is norm-bounded.

Proof: Suppose $x_n \xrightarrow{w} x$. Define $z_n := J(x_n) \in X^{**}$.
For all $\lambda \in X^*$ we know that $\underbrace{\{\lambda(x_n)\}_n}_{= z_n(\lambda)} \subseteq \mathbb{C}$ converges in $\mathbb{C}$, and so it is bounded.
So, for each $\lambda \in X^*$,
$$\underbrace{\sup_n |z_n(\lambda)| < \infty}_{\text{by } \mathbb{C}\text{-convergence}} \xRightarrow{\text{uniform boundedness}} \sup_n \|z_n\|_{op} < \infty$$
Since J is an isometry, $\sup_n \|x_n\|_X < \infty$.

$\square$

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Weak* Topology

We had that the weak topology on X is the initial topology generated by X^*.

Defn:

The weak-* topology on X^* is the initial topology generated by $J(X) \subseteq X^{**}$. That is, it is the weakest topology on X^* s.t. point evaluations are continuous w.r.t. the functional being evaluated.

From HW3, we know that if X is an infinite-dim Banach space then $\overline{B_1(0)}$ is not compact in the norm topology.

Theorem: (Banach-Alaoglu)

Let X^* be the dual of a Banach space X and $B := \{\lambda \in X^* : \|\lambda\|_{op} \leq 1\}$. Then, B is weak-* compact.

Proof: $\forall x \in X$, define $B_x := \overline{B_{\|x\|}(0_{\mathbb{C}})} \subseteq \mathbb{C}$. We know B_x is compact in $\mathbb{C}$, and so by Tychonoff's Theorem we know

$$\mathcal{B} := \prod_{x \in X} B_x \quad \text{is compact in the product topology on } \mathbb{C}^X.$$

We may think of elements in $\mathcal{B}$ as functionals, though they are not necessarily linear. However, we know that $\forall (b: X \to \mathbb{C}) \in \mathcal{B}$,

$$|b(x)| \leq \|x\|$$

So, $B \subseteq \mathcal{B}$ (i.e. $B = \mathcal{B} \cap (\text{linear})$). We should first show that the subspace topology of linear functionals $\leq \mathcal{B}$ and $(X^*, \text{weak-*})$ agree. Note that $\text{Open}(\mathcal{B})$ is the initial topology generated by the projection maps p_x sending $b \mapsto b(x)$. Since $p_x(b) = J(x)(b)$ and $\text{Open}_{\text{weak-*}}(X^*)$ is the initial topology generated by the $J(x)$'s, we know that these are the same topology. Thus, $\mathcal{B}$ is also weak-* compact.

Now, we know $\mathcal{B}$ is weak-* compact, and so we must show B is weak-* closed. We will construct a continuous map whose kernel is B.

For $x,y \in X$ and $z \in \mathbb{C}$, define $\varphi_{xyz} : \mathcal{B} \to \mathbb{C}$ by

$$\varphi_{xyz}(b) := b(x+zy) - b(x) - z\,b(y)$$

We know φ_{xyz} is weak-* continuous since it is a combination of point evaluations, which are weak-* continuous by definition.
Furthermore,

$$B = \mathcal{B} \cap (\text{linear}) = \bigcap_{\substack{x,y \in X \\ z \in \mathbb{C}}} \underbrace{\varphi_{xyz}^{-1}(\{0_{\mathbb{C}}\})}_{\text{closed}} \Rightarrow B \text{ weak-* closed}$$

$\square$

6. Banach Algebras & Spectral Analysis

Recall that if X is a Banach space, then $\mathcal{B}(X \to X)$ is a Banach space with $\|\cdot\|_{op}$. Also, we have a natural multiplicative structure via composition of linear maps. So, $\boxed{\mathcal{B}(X \to X) \text{ is a } \mathbb{C}\text{-algebra}}$

We also had that $\|AB\|_{op} \leq \|A\|_{op} \|B\|_{op}$. We will define an abstract notion of Banach spaces that are $\mathbb{C}$-algebras with submultiplicative norm.

Defn:

A **Banach algebra** $\mathcal{A}$ is a Banach space that is also a $\mathbb{C}$-algebra for which

① $\forall a,b \in \mathcal{A}$, $\|ab\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}} \|b\|_{\mathcal{A}}$

② $\exists \mathbb{1} \in \mathcal{A}$ s.t. $a\mathbb{1} = \mathbb{1}a = a \;\; \forall a \in \mathcal{A}$ and $\|\mathbb{1}\|_{\mathcal{A}} = 1$.

Prop:

$\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous

Proof: $\|xy - ab\| \leq (\|x\| + \|a-x\|)\|b-y\| + \|b\|\cdot\|a-x\|$

$\square$

Examples:

① $C([0,1] \to \mathbb{C})$ is a Banach space with the supremum norm. With pointwise multiplication, it becomes a (commutative) Banach algebra.

② $\mathbb{C}^n$ with elementwise multiplication is a commutative Banach algebra.

③ $\mathcal{B}(X)$ is in general a non-commutative Banach algebra. Note that $\mathcal{B}(\mathbb{C}^n) \cong \mathrm{Mat}_{n\times n}(\mathbb{C})$.

6.1 Invertible Elements

Def:

An element $x \in \mathcal{A}$ has a left inverse if $\exists a \in \mathcal{A}$ s.t. $ax = \mathbb{1}$.
" right inverse if $\exists b \in \mathcal{A}$ s.t. $xb = \mathbb{1}$.
If both exist, then x is invertible, $x = y$, and so inverses are unique.
We call the set of invertible elements $G_{\mathcal{A}} \equiv G(\mathcal{A})$.

Remark: What separates this discussion from usual group theory is that we have topological information via the norm.

Lemma:

If $x \in \mathcal{A}$ obeys $\|x - \mathbb{1}\| < 1$, then $x \in G(\mathcal{A})$ and

- $\boxed{x^{-1} = \sum_{n=0}^{\infty} (\mathbb{1}-x)^n}$ (von Neumann series) So important
- $\|x^{-1}\| \leq \dfrac{1}{1 - \|\mathbb{1}-x\|}$

Proof: Let $y := \mathbb{1} - x$, and $r := \|y\| < 1$. Then, submultiplicativity grants $\|y^n\| \leq \|y\|^n = r^n$.
Define $\{z_N\}_N$ via $z_N := \sum_{n=0}^{N} (\mathbb{1}-x)^n = \sum_{n=0}^{N} y^n$

So, $\|z_N - z_m\| \leq \frac{r^{N+1}}{1-r}(1 - r^{m-N})$ $(m \geq N)$, and so it is Cauchy.

By completeness, $\exists z$ s.t. $z_N \to z$ in norm. So,
$$z(\mathbb{1}-y) = (\mathbb{1}-y)z = \lim_{N\to\infty} \sum_{n=0}^{N} (\mathbb{1}-y)y^n = \lim_{N\to\infty}\left[\sum_{n=0}^{N} y^n - y^{n+1}\right] \overset{\text{telescoping}}{=} \lim_{N\to\infty}\left(\mathbb{1} - \underset{\|\cdot\|<1}{y^{N+1}}\right) = \mathbb{1}$$

So, $x^{-1} = z = \sum_{n=0}^{\infty} (\mathbb{1}-x)^n$. Next,
$$\|x^{-1}\| = \lim_{N\to\infty} \left\| \sum_{n=0}^{N} (\mathbb{1}-x)^n \right\| \leq \lim_{N\to\infty} \sum_{n=0}^{\infty} \|\mathbb{1}-x\|^n = \frac{1}{1 - \|\mathbb{1}-x\|}.$$

$\square$

Prop:

$G(\mathcal{A}) \in \mathrm{Open}(\mathcal{A})$ and $(\cdot)^{-1} : G(\mathcal{A}) \to G(\mathcal{A})$ is continuous.

Proof: Let $a \in G(\mathcal{A})$. We claim $B_{\frac{1}{\|a^{-1}\|}}(a) \subseteq G(\mathcal{A})$. So, let $\tilde{a} \in B_{\frac{1}{\|a^{-1}\|}}(a)$
So,
$$\|a-\tilde{a}\| < \frac{1}{\|a^{-1}\|} \Rightarrow 1 > \|a-\tilde{a}\| \cdot \|a^{-1}\| \geq \|(a-\tilde{a})a^{-1}\| = \|1 - \tilde{a}a^{-1}\|.$$

By the above lemma, $\tilde{a}a^{-1} \in G(\mathcal{A})$. So, $\underbrace{\tilde{a}\,a^{-1}(\tilde{a}a^{-1})^{-1}}_{\text{right inverse for } \tilde{a}} = 1$

Similarly, $\underbrace{(a^{-1}\tilde{a})^{-1}a^{-1}}_{\text{left inverse}}\tilde{a} = 1$. So, $\tilde{a} \in G(\mathcal{A}) \Rightarrow G(\mathcal{A})$ open.

Next, we have the resolvent identity
$$\boxed{\begin{aligned} a^{-1} - b^{-1} &= a^{-1}(b-a)b^{-1} \\ &= b^{-1}(b-a)a^{-1} \end{aligned}}$$
So important!

So, $\|a^{-1} - b^{-1}\| \leq \|b^{-1}\|\,\|b-a\|\,\|a^{-1}\|$. Also, $\|b^{-1}\| = \|b^{-1}aa^{-1}\| \leq \|b^{-1}a\|\,\|a^{-1}\|$

nonsense $\Big[$ Let a, b be such that $\|a-b\| < \frac{1}{2\|a^{-1}\|} \Rightarrow \|1 - b^{-1}a\| < \frac{1}{2} \overset{\text{lemma}}{\Rightarrow} \underset{=\|a^{-1}b\|}{\|(b^{-1}a)^{-1}\|} \leq \frac{1}{1 - \|1 - b^{-1}a\|}$

$\Rightarrow \|b^{-1}\| \leq 2\|a^{-1}\|$, and so $\|a^{-1} - b^{-1}\| \leq 2\|a^{-1}\|^2\|a-b\| \Rightarrow$ inverse map is continuous! $\square$

6.2: Banach-Valued complex functions (Rudin pg. 82, Conway pg. 196)

We ask about functions $f: \mathbb{C} \to X$ for a $\mathbb{C}$-Banach space X. Recall the magic of $\mathbb{C}$-differentiability from complex analysis. We do a similar thing below.

Defn:

$f: \mathbb{C} \to X$ is $\mathbb{C}$-differentiable (holomorphic) at some $z_0 \in \mathbb{C}$ if
$$\lim_{z \to 0} \frac{f(z_0+z) - f(z_0)}{z} \text{ exists (in } \|\cdot\|_X)$$

Defn:

$f: \mathbb{C} \to X$ is Frechet differentiable at $z_0 \in \mathbb{C}$ if $\exists\, L \in \mathcal{B}(\mathbb{C} \to X)$ s.t. ($L = f'(z_0)$)
$$\lim_{z \to 0} \frac{\|f(z_0+z) - f(z_0) - Lz\|_X}{|z|} = 0$$
This is equivalent to $\mathbb{C}$-differentiability.

Defn

$f: \mathbb{C} \to X$ is weakly-$\mathbb{C}$-differentiable if $\Lambda \circ f: \mathbb{C} \to \mathbb{C}$ is holomorphic for all $\Lambda \in X^*$.

Theorem:

If X is a Banach space, then $\mathbb{C}$-diff'ability and weak-$\mathbb{C}$-diff'ability are equivalent!

Proof: in Rudin. $\square$

Integration

Def: (Riemann integration)

Let $f:[a,b]\to X$, where X is a $\mathbb{C}$-Banach space. Define $\int_{[a,b]} f$ as follows:
For any partition P given by $a=x_1<\dots<x_n=b$, define

$$S(f,P):=\sum_{j=1}^{n-1}(x_{j+1}-x_j)f(x_j) \quad \text{and} \quad \omega(f,P):=\sum_{j=1}^{n-1}(x_{j+1}-x_j)\sup_{s,t\in(x_j,x_{j+1})}\{\|f(s)-f(t)\|\}$$

We want $\forall\varepsilon>0$ to find a partition P s.t. $\omega(f,P)<\varepsilon$, since then we can proceed as usual.

Importantly, if f is continuous then it is Riemann-integrable!

Def: (Bochner Integral)

Check Rudin.

Def: (Contour integral)

Let $\gamma:[a,b]\to\mathbb{C}$ be piecewise smooth and $f:\mathbb{C}\to X$ continuous. We define

$$\int_\gamma f := \int_{[a,b]} \underbrace{(f\circ\gamma)\gamma'}_{\substack{\text{continuous fn. from}\\ [a,b]\to X}} \in X$$

It turns out that $\int_\gamma f$ does not depend on the parameterization of γ.

Recall the following facts from complex analysis:

Lemma: (ML)

$$\left\|\int_\gamma f\right\| \le \left(\sup_{t\in[a,b]}\|f(\gamma(t))\|\right) L(\gamma)$$

← length of contour

Cauchy Integral Formula: (Rudin 1.31)

Let $\Omega\in\mathrm{Open}(\mathbb{C})$ be simply-connected, $f:\Omega\to X$ holomorphic, $\gamma:[a,b]\to\Omega$ a simple CCW contour, and z_0 in the interior of γ. Then, $\forall n\in\mathbb{N}\cup\{0\}$,

$$f^{(n)}(z)=\frac{n!}{2\pi i}\oint\frac{1}{(w-z)^{n+1}}f(w)\,dw$$

So, holomorphic $\Rightarrow$ smooth.

Cauchy's Inequality:

Suppose that $f:\mathbb{C}\to X$ is holomorphic on $\overline{B_R(z_0)}$. Then,

$$\|f^{(n)}(z_0)\| \le \frac{n!}{R^n}\left(\sup_{z\in\overline{B_R(z_0)}}\|f(z)\|\right)$$

6.3: The Spectrum

Def:

Given $a \in \mathcal{A}$ of a Banach algebra, the spectrum of a (denoted $\sigma(a)$) is

$$\sigma(a) := \left\{ z \in \mathbb{C} : \ (a - z\mathbb{1}) \notin G(\mathcal{A}) \right\}$$

We also define:

- the resolvent set $\rho(a) := \mathbb{C} \setminus \sigma(a)$
- the spectral radius $r : \mathcal{A} \to [0, \infty]$ sending $a \mapsto \sup_{z \in \sigma(a)} |z|$

Theorem:

The spectrum $\sigma(a)$ of some $a \in \mathcal{A}$ is a non-empty, compact subset of $\mathbb{C}$.

Proof: Let $a \in \mathcal{A}$. We want to show $\sigma(a) \in \mathrm{Closed}(\mathbb{C}) \iff \rho(a) \in \mathrm{Open}(\mathbb{C})$.
Define $\psi : \mathbb{C} \to \mathcal{A}$ sending $z \mapsto a - z\mathbb{1}$. Then, ψ is continuous.
Furthermore, $\rho(a) = \psi^{-1}(G(\mathcal{A}))$ is a preimage of an open set, and so $\sigma(a)$ is closed.

Next we WTS $r(a) \leq \|a\|$. Let $z \in \mathbb{C}$ s.t. $|z| > \|a\|$. Then,

$$1 > \frac{\|a\|}{|z|} = \left\| \frac{a}{z} \right\| = \left\| \mathbb{1} - \left(\mathbb{1} - \frac{a}{z}\right) \right\|, \text{ and so } \mathbb{1} - \frac{a}{z} \in G(\mathcal{A}) \Rightarrow a - z\mathbb{1} \in G(\mathcal{A}).$$

So, $z \notin \sigma(a)$ for all $|z| > \|a\|$, and therefore $r(a) \leq \|a\|$.
So, $\sigma(a) \subseteq \mathbb{C}$ is closed and bounded, which grants compactness by Heine-Borel.

To see nonemptiness, define the resolvent map $\varphi : \rho(a) \to \mathcal{A}$ sending $z \mapsto (a - z\mathbb{1})^{-1}$.
φ has an open domain, and

$$\frac{\varphi(z_0 + z) - \varphi(z_0)}{z} = \frac{(a - (z_0+z)\mathbb{1})^{-1} - (a - z_0\mathbb{1})^{-1}}{z} = \frac{(a - (z_0+z)\mathbb{1})^{-1}\left(\cancel{z_0 - (z_0+z)}\right)(a - z\mathbb{1})^{-1}}{\cancel{z}}$$

$$= -\varphi(z+z_0)\,\varphi(z_0)$$

As $z \to 0$, continuity of φ guarantees this $\to -[\varphi(z_0)]^2$, and so φ is holomorphic on $\rho(a)$. So, $\lambda \circ \varphi$ is holomorphic on $\rho(a)$ $\forall \lambda \in \mathcal{A}^*$.
We claim φ decays as $|z| \to \infty$. For any $|z| > \|a\|$, we know

$$\|\varphi(z)\| = \|(a - z\mathbb{1})^{-1}\| = |z|^{-1} \left\| \left(\frac{a}{z} - \mathbb{1}\right)^{-1} \right\| \overset{\text{last time}}{\leq} |z|^{-1}\left(1 - \left\|\mathbb{1} - \left(\mathbb{1} - \frac{a}{z}\right)\right\|\right)^{-1} = \frac{1}{|z|\left(1 - \left\|\frac{a}{z}\right\|\right)} = \frac{1}{|z| - \|a\|}$$

As $|z| \to \infty$, we see that $\|\varphi(z)\| \to 0$. So, $\|\lambda \circ \varphi\|_\infty \leq \|\lambda\|_{op} \sup_z \|\varphi(z)\| < \infty$
Thus, $\lambda \circ \varphi$ is bounded and meromorphic. Suppose BWOC $\rho(a) = \mathbb{C}$.
By Liouville's theorem, $\lambda \circ \varphi$ is constant $\forall \lambda$. However, $\varphi'(z) = -\varphi(z)^2 \neq 0$. $\times$

$\square$

Lemma (Fekete):

If a sequence $\{a_n\}_n \subseteq \mathbb{R}$ is subadditive ($a_{n+m} \le a_n + a_m \ \forall n,m \in \mathbb{N}$), then $\lim_{n\to\infty} \frac{1}{n} a_n$ exists and equals $\inf_{n\in\mathbb{N}} \frac{a_n}{n}$.

Proof: HW! $\square$

Lemma: (Gelfand's formula)

Let $a \in \mathcal{A}$. Then, $\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$ exists and equals $\underset{\swarrow \text{ when } a \text{ is self-adjoint, we will see that } r(a)=\|a\|}{r(a)} = \inf_n \|a^n\|^{\frac{1}{n}}$.

Proof: Submultiplicativity of $\|\cdot\|$ gives that $b_n = \log(\|a^n\|)$ is subadditive, and so Fekete's lemma gives that the limit exists and equals its inf.

Now, let $z \in \mathbb{C}$ be s.t. $|z| > \|a\|$. Then,

$$\varphi(z) = (a - z\mathbb{1})^{-1} \underset{\text{von Neumann series}}{=} -\sum_{n=0}^{\infty} z^{-n-1} a^n .$$

which converges uniformly on $\partial B_R(0)$ for $R > \|a\|$. Thus,

$$\oint_{\partial B_R(0)} \varphi(z) z^m \, dz = -\oint_{\partial B_R(0)} \sum_{n=0}^{\infty} z^{m-n-1} a^n \, dz \overset{\text{uniform convergence}}{=} -\sum_{n=0}^{\infty} \underset{\substack{\uparrow \\ \text{we may pull out} \\ a^n \text{ by continuity of} \\ \text{scalar mult.}}}{a^n} \overbrace{\oint_{B_R(0)} z^{m-n-1} \, dz}^{2\pi i \, \delta_{nm}}$$
$$= -2\pi i \, a^m,$$

and so

$$a^m = -\frac{1}{2\pi i} \oint_{B_R(0)} \varphi(z) z^m \, dz \qquad (R > \|a\|, \ m \in \mathbb{N} \cup \{0\})$$

Since $\rho(a) = \mathbb{C} \setminus \sigma(a)$ contains all $|z| > \|a\|$ and φ is holomorphic on $\rho(a)$, we may slightly decrease the radius to get

$$a^n = -\frac{1}{2\pi i} \oint_{\partial B_R(0)} \varphi(z) z^n \, dz \qquad (R > r(a), \ n \in \mathbb{N} \cup \{0\})$$

Taking the norm and applying ML lemma,

$$\|a^n\| \le R^{n+1} \underbrace{\sup_{z \in \partial B_R(0)} \|\varphi(z)\|}_{\text{bounded}} \Rightarrow \limsup_n \|a^n\|^{\frac{1}{n}} \le r(a)$$

Conversely, for $z \in \sigma(a)$,

$$(z^n \mathbb{1} - a^n) = (z\mathbb{1} - a)(z^{n-1}\mathbb{1} + \dots + a^{n-1}) \Rightarrow z^n \in \sigma(a^n) \Rightarrow |z|^n \le \|a^n\|$$
$$\Rightarrow r(a) \le \inf_n \|a^n\|^{\frac{1}{n}}$$

$\square$

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Recall the following useful facts.

① $x \in \mathcal{A}$ s.t. $\|x - \mathbb{1}\| < 1 \Rightarrow x \in G(\mathcal{A})$, $x^{-1} = \sum_{n=0}^{\infty} (\mathbb{1}-x)^n$, $\|x^{-1}\| \leq \frac{1}{1 - \|\mathbb{1} - x\|}$

② $G(\mathcal{A})$ open (since $B_{\frac{1}{\|a^{-1}\|}}(a) \subseteq G(\mathcal{A})$ $\forall a \in G(\mathcal{A})$) and

$\cdot^{-1} : G(\mathcal{A}) \to G(\mathcal{A})$ continuous

③ Let $b \in B_{\frac{1}{\|a^{-1}\|}}(a)$. Then,

- $\|b^{-1}\| \leq \dfrac{\|a^{-1}\|}{1 - \|a^{-1}\| \|a-b\|}$
- $a^{-1} - b^{-1} = a^{-1}(b-a)b^{-1} = b^{-1}(b-a)a^{-1}$
- $\|a^{-1} - b^{-1}\| \leq \dfrac{\|a^{-1}\|^2 \|a-b\|}{1 - \|a^{-1}\| \|a-b\|}$

Theorem: (Gelfand-Mazur)

If $\mathcal{A} \setminus \{0\} \subseteq G(\mathcal{A})$, then $\mathcal{A} \cong \mathbb{C}$.

Proof: Let $a \in \mathcal{A}$, $z_1 \neq z_2 \in \mathbb{C}$. We cannot have both $\begin{cases} a - z_1 \mathbb{1} = 0 \\ a - z_2 \mathbb{1} = 0 \end{cases}$, and so one of $a - z_j \mathbb{1}$ is invertible.
Since $\sigma(a) \neq \emptyset$, we find $\sigma(a)$ consists of just one point (the $z \in \mathbb{C}$ s.t. $a - z\mathbb{1} = 0$).
This is the desired map from $\mathcal{A} \to \mathbb{C}$.

□

Lemma:

Let $\{x_n\}_n \subseteq G(\mathcal{A})$ s.t. $x_n \to x$ for some $x \in \partial G(\mathcal{A})$. Then, $\|x_n^{-1}\| \to \infty$.

Proof: Suppose BWOC that $\exists M < \infty$ s.t. $\|x_n^{-1}\| < M$ for infinitely many n.
Pick an n s.t. $\|x_n^{-1}\| < M$ and $\|x - x_n\| < \frac{1}{M}$. Then,

$$\|\mathbb{1} - x_n^{-1} x\| = \|x_n^{-1} x_n - x_n^{-1} x\| \leq \|x_n^{-1}\| \, \|x_n - x\| < 1$$
$$\Rightarrow x_n^{-1} x \in G(\mathcal{A}) \Rightarrow x \in G(\mathcal{A}).$$

Contradiction!

□

Analyzing spectrums of $\mathcal{A}$ leads to questions about whether wiggling $a \in \mathcal{A}$ will change $\sigma(a)$ by a tiny amount. We answer this below.

Theorem: (Continuity of spectrum)

Let $a \in \mathcal{A}$, $\Omega \in \mathrm{Open}(\mathbb{C})$ be s.t. $\sigma(a) \subseteq \Omega$.
If b is sufficiently close to a, in particular $\|a-b\| < \sup_{z \in \Omega^c} \|(a-z\mathbb{1})^{-1}\|$
then $\sigma(b) \subseteq \Omega$.

Proof: We know the map sending $z \in \Delta(a)$ to $z \mapsto \|(a-z\mathbb{1})^{-1}\|$ is continuous, then the map sending $z \in \Omega^c$ to $z \to \|(a-z\mathbb{1})^{-1}\|$ is the restriction of a continuous map to a closed set, it is continuous (it is also bounded).
Then, $\forall z \in \Omega^c$,

$$b - z\mathbb{1} = \underbrace{(a-z\mathbb{1})}_{\text{invertible}}\underbrace{\Big(\underbrace{(a-z\mathbb{1})^{-1}(b-a)}_{\substack{\text{smaller than 1 in} \\ \text{norm by assumption} \\ \Rightarrow}} + \mathbb{1}\Big)}_{\text{invertible}}$$

So, $b - z\mathbb{1}$ is invertible, and so $\sigma(b) \subseteq \Omega$.

$\square$

6.4 Polynomial Functional Calculus

Let p be a polynomial in $\mathbb{C}$ ($p(z) = c_n z^n + \dots + c_1 z + c_0$, $c_j \in \mathbb{C}$, $n \in \mathbb{N}$).
For any $a \in \mathcal{A}$, there is a mapping from $\mathbb{C}[z] \to \mathcal{A}$; it is a $\mathbb{C}$-algebra morphism (i.e. $p_1(a)p_2(a) = (p_1 p_2)(a)$).

6.5: Holomorphic Functional Calculus

Define $\exp: \mathcal{A} \to \mathcal{A}$ via $\exp(a) = \sum_{n=0}^{\infty} \frac{1}{n!} a^n$.

We will show that this limit converges by showing the partial sums are Cauchy:

$$\left\| \sum_{n=0}^{N} \frac{1}{n!} a^n - \sum_{n=0}^{M} \frac{1}{n!} a^n \right\| \le \sum_{n=N+1}^{M} \frac{1}{n!} \|a\|^n \xrightarrow{M,N \to \infty} 0$$

Indeed, this reasoning holds $\forall$ entire functions (analytic on all of $\mathbb{C}$).

The next generalization is for $f: B_R(0) \to \mathbb{C}$ holomorphic, $R > 0$.
From complex, we may write $f(z) = \sum_{n=0}^{\infty} c_n z^n$ $(|z| \le R)$.

We claim that if $a \in \mathcal{A}$ and $r(a) < R$, then $f(a)$ converges.
We show it is again Cauchy:

$$\left\| \sum_{n=0}^{N} c_n a^n - \sum_{n=0}^{M} c_n a^n \right\| \le \sum_{n=N+1}^{M} |c_n| \underbrace{\|a^n\|}_{(\|a^n\|^{\frac{1}{n}})^n} \lesssim \sum_{n=N+1}^{M} |c_n| r(a)^n \xrightarrow{\text{as } r(a) < R} 0$$

The final generalization is for functions holomorphic on open sets containing the spectrum. We now work toward this.

Lemma:

Let $a \in \mathcal{A}$, $\alpha \in \rho(a)$, $\Omega := \mathbb{C} \setminus \{\alpha\} \in \mathrm{Open}(\mathbb{C})$. Let $\gamma : [s,t] \to \Omega$ be a ccw simple contour in Ω which surrounds $\sigma(a)$. Then,

$$\frac{1}{2\pi i} \oint_\gamma (\alpha - z)^n (z\mathbb{1} - a)^{-1} dz = (\alpha \mathbb{1} - a)^n \qquad \forall n \in \mathbb{Z}$$

Proof: $(n=0)$ We WTS $\frac{1}{2\pi i} \oint_\gamma (z\mathbb{1} - a)^{-1} dz = \mathbb{1}$. (this is a special case of "Riesz projection")

Let $R > \|a\|$, replace $\oint_\gamma$ by $\oint_{\partial B_R(0)}$, and apply the von Neumann formula $(z\mathbb{1} - a)^{-1} = \sum_{n=0}^{\infty} z^{-n-1} a^n$

$$\Rightarrow \oint_{\partial B_R(0)} (z\mathbb{1} - a)^{-1} dz = \oint_{\partial B_R(0)} \sum_{n=0}^{\infty} z^{-n-1} a^n dz \overset{\text{uniform converge}}{=} \sum_{n=0}^{\infty} a^n \oint_{\partial B_R(0)} z^{-n-1} dz$$
$$= 2\pi i \, a^0 = 2\pi i \, \mathbb{1}.$$

$(n \neq 0)$ We use a recursion formula. Let $y_n :=$ LHS. We claim
$$(\alpha \mathbb{1} - a) y_n = y_{n+1}$$

To see this, note that $\forall z \notin \sigma(A)$,

$$(z\mathbb{1} - a)^{-1} = (\alpha \mathbb{1} - a)^{-1} + (\alpha - z)(z\mathbb{1} - a)^{-1}(\alpha \mathbb{1} - a)^{-1}, \text{ and so}$$

$$y_n \equiv \frac{1}{2\pi i} \oint_\gamma (\alpha - z)^n (z\mathbb{1} - a)^{-1} dz$$
$$= \frac{1}{2\pi i} \oint_\gamma (\alpha - z)^n dz \, (\alpha \mathbb{1} - a)^{-1} + \frac{1}{2\pi i} \oint_\gamma (\alpha - z)^{n+1} (z\mathbb{1} - a)^{-1} dz \, (\alpha \mathbb{1} - a)^{-1}$$
$$\equiv y_{n+1} (\alpha \mathbb{1} - a)^{-1}$$

$\square$

Corollary:

If $R : \mathbb{C} \to \mathbb{C}$ is a rational function with poles $\{z_j\}_j \subseteq \rho(a)$, $\sigma(a) \subseteq \Omega \in \mathrm{Open}(\mathbb{C})$, R holomorphic on Ω, and $\gamma : [s,t] \to \mathbb{C}$ ccw simple contour surrounding $\sigma(a)$ in Ω, then

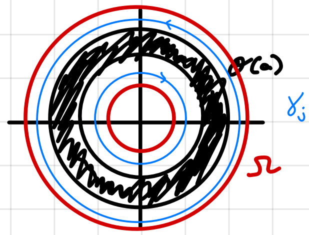
$$R(a) = \frac{1}{2\pi i} \oint_\gamma R(z) (z\mathbb{1} - a)^{-1} dz$$

Theorem: (general holo. functional calculus)

Let $a \in \mathcal{A}$, $\sigma(a) \subseteq \Omega \in \mathrm{Open}(\mathbb{C})$ ← not necessarily connected, and $f : \mathbb{C} \to \mathbb{C}$ holomorphic on Ω.

might need multiple contours → Let $\{\gamma_j\}_j : [s,t] \to \Omega$ be a finite collection of CCW simple contours that together encircle $\sigma(a)$ within Ω, i.e.

$$\frac{1}{2\pi i}\sum_{j=1}^{n} \oint_{\gamma_j} \frac{1}{z-\lambda}\, dz = \begin{cases} 1 & \lambda \in \sigma(a) \\ 0 & \lambda \notin \Omega \end{cases}$$



Define
$$f(a) := \frac{1}{2\pi i} \sum_j \oint_{\gamma_j} f(z)(z\mathbb{1}-a)^{-1}\, dz$$

Then, this definition preserves the algebraic properties; i.e. this map from $(f : \mathbb{C}\to\mathbb{C}) \mapsto (f : \mathcal{A}\to\mathcal{A})$ is a continuous unital algebra $\underbrace{\text{monomorphism}}_{\text{injective}}$

$$\left.\begin{aligned} &- f(a)g(a) = (fg)(a) \\ &- f(a)+g(a) = (f+g)(a) \\ &- (z\mapsto 1)(a) = \mathbb{1} \\ &- (z\mapsto z)(a) = a \end{aligned}\right\} \text{algebra morphism}$$

- continuous w.r.t. uniform convergence topology on the algebra of holomorphic fns.

Therefore, if $\{f_n\}_n$ is a sequence of holomorphic fns converging uniformly in compact subsets of Ω, then $f_n(a) \xrightarrow{\|\cdot\|} f(a)$.

Remark: In general, if the poles of a function f don't lie in the spectrum of an element of $\mathcal{A}$, we can give meaning to f acting on that element.

Proof: ($(z\mapsto 1)(a) = \mathbb{1}$) We have that $(z\mapsto 1)(a) = -\frac{1}{2\pi i}\sum_{j=1}^{n}\oint_{\gamma_j}(a-z\mathbb{1})^{-1}\, dz$

Note that $(a-z\mathbb{1})^{-1}(a-z\mathbb{1}) = \mathbb{1} \Rightarrow a(a-z\mathbb{1})^{-1} - z(a-z\mathbb{1})^{-1} = \mathbb{1} \Rightarrow (a-z\mathbb{1})^{-1} = \frac{a(a-z\mathbb{1})^{-1}}{z} - \frac{1}{z}\mathbb{1}$

Using the invariance $\sum_j \oint_{\gamma_j} \mapsto \oint_{\partial B_R(0)}$ for $R \gg \|a\|$, we see that

$$= \frac{1}{2\pi i}\oint_{\partial B_R(0)} \underbrace{\frac{1}{z}a(a-z\mathbb{1})^{-1}}_{\to 0 \text{ as } R\to\infty} - \frac{1}{2\pi i}\oint_{\partial B_R(0)} \frac{1}{z}\mathbb{1} = \mathbb{1}$$

($(z\mapsto z)(a) = a$) We see that

$$(z\mapsto z)(a) = -\frac{1}{2\pi i}\sum_j \oint_{\gamma_j} \underbrace{z(a-z\mathbb{1})^{-1}}_{= a(a-z\mathbb{1})^{-1} - \mathbb{1}}\, dz = a\Big(\underbrace{-\frac{1}{2\pi i}\sum_j \oint_{\gamma_j} (a-z\mathbb{1})^{-1}\, dz}_{=\mathbb{1}}\Big) + \underbrace{\frac{1}{2\pi i}\sum_j \oint_{\gamma_j} \mathbb{1}\, dz}_{=0}$$

$$= a.$$

(continuity) The proof would use continuity of the spectrum and a bound on the resolvent norm. For the rest see Rudin.

∎

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One consequence of the above is that $f(a)g(a) = g(a)f(a)$, and in particular that $af(a) = f(a)a$.

Lemma:

Let $a \in \mathcal{A}$, $\sigma(a) \subseteq \Omega \in \mathrm{Open}(\mathbb{C})$, and $f : \Omega \to \mathbb{C}$ holomorphic.
Then,
$$f(a) \in G_{\mathcal{A}} \iff 0 \notin \mathrm{im}(f|_{\sigma(a)})$$

Proof: Let $0 \notin \mathrm{im}(f|_{\sigma(a)})$. Then, $g := \frac{1}{f}$ is holo on some $\tilde{\Omega} \in \mathrm{Open}(\Omega)$ with $\sigma(a) \subseteq \tilde{\Omega}$.
We know $f(a)g(a) = g(a)f(a) = \mathbb{1}$ by the functional calculus, and so $f(a) \in G_{\mathcal{A}}$.

Now, let $0 \in \mathrm{im}(f|_{\sigma(a)})$. Then, $\exists \alpha \in \sigma(a)$ s.t. $f(\alpha) = 0$. So, $\exists h : \Omega \to \mathbb{C}$ s.t.
$$f(\lambda) = h(\lambda)(\lambda - \alpha) \quad \forall \lambda \in \Omega, \text{ and } h \text{ is holo.}$$
So,
$$f(a) = h(a)\underbrace{(a - \alpha\mathbb{1})}_{\notin G_{\mathcal{A}}} \Rightarrow f(a) \notin G_{\mathcal{A}}.$$
$\square$

Theorem (Spectral Mapping):

Let $a \in \mathcal{A}$, $\sigma(a) \subseteq \Omega \in \mathrm{Open}(\mathbb{C})$, $f : \Omega \to \mathbb{C}$ holo.
Then,
$$\sigma(f(a)) = f(\sigma(a))$$

Proof: $\forall z \in \mathbb{C}$,
$$z \in \sigma(f(a)) \iff f(a) - z\mathbb{1} \notin G_{\mathcal{A}} \iff 0 \in \mathrm{im}(\sigma(a) \ni \lambda \mapsto f(\lambda) - z)$$
$$\iff z \in \mathrm{im}(\sigma(a) \ni \lambda \mapsto f(\lambda)) = f(\sigma(a)).$$
$\square$

Remark: The spectral mapping theorem now allows us to describe a composition in the functional calculus!

Lemma:

If $a \in \mathcal{A}$ and $\sigma(a)$ does not separate $0 \leftrightarrow \infty$,
then
$$\log(a) \in \mathcal{A} \quad (\text{i.e. } \exists b \in \mathcal{A} \text{ s.t. } \exp(b) = a).$$

Proof: Define $\log : \Omega \to \mathbb{C}$ via a branch cut along the given path, and so it is holomorphic.
Apply the functional calculus.
$\square$

7. Hilbert Space

We go from Banach spaces $\rightsquigarrow$ Hilbert spaces, which are Banach spaces whose norm obeys $\square$-law.

Defn:

A **Hilbert space** is a $\mathbb{C}$-vector space w/ sesquilinear form
$$\underset{\text{anti-}\mathbb{C}\text{-linear}}{\langle \cdot,} \underset{\mathbb{C}\text{-linear}}{\cdot \rangle} : \mathcal{H}^2 \to \mathbb{C}$$
such that the associated norm induces a complete metric.

Def:

We say $\psi, \varphi \in \mathcal{H}$ are **orthogonal** if $\langle \varphi, \psi \rangle = 0$, also denoted $\varphi \perp \psi$.
$\{\varphi_i\}_i$ is **orthonormal** if $\langle \varphi_i, \varphi_j \rangle = \delta_{ij} \equiv \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Prop: $\varphi \perp \psi \iff \|\psi\| \leq \|z\varphi + \psi\| \quad \forall z \in \mathbb{C}$.

Proof: $(\Rightarrow)$ $\quad 0 \leq \|z\varphi + \psi\|^2 = |z|^2\|\varphi\|^2 + \|\psi\|^2 + 2\mathrm{Re}\langle z\varphi, \psi \rangle$.
If $\varphi \perp \psi$, then $\quad 0 \leq |z|^2\|\varphi\|^2 + \|\psi\|^2$.

$(\Leftarrow)$ If estimate holds, let $z := -\dfrac{\langle \varphi, \psi \rangle}{\|\varphi\|^2} \Rightarrow 0 \leq \|\psi\|^2 - \dfrac{|\langle \psi, \varphi \rangle|^2}{\|\psi\|^2} \Rightarrow \|z\varphi + \psi\|^2 \geq \|\psi\|^2$ if $\varphi \not\perp \psi$.

$\square$

Prop:

Let $E \subseteq \mathcal{H}$ be closed, convex, and nonempty.
Then, E contains a unique element of minimum norm.

Proof: Write $d := \inf_{x \in E} \|x\|$. Take a sequence $\{x_n\}_n \subseteq E$ s.t. $\|x_n\| \to d$ in $\mathbb{R}$.

Convexity of E gives that $\frac{1}{2}(x_n + x_m) \in E$, and so
$$\|x_n + x_m\|^2 = 4\left\|\tfrac{1}{2}(x_n + x_m)\right\|^2 \geq 4d^2$$
The parallelogram law gives
$$\underbrace{\|x_n + x_m\|^2}_{\to 4d \text{ as } n,m\to\infty} = \underbrace{2\|x_n\|^2 + 2\|x_m\|^2}_{\to 4d \text{ as } n,m\to\infty} - \|x_n - x_m\|^2$$

So, $\|x_n - x_m\| \to 0$ as $n, m \to \infty$, and it is Cauchy. So, $\exists x \in E$ s.t. $x_n \to x$.
By continuity of the norm, $\|x\| = d$. To see uniqueness, HW :).

$\square$

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We have played with Banach algebras modeled after $\mathcal{B}(X)$ with X Banach. When given the Hilbert structure, we are also given another piece of structure: the adjoint $*: \mathcal{A} \to \mathcal{A}$. This makes $\mathcal{A}$ into a C*-algebra, we will build to today.

Theorem:

Let $M \subseteq \mathcal{H}$ be a closed linear subspace. Then,

1. $M^\perp$ is also a closed linear subspace
2. $M \cap M^\perp = \{0\}$
3. $\mathcal{H} = M \oplus M^\perp$ (a $\mathbb{Z}^2$ grading)

Proof: ① $\langle \varphi, \cdot \rangle$ is linear $\Rightarrow M^\perp$ is a subspace. Also, since $\langle \varphi, \cdot \rangle$ continuous (Cauchy-Schwartz),

$$M^\perp = \bigcap_{\varphi \in M} \langle \varphi, \cdot \rangle^{-1}(\{0\}) \Rightarrow M^\perp \text{ closed}$$ (this holds even when M isn't closed)

② Let $\varphi \in M \cap M^\perp \Rightarrow \langle \varphi, \varphi \rangle = 0 \Rightarrow \|\varphi\|^2 = 0 \Rightarrow \varphi = 0$.

③ Let $\varphi \in \mathcal{H}$. Clearly, $\varphi - M$ is convex and closed, and so $\exists! \, \tilde{\varphi} \in (\varphi - M)$ of min norm. So, $\exists \psi = \varphi - \tilde{\varphi} \in M$ s.t. $\|\underbrace{\varphi - \psi}_{=:\, \zeta}\|$ is min.

Minimality gives $\|\zeta\| \le \|\zeta + \xi\| \;\; \forall \xi \in M \overset{\text{Claim 7.5}}{\Rightarrow} \zeta \in M^\perp$. So, $\varphi = \psi + \zeta \in M \oplus M^\perp$.

$\square$

Prop:

Let $W \subseteq \mathcal{H}$ be a linear subspace. Then, $(\overline{W})^\perp = W^\perp$.

Proof: $W \subseteq \overline{W} \Rightarrow (\overline{W})^\perp \subseteq W^\perp$. For the other direction, let $v \in W^\perp$ and $\{w_n\}_n \subseteq W$ converging to some $w \in \overline{W}$. Then, $\langle v, w \rangle = \langle v, \lim_{n\to\infty} w_n \rangle = \lim_{n\to\infty} \langle v, w_n \rangle = \lim_{n\to\infty} 0 = 0$.

$\square$

Prop:

Let $W \subseteq \mathcal{H}$ be a linear subspace. Then, $(W^\perp)^\perp = \overline{W}$.

Proof: ($\supseteq$) Let $w \in \overline{W}$. Then, $\langle w, v \rangle = 0 \;\; \forall v \in (\overline{W})^\perp = W^\perp$. So, $w \in (W^\perp)^\perp$.

(c) Write $\mathcal{H} = \overline{W} \oplus (\overline{W})^\perp = \overline{W} \oplus W^\perp$. Since $W^\perp$ closed, we may also write $\mathcal{H} = (W^\perp)^\perp \oplus W^\perp$. So, it cannot be that $\overline{W} \subsetneq (W^\perp)^\perp$.

$\square$

7.1: Duality in Hilbert Spaces

Theorem (Riesz Representation)

$|\varphi\rangle \leftrightarrow \langle\varphi|$

$\exists$ anti-$\mathbb{C}$-linear isometric bijection $K: \mathcal{H} \to \mathcal{H}^*$ sending $\varphi \mapsto \langle \varphi, \cdot \rangle$

Proof: It's clearly anti-$\mathbb{C}$-linear. To see isometric, we WTS $\|K(\varphi)\|_{op} = \|\varphi\|$.
By definition, $\|K(\varphi)\|_{op} = \sup_{\substack{\psi \in \mathcal{H} \\ \|\psi\| = 1}} \Big\{ \underbrace{|K(\varphi)(\psi)|}_{\substack{\leq \|\varphi\|\,\|\psi\| \\ \text{by C.S.}}} \Big\} \leq \|\varphi\|$

For the other direction, $K(\varphi)\left(\frac{\varphi}{\|\varphi\|}\right) = \frac{1}{\|\varphi\|} \langle \varphi, \varphi \rangle = \|\varphi\| \Rightarrow \|\varphi\| \leq \|K(\varphi)\|_{op}$.

Now, we know all linear isometries are injective. For surjectivity, let $\lambda \in \mathcal{H}^*$.
If $\lambda \equiv 0$, then $\lambda(\psi) = \langle 0, \psi \rangle$ $\forall \psi$. So, suppose $\lambda \neq 0$. Since λ is continuous, $N := \ker(\lambda) \neq \mathcal{H}$ is a closed linear subspace. Write $\mathcal{H} = N \oplus N^\perp$ and let $z \in N^\perp \setminus \{0\}$.
For all $\psi \in \mathcal{H}$ we see $[(\lambda\psi)z - (\lambda z)\psi] \in N$. Since $z \in N^\perp$, we see

$$0 = \langle z, (\lambda\psi)z - (\lambda z)\psi \rangle = (\lambda\psi)\|z\|^2 - (\lambda z)\langle z, \psi \rangle \Rightarrow \lambda\psi = \left\langle \frac{\overline{(\lambda z)}}{\|z\|^2} z, \psi \right\rangle.$$

So, $\lambda = K\left(\frac{\overline{(\lambda z)}}{\|z\|^2} z\right)$.

$\square$

This exhibits a $\mathbb{C}$-linear isometric bijection from $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^*)$ sending
$$A \mapsto \underbrace{K A K^{-1}}_{\text{adjoint}}$$

Alternatively, we can characterize A by its matrix elements.

Prop:

If $A \in \mathcal{B}(\mathcal{H})$ and $\langle \varphi, A\varphi \rangle = 0$ $\forall \varphi \in \mathcal{H}$, then $A = 0$.

Proof: We have $\langle \varphi + \psi, A(\varphi + \psi) \rangle = 0 \Rightarrow \langle \varphi, A\psi \rangle + \langle \psi, A\varphi \rangle = 0$.
Setting $\varphi = i\varphi$,
$$-i\langle \varphi, A\psi \rangle + i\langle \psi, A\varphi \rangle = 0.$$
Together, the above equations imply $\langle \varphi, A\psi \rangle = 0$ $\forall \varphi, \psi$.
Taking $\varphi = A\psi$, $\|A\psi\|^2 = 0$ $\forall \psi \Rightarrow A = 0$.

$\square$

Corollary: If $A, B \in \mathcal{B}(\mathcal{H})$ s.t. $\langle \varphi, A\varphi \rangle = \langle \varphi, B\varphi \rangle$ $\forall \varphi$, then $A = B$.

Theorem:

If $f: \mathcal{H}^2 \to \mathbb{C}$ is a bounded, sesquilinear map s.t.

$$S := \sup_{\|\varphi\| = \|\psi\| = 1} \left\{ |f(\varphi, \psi)| \right\} < \infty$$

then $\exists F \in \mathcal{B}(\mathcal{H})$ s.t. $f(\varphi, \psi) = \langle F\varphi, \psi \rangle$ $\forall \varphi, \psi \in \mathcal{H}$. Furthermore, $\|F\|_{op} = S$.

Proof: $|f(\varphi, \psi)| \leq S \, \|\varphi\| \, \|\psi\|$. So, $f(\varphi, \cdot) \in \mathcal{H}^*$ with $\|f(\varphi, \cdot)\|_{op} \leq S \|\varphi\|$ $\forall \varphi$.

By Riesz, $\exists z \in \mathcal{H}$ s.t. $\langle z, \cdot \rangle = f(\varphi, \cdot)$. Call $z =: F\varphi = K^{-1}(f(\varphi, \cdot))$.
So, $F: \mathcal{H} \to \mathcal{H}$ is $\mathbb{C}$-linear and bounded with $\|F\|_{op} \leq S$.
Also, $|f(\varphi, \psi)| = |\langle F\varphi, \psi \rangle| \underset{\text{C.S.}}{\leq} \|F\varphi\| \, \|\psi\| \leq \|F\|_{op} \|\varphi\| \, \|\psi\| \Rightarrow S \leq \|F\|_{op}$.

$\square$

Now, for any $A \in \mathcal{B}(\mathcal{H})$, we may define $f(\varphi, \psi) := \langle \varphi, A\psi \rangle$ as a bounded, sesquilinear map w/ $S = \|A\|_{op}$. By the above, we get some $F \in \mathcal{B}(\mathcal{H})$ s.t.

$$\langle \varphi, A\psi \rangle = \langle F\varphi, \psi \rangle \quad \forall \varphi, \psi \in \mathcal{H}.$$

Define the adjoint of A to be $A^* = F$.

We have exhibited an anti-$\mathbb{C}$-linear, isometric involution $*: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ s.t.

① $(A+B)^* = A^* + B^*$ ② $(\alpha A)^* = \bar{\alpha} A^*$ ③ $(AB)^* = B^* A^*$ ④ $(A^*)^* = A$

We call Banach algebras with an anti-$\mathbb{C}$-linear involution $*$-algebras.
There is one more piece of structure.

Theorem: (the C-$*$ identity)

If $A \in \mathcal{B}(\mathcal{H})$, then $\|A\|^2 = \| \underbrace{A^*A}_{=: |A|^2} \|$.

Proof. $\|A\varphi\|^2 = \langle A\varphi, A\varphi \rangle = \langle \varphi, |A|^2 \varphi \rangle \leq \|\varphi\|^2 \, \| |A|^2 \|$
But also $\| |A|^2 \| = \|A^*A\| \leq \|A^*\| \, \|A\| = \|A\|^2$.

$\square$

This leads us to the following definition.

Def:
A C-$*$-algebra is a Banach algebra w/ an anti-$\mathbb{C}$-linear involution obeying the C-$*$ identity $\|a\|^2 = \|a^*a\|$.

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The additional structure of a C-*-algebra allows for a continuous functional calculus, which is the direction we are moving toward.

7.2: Kernels and Images

Prop:

$$\ker(A^*) = \operatorname{im}(A)^\perp \quad \text{and} \quad \ker(A) = \operatorname{im}(A^*)^\perp$$

Proof: $A^*\psi = 0 \iff \langle \varphi, A^*\psi\rangle = 0 \ \forall \varphi \iff \langle A\varphi, \psi\rangle = 0 \ \forall \varphi \iff \psi \in \operatorname{im}(A)^\perp$
Since $A^{**} = A$, the other holds.

$\square$

Prop:

$$\ker(A) = \ker(|A|^2)$$

$|A|^2 = A^*A$

Proof: $\varphi \in \ker(A) \iff A\varphi = 0 \Rightarrow A^*A\varphi = 0 \Rightarrow \varphi \in \ker(|A|^2)$
$\varphi \in \ker(|A|^2) \iff A^*A\varphi = 0 \Rightarrow \langle \varphi, A^*A\varphi\rangle = 0 \Rightarrow \langle A\varphi, A\varphi\rangle = 0$
$\Rightarrow \|A\varphi\| = 0 \Rightarrow \varphi \in \ker(A)$

$\square$

Defn: (C-*-algebra stuff)

$a \in \mathcal{A}$ is positive iff $\exists b \in \mathcal{A}$ s.t. $a = |b|^2$.
$a \in \mathcal{A}$ is self adjoint iff $a = a^*$.
$a \in \mathcal{A}$ is normal iff $|a|^2 = |a^*|^2 \iff [a, a^*] = 0$
$p \in \mathcal{A}$ is idempotent iff $p^2 = p$.
$p \in \mathcal{A}$ is an orthogonal projection iff $p^2 = p^* = p$.
$u \in \mathcal{A}$ is unitary iff $|u|^2 = |u^*|^2 = \mathbb{1}$.

Prop:
① In a C-*-algebra, $a \in \mathcal{A}$ is positive iff $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$
② $A \in \mathcal{B}(\mathcal{H})$ is positive iff the map $\varphi \mapsto \langle \varphi, A\varphi\rangle$ is positive (i.e. $\langle \varphi, A\varphi\rangle \geq 0 \ \forall \varphi$).

Prop:
① a self-adjoint $\Rightarrow \sigma(a) \subseteq \mathbb{R}$ ② $p \in \mathcal{A}$ idempotent $\Rightarrow \sigma(p) \subseteq \{0, 1\}$
③ u unitary $\Rightarrow \sigma(u) \subseteq S^1$

Proofs: HW!

$\square$

Lemma:

The following are equivalent:

(1) $\mathrm{im}(A) \in \mathrm{Closed}(\mathcal{H})$

(2) $0 \notin \sigma(|A|^2)$ or 0 is an isolated point of $\sigma(|A|^2)$

(3) $\exists\, \varepsilon > 0$ s.t. $\|A\varphi\| \geq \varepsilon \|\varphi\|$ $\forall\, \varphi \in \ker(A)^\perp$

Proof: (1) $\Rightarrow$ (3) Define $\underset{\text{bijective}}{\tilde{A}} : \ker(A)^\perp \to \mathrm{im}(A)$ (closed, and so Hilbert spaces) sending $\varphi \mapsto A\varphi$. Then, $\|\tilde{A}\| \leq \|A\|$. By the inverse mapping theorem, $\tilde{A}^{-1}$ is bounded $\Rightarrow \exists\, c > 0$ s.t. $\|\tilde{A}^{-1}\varphi\| \leq c\|\varphi\|$ $\forall \varphi \in \mathrm{im}(A)$. Let $\varphi \in \ker(A)^\perp \Rightarrow \|\varphi\| \leq c\|A\varphi\|$.
Let $\varepsilon = \frac{1}{c}$.

(3) $\Rightarrow$ (1) Take $\{\varphi_n\}_n \subseteq \mathrm{im}(A)$ converging to $\varphi \in \mathcal{H}$. If $\varphi = 0$ we are done; so, suppose $\varphi \neq 0$. $\exists \{z_n\}_n \subseteq \mathcal{H}$ s.t. $A z_n = \varphi_n$. If we take a subsequence of $\{z_n\}_n$ not in the kernel, (slightly wrong. need to project)

$$\|z_n - z_m\| \leq \frac{1}{\varepsilon}\|A(z_n - z_m)\| \leq \frac{1}{\varepsilon}\underbrace{\|\varphi_n - \varphi_m\|}_{\text{cauchy}}$$

So, $\{z_n\}_n$ is Cauchy and converges to some $z \in \mathcal{H}$. Then,

$$Az = A\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} A z_n = \varphi \quad \text{by (3). So, } \varphi \in \mathrm{im} A.$$

(2) $\Leftrightarrow$ (3) We know $\ker A = \ker |A|^2$ and $|A|^2 \geq 0$. (2) gives $\exists\, \varepsilon > 0$ s.t. $|A|^2 \geq \varepsilon \mathbb{1}$ over $\ker(A)^\perp$, i.e. $|A|^2 - \varepsilon\mathbb{1} \geq 0$ over $\ker(A)^\perp$
$\Leftrightarrow \langle \varphi, (|A|^2 - \varepsilon\mathbb{1})\varphi\rangle \geq 0$ $\forall \varphi \in \ker(A)^\perp \Leftrightarrow \|A\varphi\|^2 \geq \varepsilon\|\varphi\|^2 \Leftrightarrow$ (3). $\square$

7.3: Bases

Prop: (Pythagoras)

If $\{\varphi_i\}_{i=1}^{\hat{n}} \subseteq \mathcal{H}$ is orthonormal, then $\|\varphi\|^2 = \sum_{i=1}^{\hat{n}} |\langle \varphi_i, \varphi\rangle|^2 + \left\|\varphi - \sum_{i=1}^{\hat{n}} \langle \varphi_i, \varphi\rangle \varphi_i\right\|^2$ $\forall \varphi \in \mathcal{H}$.

Proof: Define $P := \sum_{i=1}^{\hat{n}} \overbrace{\varphi_i \otimes \varphi_i^*}^{|\varphi_i\rangle\langle\varphi_i|}$. Then, $P = P^*$ clearly. To see idempotent,

$$P^2 = \sum_{i,j=1}^{\hat{n}} \varphi_i \otimes \varphi_i^* \varphi_j \otimes \varphi_j^* = \sum_{i,j=1}^{\hat{n}} \overbrace{\langle \varphi_i, \varphi_j\rangle}^{\delta_{ij}} \varphi_i \otimes \varphi_j^* = \sum_{i=1}^{\hat{n}} \varphi_i \otimes \varphi_i^* = P.$$

Now, $\|\varphi\|^2 = \|(P + \underbrace{(\mathbb{1} - P)}_{=: P^\perp})\varphi\|^2 = \|P\varphi\|^2 + \|P^\perp\varphi\|^2 + 2\,\mathrm{Re}\{\underbrace{\langle P\varphi, P^\perp\varphi\rangle}_{\substack{= \langle \varphi, PP^\perp\varphi\rangle \\ = \langle \varphi, (P - P^2)\varphi\rangle = 0}}\}$ $\square$

Defn:

Let $\{E_n\}_n \subseteq \mathrm{Closed}(\mathcal{H})$ be a sequence of closed vector subspaces. Then

$$\mathcal{H} = \bigoplus_{n\in\mathbb{N}} E_n \iff E_n \perp E_m \ \forall n \neq m \text{ and } \overline{\mathrm{Span}\left(\bigcup_n E_n\right)} = \mathcal{H}.$$

Lemma:

Let $\{\varphi_i\}_i \subseteq \mathcal{H}$ be mutually orthogonal with $\sum_{i=1}^{\infty} \|\varphi_i\|^2 < \infty$. Then, $\varphi := \lim_{n\to\infty} \sum_{i=1}^{n} \varphi_i$ exists and $\|\varphi\|^2 = \sum_{i=1}^{\infty} \|\varphi_i\|^2$

Proof: By assumption, $\{\sum_{i=1}^{n} \varphi_i\}_n$ is Cauchy. The second part follows by continuity of the norm. $\square$

Theorem:

If $\{E_n\}_n \subseteq \text{Closed}(\mathcal{H})$ is a seq. of vector subspaces s.t. $\mathcal{H} = \bigoplus_n E_n$, then $\forall \varphi \in \mathcal{H}$,

$$\varphi = \lim_{n\to\infty} \sum_{i=1}^{n} P_{E_i}\varphi \quad \text{and} \quad \|\varphi\|^2 = \sum_{i=1}^{\infty} \|P_{E_i}\varphi\|^2$$

Proof: $\|\varphi\|^2 \geq \sum_{i=1}^{n} \|P_{E_i}\varphi\|$ $\forall n$, so apply the above lemma to see that $\sum_{j=1}^{\infty} P_{E_j}\varphi$ $\exists$.

By pairwise orthogonality, $\left(\varphi - \sum_{i=1}^{n} P_{E_i}\varphi\right) \perp E_m$ $\forall m \leq n$.
As $n\to\infty$, $\left(\varphi - \sum_{j=1}^{\infty} P_{E_j}\varphi\right) \perp E_m$ $\forall m \in \mathbb{N}$ $\Rightarrow$ $\left(\varphi - \sum_{j=1}^{\infty} P_{E_j}\varphi\right) \perp \mathcal{H}$ since $\mathcal{H} = \bigoplus_m E_m$. $\square$

Defn:

contains all other orthogonal sets

An orthogonal basis of $\mathcal{H}$ is a maximal orthogonal set.

Prop:

Every Hilbert space $\mathcal{H}$ contains an orthogonal basis.

Proof: HW! $\square$

Prop:

If $\{\varphi_\alpha\}_{\alpha\in A}$ is an orthonormal basis, then $\forall \psi \in \mathcal{H}$,

$$\psi = \sum_{\alpha\in A} \langle \varphi_\alpha, \psi\rangle \varphi_\alpha \quad \text{and} \quad \|\psi\|^2 = \sum_{\alpha\in A} |\langle \varphi_\alpha, \psi\rangle|^2$$

Proof: Uncountable case is in Reed & Simon

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We call the isomorphism in the category of Hilbert spaces to be unitary. These are linear bijections that preserve the inner product.

Def:

A metric space X is separable iff there exists a countable dense subset.

Theorem:

A Hilbert space $\mathcal{H}$ is separable $\Leftrightarrow$ it has a countable ONB

Proof: ($\Leftarrow$) Let $\{\varphi_n\}_n$ be a countable ONB. Any $\psi \in \mathcal{H}$ may be approximated as a finite linear combination with rational coefficients. So, the set

$$\left\{ \psi \in \mathcal{H} \ : \ \psi = \sum_{n \in I} q_n \varphi_n \ , \ |I| < \infty, \ q_n \text{ rational} \right\}$$

is countable and dense.

($\Rightarrow$) Let $\{\varphi_n\}_n$ be countable and dense. We may reduce it to a countable, linearly independent, dense set. Apply Gram-Schmidt.

$\square$

Remark: If the basis has finitely many elements, say n, then $\mathcal{H} \cong \mathbb{C}^n$. If the basis is infinitely countable, then $\mathcal{H} \cong \ell^2(\mathbb{N} \to \mathbb{C}) \cong \mathbb{C}^\infty$. The unitary map

$$\psi \mapsto \left(\langle \varphi_1, \psi \rangle, \langle \varphi_2, \psi \rangle, \ldots \right)$$

realizes this relation. It's square summable since $\|\psi\|^2 < \infty$. To see it preserves the inner product, we observe

$$\langle U\psi, U\varphi \rangle_{\ell^2(\mathbb{N} \to \mathbb{C})} = \sum_{n \in \mathbb{N}} \overline{(U\psi)_n} \, (U\varphi)_n = \sum_{n \in \mathbb{N}} \langle \psi, \varphi_n \rangle \langle \varphi_n, \varphi \rangle$$

$$= \Big\langle \psi, \underbrace{\sum_{n \in \mathbb{N}} \mathbb{P}_{\varphi_n}}_{= \mathbb{1}} \varphi \Big\rangle = \langle \psi, \varphi \rangle$$

Defn:

- $B \subseteq X$ is a Hamel basis if $\forall \psi \in X$, $\psi = \sum_{j=1}^{n} \alpha_j b_j$ for some $n \in \mathbb{N}$, $\alpha_j \in \mathbb{C}$, $b_j \in B$.

⚠ Any ∞-dim Banach space has only uncountable Hamel bases.

- $B \subseteq X$ is a Schauder basis if $\forall \psi \in X$, $\psi = \sum_{i \in I} \alpha_i b_i$ for some $\alpha_i \in \mathbb{C}$, $b_i \in B$.

7.4: Direct Sums & Tensor Products

Def:

Given a sequence of Hilbert spaces $\{\mathcal{H}_n\}_{n=1}^{\infty}$, define

$$\mathcal{H} := \left\{ (x_n)_n \; : \; x_n \in \mathcal{H}_n \;\; \forall n \in \mathbb{N} \text{ and } \sum_{n \in \mathbb{N}} \|x_n\|_{\mathcal{H}_n}^2 < \infty \right\}$$

to be the direct sum. On $\mathcal{H}$ we define the inner product

$$\langle x, y \rangle_{\mathcal{H}} := \sum_{n \in \mathbb{N}} \langle x_n, y_n \rangle_{\mathcal{H}_n}$$

Prop:

$\mathcal{H}$ is complete.

Proof:

Let $\{x_\ell\}_{\ell \in \mathbb{N}}$ be Cauchy in $\mathcal{H}$. Then, $\forall \varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall k, \ell \geq N_\varepsilon$

$$\varepsilon^2 \geq \| x_\ell - x_k \|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \|x_{n\ell} - x_{nk}\|_{\mathcal{H}_n}^2 \Rightarrow \|x_{n\ell} - x_{nk}\|_{\mathcal{H}_n} \leq \varepsilon$$

So, $\forall n \in \mathbb{N}$ we know $\{x_{n\ell}\}_\ell$ is Cauchy $\Rightarrow \exists y_n \in \mathcal{H}_n$ s.t. $x_{n\ell} \to y_\ell$.
Define $y := (y_1, y_2, \ldots)$. Then,

$$\| x_\ell - y \|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \|x_{n\ell} - y_n\|_{\mathcal{H}_n}^2 = \sum_{n \in \mathbb{N}} \left\| x_{n\ell} - \left(\lim_{k \to \infty} x_{nk}\right)\right\|_{\mathcal{H}_n}^2$$

continuity of $\|\cdot\|_{\mathcal{H}_n}$

$$= \sum_{n=1}^{\infty} \lim_{k \to \infty} \|x_{n\ell} - x_{nk}\|_{\mathcal{H}_n}^2 \underset{\text{Fatou on counting measure}}{\leq} \liminf_{k \to \infty} \sum_{n \in \mathbb{N}} \|x_{n\ell} - x_{nk}\|_{\mathcal{H}_n}^2 = \liminf_{k \to \infty} \|x_\ell - x_k\|^2$$

This can be made arbitrarily small because Cauchy, and so $x_\ell \to y$.
Also,

$$\|y\|^2 = \sum_n \|y_n - x_{n\ell} + x_{n\ell}\|_{\mathcal{H}_n}^2 \leq \sum_n \left(\|y_n - x_{n\ell}\| + \|x_{n\ell}\|\right)^2 < \infty,$$

sum of two square-summable sequences is square-summable

and so $y \in \mathcal{H}$. ∎

Lemma:

If A, B are two disjoint & countable sets, then $\ell^2(A \sqcup B) \cong \ell^2(A) \oplus \ell^2(B)$

Proof. $\{e_i\}_{i \in A \sqcup B}$ is an ONB for LHS (the Kronecker delta basis). They map to $(e_i, 0)$ or $(0, e_i)$ if $i \in A$ or $i \in B$, respectively.

$\square$

Defn:

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Define the vector space

$$\tilde{\mathcal{H}} := \mathcal{H}_1 \otimes \mathcal{H}_2 = \left\{ \varphi : \varphi = \sum_{i,j=1}^{n} \alpha_{ij}\, e_i \otimes f_j \quad \text{where } e_i, f_j \text{ bases of } \mathcal{H}_1 \text{ and } \mathcal{H}_2 \right\}$$

Define $\langle e_i \otimes f_j, e_k \otimes f_\ell \rangle_{\tilde{\mathcal{H}}} := \langle e_i, e_k \rangle_{\mathcal{H}_1} \langle f_j, f_k \rangle_{\mathcal{H}_2}$ and extend linearly.

This may not be complete, so let $\mathcal{H} :=$ completion of $\tilde{\mathcal{H}}$ w.r.t. $\langle \cdot, \cdot \rangle_{\tilde{\mathcal{H}}}$.
We call $\mathcal{H}$ the Hilbert tensor product.

So, $\ell^2(\mathbb{Z}^2) \cong \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$.
particle in 2D looks like two particles in 1D

Lemma:

Let A, B be two countable sets. Then, $\ell^2(A \times B) \cong \ell^2(A) \otimes \ell^2(B)$

Proof: Map $e_{(a,b)} \mapsto e_a \otimes e_b$.

$\square$

Def:

Given a Hilbert space $\mathcal{H}$, we can form the Fock space $\mathcal{F}(\mathcal{H})$ via

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}, \quad \text{where } \mathcal{H}^{\otimes 0} \equiv \mathbb{C}.$$

We think of $\mathcal{F}(\mathcal{H})$ as the space to describe having countably many particles.
There are two important subspaces

$\mathcal{H}_1 \wedge \mathcal{H}_2$ is $(\mathcal{H}_1 \otimes \mathcal{H}_2) \,/\,$ anything not antisymmetric in $\mathcal{H}_1 \otimes \mathcal{H}_2$

① Exterior algebra $\quad \Lambda \mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\wedge n}$, where $\mathcal{H}^{\wedge n} = \mathcal{H} \wedge \dots \wedge \mathcal{H}$

As an example $\ell^2(A) \otimes \ell^2(A) \cong \ell^2(A^2) \ni \varphi$ being antisymmetric means $\varphi(a, \tilde{a}) = -\varphi(\tilde{a}, a)$. This describes the space of identical fermions.

② Symmetric subspace $\quad$ Same thing, but not antisymmetric

9: Bounded Operators on Hilbert Spaces

Weak & strong topologies on $\mathcal{B}(\mathcal{H})$

Def:

The strong operator topology is the initial topology generated by all maps
$$E_\psi : \mathcal{B}(\mathcal{H}) \to \mathcal{H} \quad \text{sending} \quad A \mapsto A\psi, \quad \psi \in \mathcal{H}$$
In words, this is the weakest topology s.t. point evaluation is continuous.

Lemma: $A_n \xrightarrow{s} A$ strongly iff $A_n\psi \to A\psi \quad \forall \psi \in \mathcal{H}$

Def:

The weak operator topology is the initial topology generated by all maps
$$E_{\varphi\psi} : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \quad \text{sending} \quad A \mapsto \langle \varphi, A\psi \rangle, \quad \varphi, \psi \in \mathcal{H}$$
In words, this is the weakest topology s.t. the inner product is continuous.

Lemma: $A_n \xrightarrow{w} A$ weakly iff $\langle \varphi, A_n \psi \rangle_{\mathcal{H}} \to \langle \varphi, A\psi \rangle_{\mathcal{H}}$ in $\mathbb{C}$ $\forall \varphi, \psi \in \mathcal{H}$.

Remark: We still have the weak topology (in the Banach sense): the initial topology generated by $(\mathcal{B}(\mathcal{H}))^*$. The $E_{\varphi\psi}$'s are indeed $\in (\mathcal{B}(\mathcal{H}))^*$, but not all continuous linear functionals can be written this way.

Claim:

Norm convergence (uniform) $\overset{\text{submultiplicativity of norm}}{\Longrightarrow}$ strong op convergence (norm pointwise) $\overset{\text{C.S.}}{\Longrightarrow}$ weak op convergence (weak pointwise)

Let's look at some examples where the converse is false!

Prop:

Take $\ell^2(\mathbb{N})$ and define $P_j := e_j \otimes e_j^*$ to be the orthogonal projections.
Then, $P_j \to 0$ strongly but not in norm.

Proof: $\|(P_j - 0)\psi\|^2 = \|P_j \psi\|^2 = |\psi_j|^2 \to 0 \quad \forall \psi \in \ell^2(\mathbb{N})$. So, strong.

However, $\|P_j - 0\| = \|P_j\| = 1$.

$\square$

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Remark: We have that $\langle \varphi, u \otimes v^* \psi \rangle \equiv \langle \varphi, u\rangle \langle v, \psi\rangle$ definitionally.

Let's see a counterexample to the converse of strong $\Rightarrow$ weak!

Prop:

Take $\ell^2(\mathbb{N})$ and define the unilateral right shift operator

$$R(\varphi_1, \varphi_2, \dots) := (0, \varphi_1, \varphi_2, \dots) \quad \forall \varphi \in \ell^2(\mathbb{N})$$

Defined on the position basis, $Re_j = e_{j+1}$. Define $A_n := R^n$ to be shift by n.
Then, $A_n \to 0$ weakly, but not strongly.

Proof:
$$|\langle \varphi, (A_n - 0)\psi\rangle| = |\langle \varphi, R^n \psi\rangle| = \left|\sum_{m=1}^{\infty} \overline{\varphi_m} (R^n\psi)_m\right| = \left|\sum_{m=n+1}^{\infty} \overline{\varphi_m}\, \psi_{m-n}\right|$$
$$\underset{\text{C.S.}}{\leq} \underbrace{\left(\sum_{m=n+1}^{\infty} |\varphi_m|^2\right)^{\frac{1}{2}}}_{\to 0} \underbrace{\left(\sum_{m=n+1}^{\infty} |\psi_{m-n}|^2\right)^{\frac{1}{2}}}_{\|\psi\|} \to 0$$

So, $A_n \xrightarrow{w} 0$ weakly.

However, $\|A_n \varphi\|^2 = \sum_{m=1}^{\infty} |(R^n \varphi)_m|^2 = \sum_{m=n+1}^{\infty} |\varphi_{m-n}|^2 = \|\varphi\|^2 \Rightarrow \|A_n\| = 1.$

So, $A_n \not\to 0$ strongly.

$\square$

Note that $\mathcal{B}(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^*$. Each element of $\mathcal{H}^*$ is $\langle v, \cdot\rangle$ by Riesz.
The simple tensors in $\mathcal{H} \otimes \mathcal{H}^*$ are then $u \otimes v^*$ sending $w \mapsto \langle v, w\rangle u$

Recall in the finite setting that we use matrix elements to represent operators.

$$M \longleftrightarrow \{\langle e_i, M e_j\rangle\}_{i,j=1}^{n} \qquad \text{and} \qquad M = \sum_{i,j=1}^{n} M_{ij}\, e_i \otimes e_j^*$$

We ask when does the same statement hold.

Prop:

If $\mathcal{H}$ has ONB $\{\varphi_j\}_{j=1}^{\infty}$ and $A \in \mathcal{B}(\mathcal{H})$, then

$$A = \underbrace{\operatorname*{s\,lim}_{N\to\infty}}_{\text{"strong limit"}} \underbrace{\sum_{n,m=1}^{N} \langle \varphi_n, A\varphi_m\rangle\, \varphi_n \otimes \varphi_m^*}_{S_N}$$

Proof: $\|(A-S_N)\psi\|^2 = \sum_{j=1}^{\infty} |\langle \varphi_j, (A-S_N)\psi\rangle|^2$

Each
$$\langle \varphi_j, (A-S_N)\psi\rangle = \langle \varphi_j, A\psi\rangle - \sum_{n,m=1}^{N} \langle \varphi_j, \langle \varphi_n, A\varphi_m\rangle \varphi_n \otimes \varphi_m^* \psi\rangle$$
$$= \langle \varphi_j, A\psi\rangle - \sum_{n,m=1}^{N} \underbrace{\langle \varphi_j, \varphi_n\rangle}_{\delta_{jn}\chi_{\{1,\dots,N\}}(j)} \langle \varphi_n, A\varphi_m\rangle \langle \varphi_m, \psi\rangle$$

if $j \le N$:
$$= \langle \varphi_j, A\psi\rangle - \sum_{m=1}^{N} \langle \varphi_j, A\varphi_m\rangle \langle \varphi_m, \psi\rangle$$
$$= \sum_{m=1}^{\infty} \langle \varphi_j, A\varphi_m\rangle\langle \varphi_m, \psi\rangle - \sum_{m=1}^{N} \langle \varphi_j, A\varphi_m\rangle\langle \varphi_m, \psi\rangle$$
$$= \sum_{m=N+1}^{\infty} \langle \varphi_j, A_m\rangle \langle \varphi_m, \psi\rangle$$

if $j > N$: $= \langle \varphi_j, A\psi\rangle$

So,
$$\|(A-S_N)\psi\|^2 = \underbrace{\sum_{j=1}^{N} \left|\sum_{m=N+1}^{\infty} \langle \varphi_j, A_m\rangle\langle \varphi_m, \psi\rangle\right|^2}_{\to 0 \text{ by C.S.}} + \underbrace{\sum_{j=N+1}^{\infty} |\langle \varphi_j, A\psi\rangle|^2}_{\to 0}$$

9.3: Spectrum of Elements in $\mathcal{B}(\mathcal{H})$:

$\mathcal{B}(\mathcal{H})$ is still a Banach algebra, and so we have the usual stuff.

$$\sigma(A) = \{\lambda \in \mathbb{C} : (A - \lambda \mathbb{1}) \text{ is } \underline{\text{not}} \text{ invertible}\}$$

There is much more to do.

Defn: (point spectrum) this is when $A - \lambda\mathbb{1}$ fails to be injective! eigenvalues!

$$\sigma_p(A) := \{\lambda \in \mathbb{C} : \ker(A - \lambda\mathbb{1}) \neq \{0\}\}$$

$$\lambda \in \sigma_p(A) \Leftrightarrow \exists \varphi \in \mathcal{H} \setminus \{0\} \text{ s.t. } (A - \lambda\mathbb{1})\varphi = 0 \Leftrightarrow A\varphi = \lambda\varphi.$$

Defn: (continuous spectrum) this is when $A - \lambda\mathbb{1}$ fails to be surjective (but it's close)!

$$\sigma_c(A) := \left\{ \lambda \in \mathbb{C} : \begin{array}{l} \ker(A - \lambda\mathbb{1}) = \{0\}, \\ \& \operatorname{im}(A - \lambda\mathbb{1}) \neq \mathcal{H} \end{array} \text{ yet } \overline{\operatorname{im}(A - \lambda\mathbb{1})} = \mathcal{H} \right\}$$

Defn: (residual spectrum) the rest

$$\sigma_r(A) := \sigma(A) \setminus \left(\sigma_p(A) \cup \sigma_c(A)\right)$$

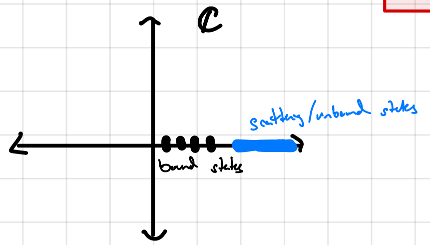
$\lambda \in \sigma_r(A) \Leftrightarrow A - \lambda\mathbb{1}$ is injective but not surjective and $\overline{\operatorname{im}(A - \lambda\mathbb{1})} \neq \mathcal{H}$.

Pictures

$\sigma_p(A)$ $\qquad$ $\sigma_c(A)$ $\qquad$ $\sigma_r(A)$

> **Aside:** for C^*-algebra information, check the following K-theory books:
> - Rørdam-Larsen
> - Murphy

H Hamiltonian of hydrogen atom



bound states are square-summable eigenvalues

unbound aren't square summable

Laplacian $-\partial^2$ on $L^2(\mathbb{R})$

$$\sigma_c(-\partial^2) = \sigma(-\partial^2) = [0, \infty)$$

with eigenvectors φ_k: $x \mapsto e^{ikx}$ with eigenvalue k^2

$$(-\partial^2)\varphi_k = k^2\varphi_k$$

Example: (multiplication operator)

Let $\mathcal{H} = \ell^2(\mathbb{N})$, and A_f be a multiplication operator

$$(A_f\psi)_n = f(n)\,\psi_n$$

Then, $\sigma_p(A) = \{f(n) : n \in \mathbb{N}\}$ and $\sigma(A) = \overline{\sigma_p(A)}$

Consider the setting $f(n) = \frac{1}{n}$. Then, $\frac{1}{n} \in \sigma(A)$ $\forall n$.

Since the spectrum is closed, $0 \in \sigma(A)$. Where is it?

<u>Claim:</u> $0 \in \sigma_c(A)$

We know that there is an "inverse" $(A^{-1}\psi)_n = n\psi_n$, but A is NOT bounded. So, A is not invertible in $\mathcal{B}(\mathcal{H})$.

<u>Example:</u> (position operator)

Let $\mathcal{H} = L^2([0,1] \to \mathbb{C})$ and X be defined by $(X\psi)(x) = x\psi(x)$ $\forall x \in [0,1]$. Since x is on compact domain, we won't ruin integrability or boundedness. So, $X \in \mathcal{B}(\mathcal{H})$. Here,

$$\sigma(X) = \sigma_c(X) = [0,1]$$

The eigenvectors are Dirac deltas, which aren't in $\mathcal{H}$! Once again, eigenvectors lying outside $\mathcal{H}$ causes eigenvalues in the continuous spectrum.

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For fun, we will next consider the adjoint of a shift operator.

~~Example~~

$$\mathcal{H} = \ell^2(\mathbb{N}) \qquad (R\psi)_n := \begin{cases} \psi_{n-1} & n > 1 \\ 0 & n = 1 \end{cases} \qquad \text{is unilateral right shift}$$

Then, $R(\psi_1, \psi_2, \ldots) = (0, \psi_1, \psi_2, \ldots)$

To see R^*,

$$\langle \psi, R^*\varphi \rangle = \langle R\psi, \varphi \rangle = \sum_{n=2}^{\infty} \overline{\psi_{n-1}}\,\varphi_n = \sum_{m=1}^{\infty} \overline{\psi_m}\,\varphi_{m+1}$$
$$= \langle \psi, L\varphi \rangle$$

where $L(\varphi_1, \varphi_2, \ldots) = (\varphi_2, \varphi_3, \ldots)$. So, $R^* = L$.

Thus, $|R|^2 = R^*R = \mathbb{1}$, yet $|R^*|^2 = RR^* = \mathbb{1} - \delta_1 \otimes \delta_1^*$

R is not unitary. The above shows it's a partial isometry.

Prop:

If $\lambda \in \mathbb{C}$ and $A \in \mathcal{B}(\mathcal{H})$, then

① $\bar{\lambda} \in \sigma_r(A^*) \Rightarrow \lambda \in \sigma_p(A)$

② $\lambda \in \sigma_p(A) \Rightarrow \bar{\lambda} \in \sigma_r(A^*) \cup \sigma_p(A^*)$

Proof: ① Let $\bar{\lambda} \in \sigma_r(A^*) \iff \overline{\operatorname{im}(A^* - \bar{\lambda}\mathbb{1})}$ is proper subset of $\mathcal{H}$

$$\iff \left(\overline{\operatorname{im}(A^* - \bar{\lambda}\mathbb{1})}\right)^\perp \neq \{0\}$$

$$\overset{(\overline{W})^\perp = W^\perp}{\iff} \left(\operatorname{im}(A^* - \bar{\lambda}\mathbb{1})\right)^\perp \overset{\text{Claim } 7.8}{=} \ker(A - \lambda\mathbb{1})$$

So, $A - \lambda\mathbb{1}$ is not injective

② For the reverse, we recall here that either $\bar{\lambda} \in \sigma_r(A^*)$ or $A^* - \bar{\lambda}\mathbb{1}$ isn't injective.

$\square$

Theorem:

If $a \in \mathcal{A}$ in a C^*-algebra has $a^* = a$, then $\sigma(a) \subseteq \mathbb{R}$.

Proof: see below :)

$\square$

Theorem: (perpendicular eigenspaces of self-adjoint operators)

If $A = A^* \in \mathcal{B}(\mathcal{H})$ then $\sigma_r(A) = \emptyset$ and if $\lambda, \mu \in \sigma_p(A)$ with $\lambda \neq \mu$, then $\ker(A - \lambda\mathbb{1}) \perp \ker(A - \mu\mathbb{1})$.

Proof: Suppose $\lambda \in \sigma_r(A)$. Then, $\lambda \in \sigma_r(A^*) \Rightarrow \bar{\lambda} \in \sigma_p(A)$.
Since $A = A^* \Rightarrow \lambda \in \mathbb{R}$, we see $\lambda \in \sigma_p(A) \cap \sigma_r(A)$.
However, these are disjoint. $\times$

Now, let $A\psi = \lambda\psi$, $A\varphi = \mu\varphi$ with $\lambda \neq \mu$. Suppose WOLOG that $\lambda \neq 0$.
Then,

$$\langle \psi, \varphi \rangle = \frac{1}{\lambda}\langle \lambda\psi, \varphi \rangle = \frac{1}{\lambda}\langle A\psi, \varphi \rangle = \frac{1}{\bar{\lambda}}\langle \psi, A\varphi \rangle = \frac{\mu}{\bar{\lambda}}\langle \psi, \varphi \rangle$$

Either $\mu/\bar{\lambda} = 1 \Rightarrow \mu = \bar{\lambda} = \lambda$, which cannot be, or $\langle \psi, \varphi \rangle = 0$.

$\square$

More about C^*-algebras

In the below, $\mathcal{A}$ is a C^*-algebra (i.e. $\|a\|^2 = \|a^*a\| = \| |a|^2 \|$)

Def:

$a \in \mathcal{A}$ is

- normal $\Leftrightarrow |a|^2 = |a^*|^2 \Leftrightarrow [a, a^*] = 0$
- self-adjoint $\Leftrightarrow a^* = a$
- positive $\Leftrightarrow a \geq 0 \Leftrightarrow \exists b \in \mathcal{A}$ s.t. $a = |b|^2$
- unitary $\Leftrightarrow |a|^2 = |a^*|^2 = \mathbb{1}$
- isometry $\Leftrightarrow |a|^2 = \mathbb{1}$
- co-isometry $\Leftrightarrow |a^*|^2 = \mathbb{1}$
- idempotent $\Leftrightarrow a^2 = a$
- orthogonal projection (or self-adjoint projection) $\Leftrightarrow a^* = a^2 = a$
- partial isometry $\Leftrightarrow |a|^2$ is an idempotent (automatically a s.a. proj)

Prop: $a = 0 \Leftrightarrow |a|^2 = 0$

Proof: $(\Rightarrow)$ duh. $(\Leftarrow)$ $\| |a|^2 \| = 0 \overset{C^*\text{-identity}}{\Longrightarrow} \|a\|^2 = 0 \Rightarrow a = 0$. $\square$

partial iso!

Lemma: a is a partial isometry $\Leftrightarrow a^*$ is a partial isometry.
For such elements,

$$a \overset{(1)}{=} a a^* a \overset{(2)}{=} a a^* a a^* a = |a^*|^2 a |a|^2$$

If we write $p = |a|^2$, $q = |a^*|^2$, then

$$a = ap = qa$$

Proof: (1) $|(\mathbb{1} - |a^*|^2) a|^2 = [(\mathbb{1} - |a^*|^2)a]^* [(\mathbb{1} - |a^*|^2)a] = a^*(\mathbb{1} - |a^*|^2)^2 a$
$= a^*(\mathbb{1} + |a^*|^4 - 2|a^*|^2) a = a^*a - 2a^*aa^*a + a^*aa^*aa^*a$
$= |a|^2 - 2|a|^2 + |a|^2 = 0$

$\Rightarrow (\mathbb{1} - |a^*|^2) a = 0$

So, $(aa^*)a = |a^*|^2 a = (\mathbb{1} - \cancel{(\mathbb{1} - |a^*|^2)})a = a$

To see $|a^*|^2$ is idempotent,

$$|a^*|^4 = \underbrace{aa^*a}_{a}a^* = aa^* = |a^*|^2 \Rightarrow a^* \text{ partial isometry.}$$

(2) $(aa^*a)a^*a = aa^*a = a$ by (1). $\square$

Remark: In $\mathcal{B}(\mathcal{H})$, we think of the partial iso. A as mapping
$\operatorname{im}|A|^2 \to \operatorname{im}|A^*|^2$.

s.a. proj! **Prop:** If $p = p^* = p^2$ and $p \neq 0$ then $\|p\| = 1$.

Proof: C^* identity. $\square$

unitary! **Prop:**

If $u \in \mathcal{A}$ is unitary, then $\|u\| = 1$ and $\sigma(u) \subseteq S^1$.

Proof: $\|u\| = 1$ from C^* identity. Suppose $\lambda \in \sigma(u)$. $\lambda \neq 0$ as u is invertible $(u^*u = uu^* = 1 \Rightarrow u^* = u^{-1})$. Also, since $(\cdot)^{-1}$ is holomorphic, the spectral mapping theorem gives

$$\lambda^{-1} \in \sigma(u^{-1}) = \sigma(u^*) = \overline{\sigma(u)} \Rightarrow \frac{1}{\bar{\lambda}} \in \sigma(u).$$

So, since $r(u) \leq \|u\| = 1$, we see $|\lambda| \leq 1$ and $|\lambda^{-1}| \leq 1$. Thus, $|\lambda| = 1$.

$\square$

self-adjoint! **Prop:**

If $a = a^*$, then $r(a) = \|a\|$.

Proof: $\|a\|^2 \underset{C^* \text{ id.}}{=} \|a^*a\| = \|a^2\| \Longrightarrow \|a\|^{2^n} = \|a^{2^n}\| \quad \forall n \in \mathbb{N}$

So, by Gelfand, $r(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|a\|^{\frac{2^n}{2^n}} = \|a\|$.

$\square$

Corollary

There is a <u>unique</u> norm on a C^* algebra.

Proof: $\|a\|^2 = \||a|^2\| = r(|a|^2)$ is independent of the norm! $\square$

Claim:

If the norm obeys $\|a\|^2 \leq \||a|^2\|$ $\forall a$, then it obeys C^* id.

Proof: HW! $\square$

Theorem:

If $a \in \mathcal{A}$ in a C^*-algebra has $a^* = a$, then $\sigma(a) \subseteq \mathbb{R}$.

Proof: Note that $z \mapsto e^{iz}$ is entire, and so by the "entire functional calculus",

$$e^{ia} = \sum_{n=0}^{\infty} \frac{i^n}{n!} a^n \in \mathcal{A}$$

We WTS e^{ia} is unitary; namely that $(e^{ia})^* = e^{-ia}$.

$$(e^{ia})^* = \left(\sum_{n=0}^{\infty} \frac{i^n}{n!} a^n \right)^* \underset{\substack{\|a\| = \|a^*\| \\ \Rightarrow (\cdot)^* : \mathcal{A} \to \mathcal{A} \\ \text{is continuous}}}{=} \sum_{n=0}^{\infty} \left(\frac{i^n}{n!} a^n \right)^* = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} a^n = e^{-ia}$$

So, e^{ia} is unitary! $e^{ia} e^{-ia} = e^{-ia} e^{ia} = \mathbb{1}$ by homomorphism of functional calculus. By the unitary prop. $\sigma(e^{ia}) \subseteq S^1$!

Let $\lambda \in \sigma(a)$. Then, $e^{i\lambda} \in \sigma(e^{ia})$ by spectral mapping theorem. Thus, $|e^{i\lambda}| = 1 \Rightarrow \lambda \in \mathbb{R}$.

$\square$

Back to $\mathcal{B}(\mathcal{H})$: polar decomposition

We seek a decomposition analogous to $\underbrace{z = e^{i\theta}|z|}_{\text{polar}}$ and in $\mathbb{R}^n$:

SVD $$A = \underbrace{W \Sigma V^*}_{\text{unitary, positive}} = \underbrace{(WV^*)}_{\text{unitary}} \underbrace{V \Sigma V^*}_{\text{positive}}$$

In infinite-dimensional $\mathcal{H}$, we will see that for any $A \in \mathcal{B}(\mathcal{H})$ we will have

$$A = U|A| = U\sqrt{|A|^2} = U\sqrt{A^*A}$$ for some partial isometry U.

If we require $\ker A = \ker U$, then U is unique!

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partial isometries!

Lemma:

$U \in \mathcal{B}(\mathcal{H})$ is a partial isometry $\iff$ U is an isometry on $\ker(U)^\perp$ (i.e. $\|U\psi\| = \|\psi\|$ $\forall \psi \in \ker(U)^\perp$)

Proof: ($\Rightarrow$) Assume $|U|^2$ is idempotent $\Rightarrow |U^*|^2$ is also idempotent. Since $\ker(U) = \ker(|U|^2)$. So, if $\psi \in \ker(U)^\perp$ then by decomp. of $\mathcal{H}$, $\psi \in \operatorname{im}(|U|^2) \underset{|U|^2 \text{ is projection}}{\Rightarrow} |U|^2\psi = \psi \Rightarrow \|U\psi\|^2 = \langle U\psi, U\psi \rangle = \langle \psi, |U|^2\psi \rangle = \|\psi\|^2$.

($\Leftarrow$) For $\varphi \in \ker(U)^\perp$,

$$\|(\mathbb{1}-|U|^2)\varphi\|^2 = \langle \varphi, |\mathbb{1}-|U|^2|^2\varphi\rangle = \langle \varphi, (\mathbb{1}-|U|^2)\varphi\rangle$$
$$= \|\varphi\|^2 - \|U\varphi\|^2 = 0.$$

So, $|U|^2 = \mathbb{1}$ over $\ker(U)^\perp$.

$\square$

So, we see that

U is partial iso. $\Rightarrow \mathcal{H} = \ker(U) \oplus \ker(U)^\perp = \operatorname{im}(U) \oplus \operatorname{im}(U)^\perp$

since $\operatorname{im}(U)$ is closed (can be seen from isometry condition).

Thus,

$\tilde{U} : \ker(U)^\perp \to \operatorname{im}(U)$ sending $\varphi \mapsto U\varphi$ is unitary

Def:

If U is a partial isometry, we say

$\ker(U)^\perp =$ initial space $\qquad \operatorname{im}(U) =$ final space

Then, $|U|^2$ is a s.a. proj onto the initial space, and
$|U^*|^2$ ——————————— final space.

Lemma: (Square root lemma)

If $A \geq 0$ then $\exists!\, B \geq 0$ s.t. $B^2 = A$ and $[B, A] = 0$.

Proof: Note that if $0 \notin \sigma(A)$ then we can apply the holomorphic functional calculus. In the general case, we could use the continuous functional calculus. We do it different.

Let $B_1(0_{\mathbb{C}}) \ni z \mapsto \sqrt{1-z} = (1-z)^{\frac{1}{2}} = \sum_{j=0}^{\infty} \binom{1/2}{j} z^j$. It turns out this converges absolutely on $\overline{B_1(0_{\mathbb{C}})}$.

So, $\forall x \in \mathcal{A}$, $\|x\| \leq 1 \Rightarrow \sqrt{1-x}$ is defined via the series in the norm limit. So, let $x := \mathbb{1} - \frac{a}{\|a\|} \Rightarrow \|x\| \leq 1$

$$\Rightarrow \sqrt{a} = \sqrt{\|a\|}\left(\mathbb{1} - \sum_{j=1}^{\infty} \binom{1/2}{j}\left(\mathbb{1} - \frac{a}{\|a\|}\right)^j\right)$$

≥ 0

$\square$

empty space :)

Theorem (Polar Decomp)

Let $A \in \mathcal{B}(\mathcal{H})$. Then, $\exists!$ partial isometry U s.t.

- $\ker(U) = \ker(A)$ and
- $A = U|A| = U\sqrt{A^*A}$

Moreover, $\operatorname{im}(U) = \overline{\operatorname{im}(A)}$.

Warmup!: Suppose first that A is invertible. Then, $|A|$ is also invertible and letting $\underbrace{U}_{\text{invertible}} := A|A|^{-1}$, $A = U|A|$. We see

$$|U|^2 = (A|A|^{-1})^* A|A|^{-1} = \underbrace{(|A|^{-1})^*}_{=(|A|^*)^{-1} = |A|^{-1}} |A|^2 |A|^{-1} = \mathbb{1} \Rightarrow U \text{ partial iso.}$$

All invertible partial iso's are unitary. So, A invertible $\Rightarrow$ $U = A|A|^{-1}$ unitary

Remark: We might try to decompose $U: \ker(U)^\perp \oplus \ker(U) \to \mathrm{im}(U) \oplus \mathrm{im}(U)^\perp$, let $\tilde{U}: \ker(U)^\perp \to \mathrm{im}(U)$ be unitary, and redefine $U = \begin{bmatrix} \tilde{U} & 0 \\ 0 & V \end{bmatrix}$ for some

$V: \ker(U) \to \mathrm{im}(U)^\perp$ (any redefining on this space won't affect the polar decomp since $\ker(U) = \ker(A)$). However, in the full generality there might never be an isomorphism $V: \ker(U) \to \mathrm{im}(U)^\perp$ since they may have different dims. So, we can't cheat and make U unitary.

Proof: Define $U': \mathrm{im}(|A|) \to \mathrm{im}(A)$ by $|A|\psi \mapsto A\psi$. To see this is well-defined, let $|A|\varphi = |A|\psi$; we WTS $A\varphi = A\psi$. We have

$$\|A\varphi - A\psi\| = \|A(\varphi - \psi)\| \overset{*}{=} \||A|(\varphi - \psi)\| = 0$$

where $(*)$ holds since $\|A\gamma\|^2 = \langle \gamma, |A|^2 \gamma \rangle \overset{|A| \text{ s.a.}}{=} \langle \gamma, |A|^* |A| \gamma \rangle = \||A|\gamma\|$
So, U' is a well-defined isometry.

Now, extend to $\tilde{U}: \overline{\mathrm{im}(|A|)} \to \overline{\mathrm{im}(A)}$ (also an iso.). To do so, let $\psi \in \overline{\mathrm{im}(|A|)}$. Then, $\exists \{\varphi_n\}_n \subseteq \mathcal{H}$ s.t. $|A|\varphi_n \to \psi$. So,

$$\tilde{U}\psi := \underbrace{\lim_n A\varphi_n}_{\substack{\text{exists since} \\ \|A(\varphi_n - \varphi_m)\| = \||A|(\varphi_n - \varphi_m)\| \to 0 \\ \text{as } |A|\varphi_n \text{ converges}}} \in \overline{\mathrm{im}\, A}$$

Now, $\mathcal{H} = \overline{\mathrm{im}(|A|)} \oplus \overline{\mathrm{im}(|A|)}^\perp$. So, we may extend $\tilde{U}$ to a partial iso. $U: \mathcal{H} \to \overline{\mathrm{im}(A)}$ by setting $U \equiv 0$ on $\overline{\mathrm{im}(|A|)}^\perp$

Hence, $\ker(U) = \left(\overline{\mathrm{im}(|A|)}\right)^\perp \overset{(\overline{\cdot})^\perp = (\cdot)^\perp}{=} \mathrm{im}(|A|)^\perp = \ker(|A|^*) = \ker(|A|) = \ker(|A|^2) = \ker(A)$

To show uniqueness, let $A = WP$ for partial iso W and $P \geq 0$. In order for W's initial space to be $\mathrm{im}(P)$, then

$$|A|^2 = A^*A = \overset{P \geq 0 \Rightarrow P \text{ s.a.}}{P} W^* W P = P|W|^2 P \overset{|W|^2 = 1 \text{ over } \mathrm{im}(P)}{=} P^2 \underset{\text{sq. root lemma}}{\Longrightarrow} |A| = P.$$

So, $U|A| = W|A| \Rightarrow U$ and W agree on their initial spaces $\mathrm{im}(|A|)$.
Since $U = W = 0$ elsewhere, we have $U = W$.

$\square$

9.8 Compact Operators

Intuitively, the compact operators are the norm-closure of the finite matrices embedded in $\mathcal{H}$. We make this rigorous.

Def: (finite rank)

We say $A \in \mathcal{B}(\mathcal{H})$ is of finite rank iff $\dim(\mathrm{im}(A)) < \infty$.

Prop:

$\Rightarrow \mathrm{im}(A) = \mathrm{span}\{\varphi_n : n \le N\}$

A is of finite rank iff $A = \sum_{n=1}^{N} \alpha_n \varphi_n \otimes \psi_n^*$, where $N = \dim(\mathrm{im}(A))$, $\underline{\alpha_n \in [0,\infty)}$, and $\{\varphi_n\}_n$, $\{\psi_n\}_n$ are ONB's.

singular values of A, i.e. eigenvalues of $|A|$

Proof: ($\Rightarrow$) Let $N = \dim(\mathrm{im}(A)) < \infty \Longrightarrow \mathrm{im}(A)$ is closed $\Longrightarrow \mathcal{H} = \mathrm{im}(A) \oplus \mathrm{im}(A)^\perp$
$= \ker(A)^\perp \oplus \ker(A)$

So, $\tilde{A} : \ker(A)^\perp \to \mathrm{im}(A)$ is an isomorphism (finite-dim linear map w/ trivial kernel).
Thus, $\dim(\ker(A)^\perp) = N < \infty$. So $\tilde{A}$ is just some square matrix.
Do SVD on that and complete the ONB's to finish.

($\Leftarrow$) Duh.

$\square$

Examples:

1. $\forall u, v \in \mathcal{H}$, $u \otimes v^*$ is a rank-1 operator with $(u \otimes v^*)(\psi) = \langle v, \psi \rangle u$

2. $\mathbb{1}$ isn't finite rank if $\dim \mathcal{H} = \infty$. In fact, anything invertible isn't finite rank.
So, $\exp(-X^2)$ on $\ell^2(\mathbb{N})$ is also not finite rank.

Def. (Compact operator)

We say $A \in \mathcal{B}(\mathcal{H})$ is compact iff $\|A - A_n\|_{\mathcal{B}(\mathcal{H})} \to 0$, where $\{A_n\}_n$ is a sequence of finite rank operators. In particular, we can always write

$$A = \lim_{N \to \infty} \sum_{n=1}^{N} \alpha_n \varphi_n \otimes \psi_n^*$$

norm-limit

We denote by $\mathcal{K}(\mathcal{H})$ the set of compact operators on $\mathcal{H}$.

Lemma:

E is Banach space

For $A \in \mathcal{B}(E)$, the following are equivalent:

(a) $A \in \mathcal{K}(E)$

(b) For any bounded sequence $\{\psi_n\}_n \subseteq E$, $\{A\psi_n\}_n$ contains a convergent subsequence.

(c) For any bounded $B \subseteq E$, $\overline{A(B)}$ is a compact subset of E.

Proof: $(a \Rightarrow c)$ Suppose $A = \lim A_n$, A_n finite rank. Finite rank ops obey (c) since $\|A_n\|_{op} < \infty$. So, B bounded $\Rightarrow A_n(B)$ bounded. Thus, $\overline{A_n(B)}$ is closed, bounded, and finite-dim (as A_n finite-rank), so $\overline{A_n B}$ is compact by Heine-Borel. So, A_n obeys (c).

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So, all we must show is that property (c) is closed under norm limits.

Lemma: (c) is closed under norm limits $(A_n \to A)$.

Proof of lemma: Let $\varepsilon > 0$. Let n large enough that $\|A_n - A\| < \frac{\varepsilon}{3|B|}$. A_n obeys (c), so $\exists\, \psi_1, \ldots, \psi_m \in B$ s.t.

$$\overline{A_n(B)} \subseteq \bigcup_{j=1}^{m} B_{\frac{\varepsilon}{3}}(A\psi_j)$$

So, $\forall \psi \in B$ $\exists j \leq m$ s.t. $\|A_n\psi - A_n\psi_j\| \leq \frac{\varepsilon}{3}$

$$\Rightarrow \|A\psi_j - A\psi\| \leq \|A\psi_j - A_n\psi_j\| + \|A_n\psi_j - A_n\psi\| + \|A_n\psi - A\psi\|$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus, $\overline{A(B)} \subseteq \bigcup_{j=1}^{m} B_\varepsilon(A\psi_j)$. Use this to cover any cover of $\overline{A(B)}$. $\square$

So, A satisfies (c).

$(c \Rightarrow a)$ Suppose B bounded $\Rightarrow \overline{A(B)}$ compact. So, $\overline{\operatorname{im}(A)}$ is separable. Thus, $\exists$ ONB $\{e_j\}_j$ of $\overline{\operatorname{im}(A)}$. Define $P_n :=$ orthogonal proj onto $\{e_j\}_{j=1}^{n}$. So, $P_n A$ is a rank-n operator. Define $A_n := P_n A$.

We know $A_n \to A$ strongly, but (c) will let us upgrade to norm convergence. Let $\varepsilon > 0$. Then, $\exists\, \psi_1, \ldots, \psi_m$ in unit ball s.t. $\overline{A(B_1)} \subseteq \bigcup_{j=1}^{m} B_{\frac{\varepsilon}{3}}(A\psi_j)$

So, $\forall \psi$ in unit ball $\exists j$ s.t. $\|A\psi - A\psi_j\| < \frac{\varepsilon}{3}$.

Then,
$$\|(A - A_n)\psi\| \leq \underbrace{\|A\psi - A\psi_j\|}_{\leq \frac{\varepsilon}{3}} + \underbrace{\|A\psi_j - A_n\psi_j\|}_{\leq \max_{j=1,\ldots,m}} + \underbrace{\|P_n(A\psi_j - A\psi)\|}_{\leq \frac{\varepsilon}{3},\ \|P_n\| \leq 1}$$
$$\leq \varepsilon$$

Since the n s.t. $\max_{j=1,\ldots,m} \|A\psi_j - A_n\psi_j\| < \frac{\varepsilon}{3}$ (guaranteed by strong conv.) is uniform in ψ, $A_n \to A$ in norm.

$\square$

Theorem:

A compact $\Rightarrow A^*, BA, AB$ compact $\forall B \in \mathcal{B}(\mathcal{H})$

$\mathcal{K}(\mathcal{H})$ is a closed, two-sided-$*$-ideal of $\mathcal{B}(\mathcal{H})$.

Proof: Closure follows since (c) from above is preserved under norm limits. $A \in \mathcal{K}(\mathcal{H}) \Rightarrow A^* \in \mathcal{K}(\mathcal{H})$ follows from the fact that $* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is norm-continuous.

Now, by boundedness of $B \in \mathcal{B}(\mathcal{H})$, $A = \lim_{n\to\infty} A_n \Rightarrow$ $BA = \lim_{n\to\infty} BA_n$, $AB = \lim_{n\to\infty} A_n B$. Since $A_n B$, BA_n are finite-rank, we're done. $\square$

Prop:

A multiplication operator in an ONB $\{e_n\}_n$ is compact if and only if $\langle e_n, Ae_n\rangle \longrightarrow 0$ as $n \to \infty$.

Proof: We know $A = \sum_{n=1}^{\infty} \langle e_n, Ae_n\rangle e_n \otimes e_n^*$ strongly.

($\Leftarrow$) Suppose $\langle e_n, Ae_n\rangle \to 0$. Define $A_N := \sum_{n=1}^{N} \langle e_n, Ae_n\rangle e_n \otimes e_n^*$.
A_N is bounded and finite-rank, and

something about no continuous spectrum?

$$\|A - A_N\| \le \sup_{n > N} |\langle e_n, Ae_n\rangle| \to 0 \quad \text{by assumption.}$$

($\Rightarrow$) Suppose BWOC $\langle e_n, Ae_n\rangle \not\to 0 \Rightarrow \exists \{n_j\}_j$ subsequence s.t. $|\langle e_{n_j}, Ae_{n_j}\rangle| > \varepsilon$. Since $\{e_{n_j}\}_j$ is bounded seq, $\{Ae_{n_j}\}_j$ has a convergent subsequence by (b). Since $\{e_{n_j}\}_j \to 0$ weakly, $\exists$ a subsequence of $\{Ae_{n_j}\}_j$ which $\to 0$ in norm. This contradicts $\langle e_{n_j}, A_{n_j}\rangle > \varepsilon$. $\square$

Example: If $\{e_n\}_n$ is an ONB and $A \in \mathcal{B}(\mathcal{H})$, then
$$A = \sum_{n=1, m=1}^{\infty} \langle e_n, Ae_m\rangle e_n \otimes e_m^*$$ converges strongly, and each partial sum is finite-rank.

Theorem:

Let $A \in \mathcal{K}(\mathcal{H})$ and let $\{\varphi_n\}_n \subseteq \mathcal{H}$ be s.t. $\varphi_n \to \varphi$ weakly. Then, $A\varphi_n \to A\varphi$ in norm.

Proof: HW :) $\square$

★ **Theorem:** (Riesz-Schauder)

Let $A \in \mathcal{K}(\mathcal{H})$. Then,

① $0 \in \sigma(A)$, and so A isn't invertible

② $\sigma(A)$ is a discrete set whose only possible limit point is 0.

③ $\forall \varepsilon > 0$, $|\sigma(A) \setminus B_\varepsilon(0_\varepsilon)| < \infty$

④ $\sigma(A) = \sigma_p(A) \cup \{0\}$, and so $\left.\begin{matrix} \ker(A - \lambda I) \neq \{0\} \\ \dim \ker(A - \lambda I) < \infty \end{matrix}\right\} \forall \lambda \neq 0$

Proof: Eventually, once we get Fredholm. There are proofs in Rudin and Reed & Simon. $\square$

Fredholm Operators

Def:

$A \in \mathcal{B}(\mathcal{H})$ is Fredholm iff

① $\dim \ker A < \infty$ ← almost injective

② $\dim \ker A^* < \infty$

③ $\operatorname{im}(A) \in \operatorname{Closed}(\mathcal{H})$ $\}$ ← almost surjective

The opposite of finite-rank ops are invertible (explores $\mathcal{H}$ fully). This is too restrictive, and so Fredholm ops are almost invertible.

Def:

The cokernel of A is $\operatorname{coker}(A) := \mathcal{H} \setminus \operatorname{im}(A)$

Prop: $\dim \operatorname{coker}(A) < \infty \iff \begin{matrix} \dim \ker(A^*) < \infty \\ \operatorname{im}(A) \in \operatorname{Closed}(\mathcal{H}) \end{matrix} \iff \operatorname{coker}(A) \cong \ker(A^*)$

Theorem: (Atkinson)

the "parametrix"

A is Fredholm $\iff \exists B$ s.t. $AB - \mathbb{1}$, $BA - \mathbb{1}$ are both compact.

Proof: If we have time, god willing :) $\square$

Remark: $\mathbb{1}$ is Fredholm, and so is $-\lambda\mathbb{1}$ if $\lambda \neq 0$. So, A compact $\Rightarrow A-\lambda\mathbb{1}$ is Fredholm, giving Riesz-Schauder (4).

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Spectral Theorem for $\mathcal{B}(\mathcal{H})$

Recall our conditions on when we may apply the functional calculus.

⊛ For $\mathcal{A}$ a Banach algebra, $f(a) \in \mathcal{A}$ $\forall a \in \mathcal{A}$ if f is holomorphic on a nbhd of $\sigma(a)$.

(we do this now) ⊛ For $\mathcal{B}(\mathcal{H})$, if $A \in \mathcal{B}(\mathcal{H})$ is normal, then $f(A) \in \mathcal{B}(\mathcal{H})$ for all f bounded and measurable.

We will start with the theory for self-adjoints. Note that any $A \in \mathcal{B}(\mathcal{H})$ may be written as the sum of two self-adjoints

$$A = \mathrm{Re}\{A\} + i\,\mathrm{Im}\{A\} = \underbrace{\tfrac{1}{2}(A+A^*)}_{=:\mathrm{Re}\{A\}} + i\Big(\underbrace{\tfrac{1}{2i}(A-A^*)}_{=:\mathrm{Im}\{A\}}\Big)$$

When A is normal they commute, and the spectral theory is inherited. So, we proceed with A selfadjoint.

Herglotz-Pick-Nevanlinna-R Functions

Let $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}\{z\} > 0\}$ be the open upper half-plane.

Defn: A map $f : \mathbb{C}^+ \to \mathbb{C}^+$ is Herglotz if it is holomorphic.

Remark: $\cong$ restricting to open unit disk via conformal maps :).

Ex/
- $z \mapsto c + dz$ for $d \geq 0$, $c \in \mathbb{R}$
- $z \mapsto \log(z)$ w/ appropriate branch
- $z \mapsto z^r$, $0 < r < 1$ w/ appropriate branch
- Möbius transform $z \mapsto \frac{a+bz}{c+dz}$ for $\begin{bmatrix} c & d \\ a & b \end{bmatrix} =: M$ with $M^* J M = J =: \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ (such as $z \mapsto -\frac{1}{z}$)

Prop: If m, n are Herglotz, then $m+n$ and $m \circ n$ are as well.

Proof: Duh. $\square$

Prop: (Resolvent Fn. is Herglotz)

If $A = A^* \in \mathcal{B}(\mathcal{H})$ and $\varphi \in \mathcal{H}$, then $f: \mathbb{C}^+ \to \mathbb{C}^+$ given by $z \mapsto \langle \varphi, (A - z\mathbb{1})^{-1}\varphi \rangle$ is Herglotz.

Proof: Since $\sigma(A) \subseteq \mathbb{R}$, then $\mathbb{C}^+ \subseteq \rho(A)$ and f holo, as

$$\frac{f(z+w) - f(z)}{w} = \frac{\langle \varphi, (A-(z+w)\mathbb{1})^{-1}\varphi\rangle - \langle \varphi, (A - z\mathbb{1})^{-1}\varphi\rangle}{w} \overset{\text{resolvent identity}}{=} \frac{1}{\cancel{w}} \langle \varphi, (A-(z+w))^{-1})(\cancel{w})(A - z\mathbb{1})^{-1}\varphi\rangle$$

$$\xrightarrow{w \to 0} \langle \varphi, (A - z\mathbb{1})^{-2}\varphi\rangle$$

Next,
$$\begin{aligned}
\operatorname{Im}\{f(z)\} &= \operatorname{Im}\{\langle \varphi, (A-z\mathbb{1})^{-1}\varphi\rangle\} \\
&= \frac{1}{2i}\left(\langle \varphi, (A-z\mathbb{1})^{-1}\varphi\rangle - \overline{\langle \varphi, (A-z\mathbb{1})^{-1}\varphi\rangle}\right) \\
&= \frac{1}{2i}\left(\langle \varphi, (A-z\mathbb{1})^{-1}\varphi\rangle - \langle \varphi, (\underbrace{A^*}_{=A} - \bar{z}\mathbb{1})^{-1}\varphi\rangle\right) \\
&\overset{\text{resolvent identity}}{=} \frac{1}{2i}\langle \varphi, (A - z\mathbb{1})^{-1}(z - \bar{z})(A - \bar{z}\mathbb{1})^{-1}\varphi\rangle \\
&= \operatorname{Im}\{z\}\langle \varphi, (A-z\mathbb{1})^{-1}(A - \bar{z}\mathbb{1})^{-1}\varphi\rangle \\
&= \underbrace{\operatorname{Im}\{z\}}_{>0} \underbrace{\|(A - z\mathbb{1})^{-1}\varphi\|^2}_{>0 \text{ since } A - z\mathbb{1} \text{ is invertible!}}
\end{aligned}$$

$\square$

Theorem: (Representation Theorem for Herglotz Functions)

Let $f: \mathbb{C}^+ \to \mathbb{C}^+$ be Herglotz. Then, $\exists!\ \mu_f$ Borel measure on $\mathbb{R}$ such that

① $\int_{\lambda \in \mathbb{R}} \frac{1}{1+\lambda^2}\, d\mu_f(\lambda) < \infty$

② $f(z) = a + bz + \overbrace{\int_{\lambda \in \mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\mu_f(\lambda)}^{\text{Borel-Stieltjes Transform}} \quad \forall z \in \mathbb{C}^+$

where $a := \operatorname{Re}\{f(i)\}$ and $b := \lim_{y \to \infty} \frac{f(iy)}{iy}$ exists and is ≥ 0.

Proof: Look it up :) $\square$

Theorem: (Representation Theorem for Special Kind of Herglotz fns)

Let $f: \mathbb{C}^+ \to \mathbb{C}^+$ be Herglotz s.t. $|f(z)| \le \frac{M}{\operatorname{Im}\{z\}}$ $\forall z \in \mathbb{C}^+$.
Then, $\exists!$ Borel measure μ_f on $\mathbb{R}$ s.t.

① $\mu(\mathbb{R}) \le M$

② $f(z) = \int_{\lambda \in \mathbb{R}} \frac{1}{\lambda - z} \, d\mu_f(\lambda) \quad \forall z \in \mathbb{C}^+$ (f is Borel transform of μ)

Proof sketch: Recall "approximations to the identity". Use Poisson kernel

$$K_\varepsilon : \quad x \longmapsto \frac{1}{\pi} \operatorname{Im}\left\{ \frac{1}{(E-x)+i\varepsilon} \right\} \approx \delta(x-E)$$

We mollify with K_ε to get ε-approximations of μ_f via

$$\mu_\varepsilon((-\infty, \lambda)) = \int_{-\infty}^{\lambda} \operatorname{Im}\{ f(\lambda + i\varepsilon)\} \, d\lambda$$

Show that $\mu_\varepsilon \xrightarrow{*} \mu_f$. Uniqueness follows separately. $\square$

Remark: Differentiation of μ_f w.r.t. Lebesgue gives $\underset{\text{abs. cont}}{\mu_c} + \underset{\text{singular}}{\mu_s} = \mu_f$.
It turns out we can recover these via the boundary values of f!

$\boxed{\text{Spectral Measures}}$ (the Herglotz way)

Defn:

For any $A = A^* \in \mathcal{B}(\mathcal{H})$ and $\varphi \in \mathcal{H}$ there is a Borel measure $\mu_{A,\varphi}$ called the spectral measure of (A, φ) obeying

① $\langle \varphi, (A - z\mathbb{1})^{-1} \varphi \rangle = \int_{\lambda \in \mathbb{R}} \frac{1}{\lambda - z} \, d\mu_{A,\varphi}(\lambda) \quad \forall z \in \mathbb{C}^+$

② $\mu_{A,\varphi}(\mathbb{R}) = \|\varphi\|^2$ (so $\|\varphi\| = 1 \Rightarrow \mu_{A,\varphi}$ is a prob. meas.)

Remark: $\forall \varphi \in \mathcal{H}$ and $A = A^* \in \mathcal{B}(\mathcal{H})$, $\operatorname{spt}(\mu_{A,\varphi}) \subseteq \sigma(A)$

Through polarization, for any $Z \in \mathcal{B}(\mathcal{H})$ we may write

$$\langle \varphi, Z\psi \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \varphi + i^k \psi, Z(\varphi + i^k \psi) \rangle$$

Def:

For any $A = A^* \in \mathcal{B}(\mathcal{H})$ and $\varphi, \psi \in \mathcal{H}$ there is a complex-valued Borel measure $\mu_{A,\varphi,\psi}$ called the spectral measure of (A, φ, ψ) given by

$$\mu_{A,\varphi,\psi} = \frac{1}{4}\left(\mu_{A,\varphi+\psi} - \mu_{A,\psi-\varphi} - i\mu_{A,\varphi+i\psi} + i\mu_{A,\varphi-i\psi}\right)$$

satisfying $\langle \varphi, (A - z\mathbb{1})^{-1}\psi \rangle = \int_{\lambda \in \mathbb{R}} \frac{1}{\lambda - z}\, d\mu_{A,\varphi,\psi}(\lambda) \quad \forall z \in \mathbb{C}^+$

Bounded & Measurable Functional Calculus

☆ Def:

Let $f: \mathbb{R} \to \mathbb{C}$ be bounded and measurable. Let $A = A^* \in \mathcal{B}(\mathcal{H})$. For all $\varphi, \psi \in \mathcal{H}$, define

$$\langle \psi, f(A)\varphi \rangle := \int_{\lambda \in \mathbb{R}} f(\lambda)\, d\mu_{A,\psi,\varphi}(\lambda)$$

Via Thm. 7.13 in notes, this uniquely determines $f(A) \in \mathcal{B}(\mathcal{H})$.

Theorem: (Properties of functional calculus)

The bounded, measurable functional calculus obeys:

① $*$-homomorphism:
- $f(A)^* = (\bar{f})(A)$
- $(f+g)(A) = f(A) + g(A)$
- $(fg)(A) = f(A)\, g(A)$

② $\|f(A)\| = \sup_{\lambda \in \sigma(A)} |f(\lambda)| \equiv \|f\|_{L^\infty(\sigma(A))}$

③ $(x \mapsto x)(A) = A$

④ $f_n \to f$ in $L^\infty \Rightarrow f_n(A) \to f(A)$ in strong op. topology

⑤ $[B, A] = 0 \Rightarrow [B, f(A)] = 0$

⑥ spectral mapping theorem $f(\mathrm{Ker}(A - \lambda\mathbb{1})) = \mathrm{ker}(f(A) - f(\lambda)\mathbb{1})$

Projection-Valued Measure (Spectral Projections)

There is another way to view spectral measures. Given any $S \subseteq \mathbb{R}$ measurable, χ_S is measurable and bounded.
So, $\chi_S(A)$ is a self-adjoint projection onto eigenspace of A within S.
Then,

① $\chi_{\mathbb{R}}(A) = \mathbb{1}$ ② $\chi_{\emptyset}(A) = 0$ ③ $\{S_j\}_{j\in\mathbb{N}}$ pairwise disjoint implies
$$\chi_{\cup S_j}(A) = \sum_j \chi_{S_j}(A)$$
strong lim for inf. sum

<u>Defn:</u> (Projection-Valued Measure)

A set function taking operator values $\chi_\cdot(A)$ obeying ①–③ is a projection-valued measure. We have $\langle \psi, \chi_\cdot(A)\psi\rangle = \mu_{A,\psi,\psi}$.

<u>Theorem:</u> (Stone's Theorem)

$$\frac{1}{2}\left(\chi_{(\lambda_1,\lambda_2)}(A) + \chi_{[\lambda_1,\lambda_2]}(A)\right) = \underset{\varepsilon\downarrow 0}{\text{s}\lim} \frac{1}{2\pi i}\int_{\lambda=\lambda_1}^{\lambda_2} \left(R_A(\lambda+i\varepsilon) - R_A(\lambda-i\varepsilon)\right) d\lambda$$

where $R_A(z) \equiv (A - z\mathbb{1})^{-1}$

11/16:

Our goal is for $A \in \mathcal{B}(\mathcal{H})$ to find a $\mathcal{U}: \mathcal{H} \to L^2(M, d\mu)$ such that
$$(\mathcal{U}A\mathcal{U}^*)(f)(x) = F(x)f(x)$$
for some $F(x)$ fixed (usually $F(x) = x$) and $\mathcal{U}$ a unitary.

Continuous Functional Calculus

<u>Theorem:</u> (BLT Theorem)

Let $T: S \to Y$ where $S \subseteq X$ is dense and X, Y Banach spaces.
<u>Then</u>, there exists a unique! $\hat{T}: X \to Y$ s.t.
$$\hat{T}|_S = T$$
"Densely defined linear maps can be uniquely extended."

Theorem: (Continuous functional calculus)

Let $a \in \mathcal{A}$ (C*-alg.) be a s.a.. Then, there is a unique $\phi : C(\sigma(a)) \to \mathcal{A}$ s.t.: ($\phi(f) \equiv f(a)$)

(a) ϕ is a $*$-homomorphism, $\phi(fg) = \phi(f)\phi(g)$, $\phi(\lambda f) = \lambda\phi(f)$, $\phi(z \mapsto 1) = 1$, $\phi(f+g) = \phi(f) + \phi(g)$, $\phi(\bar{f}) = \phi(f)^*$

(b) $\|\phi(f)\|_{\mathcal{A}} = \|f\|_\infty$ (c) $\phi(z \mapsto z) = a$ (d) $\sigma(\phi(f)) = f(\sigma(a))$

(e) $f \geq 0 \Rightarrow \phi(f) \geq 0$

Proof sketch: By BLT theorem and density of polynomials, suffices to show for polynomials. . .

$\square$

Spectral Measures version 2 (the Riesz-Markov way)

Let $A \in \mathcal{B}(\mathcal{H})$ be s.t. $A = A^*$. Then, via the continuous functional calculus we have that the map from $C(\sigma(A)) \mapsto \mathbb{C}$ defined for fixed $\varphi \in \mathcal{H}$ as

$$f \longmapsto \langle \varphi, f(A)\varphi \rangle$$

is a positive linear functional on $C(\sigma(A))$. By the Riesz-Markov theorem, $\exists!$ Borel measure μ_φ on $\sigma(A)$ s.t.

$$\langle \varphi, f(A)\varphi \rangle = \int_{\sigma(A)} f(\lambda)\, d\mu_\varphi(\lambda) \qquad (\forall f \in C(\sigma(A)))$$

This is the spectral measure of A, φ.

Theorem: (Borel-measurable functional calculus)

Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. There is a unique $\hat{\phi} : B(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ (Borel functions from $\mathbb{R} \to \mathbb{C}$) s.t.

(a) $\hat{\phi}$ is a $*$-homomorphism (b) $\|\hat{\phi}(f)\|_{\mathcal{B}(\mathcal{H})} \leq \|f\|_\infty$ (over $\sigma(A)$)

(c) $\hat{\phi}(z \mapsto z) = A$ (d) $f_n \to f$ in $\|\cdot\|_\infty$ and $\|f_n\| \leq C \Rightarrow \hat{\phi}(f_n) \to \hat{\phi}(f)$ strongly

(e) $A\psi = \lambda\psi \Rightarrow \hat{\phi}(f)\psi = f(\lambda)\psi$

Defn: (cyclic vector)

We say $\varphi \in \mathcal{H}$ is cyclic for A if $\operatorname{span}\{A^n\varphi : n \geq 0\}$ (finite linear combinations) is dense in $\mathcal{H}$.

Spectral Theorem

Theorem:

Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and let $\varphi \in \mathcal{H}$ be cyclic for A. Then, there is a unitary $\mathcal{U}: \mathcal{H} \to L^2(\sigma(A), \mu_{A,\varphi})$ s.t.

$$(\mathcal{U} A \mathcal{U}^{-1})(f)(\lambda) = \lambda f(\lambda)$$

It turns out that we may decompose $\mathcal{H}$ into a direct sum of countably many spaces which have cyclic vectors. Then, we may generalize:

Theorem: (Spectral Theorem, general)

finite or countably mf

Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then, there exist measures $\{\mu_m\}_m$ on $\sigma(A)$ and a $\mathcal{U}: \mathcal{H} \to \bigoplus_m L^2(\sigma(A), \mu_m)$ s.t.

$$(\mathcal{U} A \mathcal{U}^*)(\varphi)_m(\lambda) = \lambda \varphi_m(\lambda)$$

where $\varphi = (\varphi_1, \varphi_2, \dots) \in \bigoplus_m L^2(\sigma(A), \mu_m)$

Defn:

Let $\{\mu_m\}_m$ be a family of measures. Then, its **support** is

$$\operatorname{spt}(\{\mu_m\}_m) := \overline{\bigcup_m \operatorname{spt}(\mu_m)},$$

intersection of closed sets given full measure

Prop:

Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint and let $\{\mu_m\}_m$ be the measures given by the Spectral Theorem. Then,

$$\sigma(A) = \operatorname{spt}(\{\mu_m\}_m)$$

Proof: Go look for it. $\square$

Recall the measure theory facts:

discrete meas. pure point / $\ll \mathcal{L}$, abs. cont. / $\perp \mathcal{L}$, singular continuous

Let μ be a measure on $\mathbb{R}$. Then, $\mu = \mu_{pp} + \mu_{ac} + \mu_{sc}$

Then,
$$L^2(\mathbb{R}, \mu) = L^2(\mathbb{R}, \mu_{pp}) \oplus L^2(\mathbb{R}, \mu_{ac}) \oplus L^2(\mathbb{R}, \mu_{sc})$$

The spectral theorem then gives

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}$$

where $A|_{\mathcal{H}_{pp}}$ has only pure point spectrum,

$A|_{\mathcal{H}_{ac}}$ has only abs cont. spectrum

$A|_{\mathcal{H}_{sc}}$ has only singular cont. spectrum

and
$$\sigma(A) = \sigma_{pp}(A) \cup \underbrace{\sigma_{ac}(A) \cup \sigma_{sc}(A)}_{= \sigma_c(A)}$$

In terms of spectral projections:

Let $\Omega \subseteq \mathbb{R}$ Borel, and define $P_\Omega := \chi_\Omega(A)$

Then,

(a) P_Ω is an orthogonal (s.a.) projection

(b) $P_\emptyset = 0$, $\quad P_{(-a,a)} = \mathbb{1} \quad \forall a > \|A\|$

(c) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$

(d) If $\Omega = \bigsqcup_{n=1}^{\infty} \Omega_n$, then $P_\Omega = \text{s-}\lim_{N\to\infty} \sum_{n=1}^{N} P_{\Omega_n}$

We call such $P_\cdot : \mathcal{M}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ projection-valued measures.

<u>Theorem:</u> (Borel functional calculus again)

Let P be a projection-valued measure. Then, then $\forall f \in C(\sigma(A))$ there is a unique $B \in \mathcal{B}(\mathcal{H})$, denoted $B \equiv \int f(\lambda)\, dP_\lambda$, s.t.

$$\langle \psi, B\psi \rangle = \int_{\sigma(A)} f(\lambda)\, d\langle \psi, P_\lambda \psi \rangle \qquad (\forall \psi \in \mathcal{H})$$

Theorem: (Spectral Theorem)

There is a 1-to-1 correspondence between self-adjoint $A \in \mathcal{B}(\mathcal{H})$ and a projection-valued measure $\{P_\Omega\}_{\Omega \in \mathcal{B}(\mathbb{R})}$ s.t.

Fill in 11/27

11/30-

11. Unbounded Operators

Recall that for bounded operators $A: \mathcal{H} \to \mathcal{H}$,
$$\|A\| \equiv \sup\{\|A\varphi\| : \|\varphi\|=1\} < \infty$$

We now turn to unbounded operators, where the domain $\mathcal{D} \subseteq \mathcal{H}$ is a vector subspace (perhaps not closed), $A: \mathcal{D} \to \mathcal{H}$ linear, and $\|A\| \equiv \sup\{\|A\varphi\| : \|\varphi\|=1 \text{ and } \varphi \in \mathcal{D}(A)\}$ can be infinite.

We call an operator A closed iff
$$\Gamma(A) := \{(\varphi, A\varphi) : \varphi \in \mathcal{D}(A)\} \in \text{Closed}(\mathcal{H}^2)$$

$\mathcal{D}(B) \supseteq \mathcal{D}(A)$ and $B|_{\mathcal{D}(A)} = A$

We call an operator A closable iff $\exists$ closed extension $B \supseteq A$.
iff $\overline{\Gamma(A)}$ is the graph of some operator.

Theorem:

If $\|A\| < \infty$, then A is closed $\iff$ $\mathcal{D}(A) \in \text{Closed}(\mathcal{H})$

We call A densely defined iff $\overline{\mathcal{D}(A)} = \mathcal{H}$.

Def: (Adjoints)

Let A be densely defined. We seek A^* s.t.
$$\langle \varphi, A\psi \rangle = \langle A^*\varphi, \psi \rangle \qquad \forall \psi \in \mathcal{D}(A)$$

Equivalently, for each φ we seek a solution $\xi \in \mathcal{H}$ s.t. $\langle \varphi, A\psi \rangle = \langle \xi, \psi \rangle$ $\forall \psi \in \mathcal{D}(A)$
This doesn't exist everywhere, and so we define the domain

$$\mathcal{D}(A^*) := \{\varphi \in \mathcal{H} : \exists \xi \in \mathcal{H} \text{ s.t. } \langle \varphi, A\psi \rangle = \langle \xi, \psi \rangle \ \forall \psi \in \mathcal{D}(A)\}$$

Then, define $A^*\varphi = \xi$ on this domain.

To see uniqueness, $\langle \tilde{\xi}, \psi \rangle = \langle \varphi, A\psi \rangle = \langle \xi, \psi \rangle \Rightarrow \tilde{\xi} - \xi \in \mathcal{D}(A)^\perp = \left(\overline{\mathcal{D}(A)}\right)^\perp = \{0\}$ (by density of $\mathcal{D}(A)$)

So, A^* is a valid operator.

⚠ It may happen that $\mathcal{D}(A^*) = \{0\}$.

Similarly, we may define A^{**} only when A^* is densely defined.

Prop:

For A densely defined, $$\mathcal{D}(A^*) = \left\{ \varphi \in \mathcal{H} : \sup_{\psi \in \mathcal{D}(A)} \frac{|\langle \varphi, A\psi \rangle|}{\|\psi\|} < \infty \right\}$$

Proof: ($\subseteq$) Let $\varphi \in \mathcal{D}(A^*)$, and so $|\langle \varphi, A\psi \rangle| = |\langle \xi, \psi \rangle| \overset{\text{C.S.}}{\leq} \|\xi\| \, \|\psi\|$

($\supseteq$) Note that $\mathcal{D}(A) \ni \psi \mapsto \langle \varphi, A\psi \rangle$ is a bounded linear functional. By Riesz representation, $\langle \varphi, A\psi \rangle = \langle \xi, \psi \rangle$.

$\square$

Example: ($\mathcal{D}(A^*)$ isn't dense)

Let $f : \mathbb{R} \to \mathbb{C}$ be bounded and measurable, but not L^2. Fix $\psi_0 \in \mathcal{H} = L^2$ and define
$$A\psi := \langle f, \psi \rangle \psi_0 \quad \text{on} \quad \mathcal{D}(A) := \left\{ \psi \in L^2 : \int |f\psi| < \infty \right\}$$

Note that $\mathcal{D}(A)$ contains all functions of compact support, and so A is densely defined. However,

$$\langle \psi, A^*\varphi \rangle = \langle A\psi, \varphi \rangle = \langle (\langle f, \psi \rangle \psi_0), \varphi \rangle = \langle \psi_0, \langle \psi, f \rangle \varphi \rangle$$

So, $A^*\varphi = \langle \psi_0, \varphi \rangle f$, which lies in L^2 iff $\langle \psi_0, \varphi \rangle = 0$.

Thus,
$$\mathcal{D}(A^*) = \{\psi_0\}^\perp \quad \text{which isn't dense.}$$

Theorem:

Let A be densely defined. Then,

① A^* is closed

② A is closable $\Leftrightarrow \overline{\mathcal{D}(A^*)} = \mathcal{H}$, in which case $\overline{A} = A^{**}$

③ when A is closable, $(\overline{A})^* = A^*$ (analogy with $\perp$)

Proof: ① Define a unitary V on $\mathcal{H}^2 = \mathcal{H}\times\mathcal{H} \equiv \mathcal{H}\oplus\mathcal{H}$ via

$$V := \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}$$

A taking $*$ $\Leftrightarrow$ taking $\perp$ of graphs

By unitarity, $V(E)^\perp = V(E^\perp)$ for any vector subspace E.
We WTS A^* is closed $\Leftrightarrow \Gamma(A^*) \in \text{Closed}(\mathcal{H}^2)$, which we will do by showing that $\Gamma(A^*) = (V\Gamma(A))^\perp$. To see this, note that

$$(\varphi,\psi) \in \Gamma(A^*) \Leftrightarrow \varphi \in \mathcal{D}(A^*) \text{ and } \psi = A^*\varphi \Leftrightarrow \langle \varphi, A\zeta\rangle = \langle \psi,\zeta\rangle \quad \forall \zeta \in \mathcal{D}(A)$$

$$\Leftrightarrow \left\langle (\psi,\varphi), (\zeta, -A\zeta)\right\rangle_{\mathcal{H}^2} = 0 \quad \forall \zeta\in\mathcal{D}(A) \Leftrightarrow (\psi,\varphi) \perp V(\Gamma(A))$$

② ($\Leftarrow$) Suppose A^* is densely defined. We WTS $\overline{\Gamma(A)}$ is the graph of some operator. We know

$$\overline{\Gamma(A)} = (\Gamma(A)^\perp)^\perp \overset{V^2=\mathbb{1}}{=} (V^2\Gamma(A)^\perp)^\perp \overset{①}{=} (V\Gamma(A^*))^\perp \overset{①}{=} \Gamma(A^{**})$$

where we were able to apply ① to A^* since $\overline{\mathcal{D}(A^*)} = \mathcal{H}$.

($\Rightarrow$) Suppose A^* is not densely defined. Let $\psi \in \mathcal{D}(A^*)^\perp$, and so

$$(\psi, 0) \in \Gamma(A^*)^\perp$$

By the previous calculation, $\overline{\Gamma(A)}$ is not the graph of an operator.

③ If A is closable, then

$$A^* = \overline{A^*} \overset{②}{=} (A^*)^{**} = (A^{**})^* = (\overline{A})^*$$

$\square$

Defn: (Spectrum of closable operator)

Let A be closed (if closable, handle $\overline{A}$). We define the resolvent set

$$\rho(A) := \{ z \in \mathbb{C} : (A - z\mathbb{1}) : \mathcal{D}(A) \to \mathcal{H} \text{ is bijective} \}$$

We define the spectrum $\sigma(A) := \mathbb{C}\setminus\rho(A)$

Remark: why do we need closed ops? Let $X = \mathcal{D}(A)$ be a normed v.s. with norm $\|\varphi\| + \|A\varphi\|$, making A a Banach space. By the closed graph theorem, $f: X\to\mathcal{H}$ linear is bounded $\Leftrightarrow \Gamma(f) \in \text{Closed}(X\times\mathcal{H})$.

Then, $\forall z\in\rho(A)$, if A is closed then $(A-z\mathbb{1})^{-1}: \mathcal{H}\to\mathcal{D}(A)$ is invertible and $\|(A-z\mathbb{1})^{-1}\| < \infty \Rightarrow (A-z\mathbb{1})^{-1} \in \mathcal{B}(\mathcal{H})$.

Remark: We still have point, cont, residual spectrum and the usual theorems still hold.

Example: (spectrum depends on domain)

Recall f is absolutely continuous if $f' \in L^1$ and $f(x) = f(0) + \int_0^x f'$.
Define
$$\mathcal{A} := \{ \varphi : [0,1] \to \mathbb{C} : \varphi \text{ is absolutely continuous and } \varphi' \in L^2([0,1]) \}$$

Define two ops. A_1, A_2 via $\mathcal{D}(A_1) := \mathcal{A}$, $\mathcal{D}(A_2) := \{ \varphi \in \mathcal{A} : \varphi(0) = 0 \}$, and both act via $\varphi \mapsto -i\varphi'$ (momentum operator)

It turns out that both A_1, A_2 are closed and densely defined, but
$$\sigma(A_1) = \mathbb{C} \quad \& \quad \sigma(A_2) = \emptyset$$

Symmetric & Self-Adjoint Operators

(fill in proofs for this section later)

Defn:

A (densely defined) is symmetric iff
$$\langle \varphi, A\psi \rangle = \langle A\varphi, \psi \rangle \quad (\varphi, \psi \in \mathcal{D}(A))$$
$$\iff \underbrace{A \subseteq A^*}_{\text{defn in Reed \& Simon}}$$

Defn:

A (densely defined) is self-adjoint iff $A = A^*$. That is, A is symmetric AND $\mathcal{D}(A) = \mathcal{D}(A^*)$.

Prop:

Let A be densely defined. Then,

A symmetric $\Rightarrow$ A closable and $\overline{A} = A^{**} \subseteq A^* \Rightarrow A \subseteq A^{**} \subseteq A^*$
A closed & symmetric $\Rightarrow$ $A = A^{**} \subseteq A^*$
A self-adjoint $\Rightarrow$ $A = A^{**} = A^*$

Defn:

We say a symmetric A is essentially self-adjoint iff $(\bar{A})^* = \bar{A}$

Prop:

If A is essentially SA then it has a unique SA extension.

Proof: We know $\bar{A} = A^{**}$ is a SA extension. Let B be any other SA extension. So, $A \subseteq \underset{A^{**}}{\underset{\parallel}{\bar{A}}} \subseteq B$. Since $C \subseteq D \Rightarrow D^* \subseteq C^*$, we know

$$A^{**} \subseteq B \Rightarrow B^* \subseteq A^{***} = (\bar{A})^* = \bar{A} = A^{**}$$

Since $B = B^*$, we find $B \subseteq \bar{A} \Rightarrow B = \bar{A}$.

$\square$

Theorem:

Let A be symmetric. Then, the following are equivalent:

1. A is S.A.
2. A is closed and $\ker(A^* \pm i\mathbb{1}) = \{0\}$

   can replace with any element of $\mathbb{C} \setminus \mathbb{R}$
3. $\operatorname{im}(A \pm i\mathbb{1}) = \mathcal{H}$

Corollary:

Let A be symmetric. Then, the following are equivalent:

1. A is ess. SA
2. $\ker(A^* \pm i\mathbb{1}) = \{0\}$
3. $\overline{\operatorname{Im}(A \pm i\mathbb{1})} = \mathcal{H}$

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Theorem:

For $A = A^*$,

① $\|(A - z\mathbb{1})\psi\|^2 \overset{z=x+iy}{=} \|(A - x\mathbb{1})\psi\|^2 + y^2\|\psi\|^2 \qquad (\psi \in \mathcal{D}(A))$

② $\sigma(A) \subseteq \mathbb{R}$ and $\|(A - z\mathbb{1})^{-1}\| \leq \dfrac{1}{|\text{Im}\{z\}|} \qquad (z \in \mathbb{C}\setminus\mathbb{R})$

③ $\forall x \in \mathbb{R}$, $\lim_{y\to\infty} iy\,(A - (x+iy)\mathbb{1})^{-1}\psi = -\psi \qquad (\psi \in \mathcal{H})$

④ If $\lambda_1, \lambda_2 \in \sigma_p(A)$ with $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors are orthogonal.

Proof: ① $\|(A - z\mathbb{1})\psi\|^2 = \langle \psi, \underbrace{|A - z\mathbb{1}|^2}_{\substack{(A-\bar{z}\mathbb{1})(A-z\mathbb{1}) \\ = (A - x\mathbb{1})^2 + y^2}} \psi\rangle = \|(A - x\mathbb{1})\psi\|^2 + y^2\|\psi\|^2$

② Let $z \in \mathbb{C}\setminus\mathbb{R}$. We WTS $z \in \rho(A) \iff (A - z\mathbb{1}) : \mathcal{D}(A) \to \mathcal{H}$ bijective.
If $(A - z\mathbb{1})\psi = 0$ for some $\psi \in \mathcal{D}(A)$, $0 \geq y^2\|\psi\|^2 \underset{(y\neq 0)}{\Rightarrow} \psi = 0$, and $A - z\mathbb{1}$ injective.
Since A S.A., we know $\text{ran}(A - z\mathbb{1}) = \mathcal{H}$, and so $A - z\mathbb{1}$ is bijective.
Thus, $\mathbb{C}\setminus\mathbb{R} \subseteq \rho(A) \Rightarrow \sigma(A) \subseteq \mathbb{R}$.

Now, $\forall \psi \in \mathcal{D}(A)$ and all $z = x+iy$, $y>0$, $\|\psi\| \leq \frac{1}{|y|}\|(A - z\mathbb{1})\psi\|$
For any $\varphi \in \mathcal{H}$, since $A - z\mathbb{1} : \mathcal{D}(A) \to \mathcal{H}$ is invertible, $\exists\, \psi \in \mathcal{D}(A)$ s.t. $(A - z\mathbb{1})\psi = \varphi$, and so

$$\|(A - z\mathbb{1})^{-1}\varphi\| \leq \frac{1}{|y|}\|\varphi\| \qquad (\varphi \in \mathcal{H})$$

$$\Rightarrow \boxed{\|(A - z\mathbb{1})^{-1}\| \leq \frac{1}{|\text{Im}\{z\}|}}$$

"the trivial bound"

③ Write $B := A - x\mathbb{1}$, and so B is also S.A. with $\mathcal{D}(B) = \mathcal{D}(A)$.
Note $B - iy\mathbb{1}$ is invertible by the above, and so

$$(B - iy\mathbb{1})(B - iy\mathbb{1})^{-1} \equiv \mathbb{1} \Rightarrow -iy(B - iy\mathbb{1})^{-1} + B(B - iy\mathbb{1})^{-1} = \mathbb{1}$$

$$\Rightarrow -iy(B - iy\mathbb{1})^{-1} + \mathbb{1} = (B - iy\mathbb{1})^{-1}B$$

← note that B commutes with its resolvent

For any fixed $\psi \in \mathcal{D}(B)$, we get

$$\underbrace{\|iy(B - iy\mathbb{1})^{-1}\psi + \psi\|}_{\text{LHS}} = \|(B - iy\mathbb{1})^{-1}B\psi\| \leq \|(B - iy\mathbb{1})^{-1}\|\,\|B\psi\|$$

$$\leq \frac{1}{|y|}\|B\psi\| \to 0$$

For $\psi \in \mathcal{H}$, $\exists \{\varphi_n\}_n \subseteq \mathcal{D}(B)$ s.t. $\varphi_n \to \psi$. S.A. is densely defined. Use $\frac{\varepsilon}{3}$ argument to show that ψ is uniformly close to LHS to finish.

(4) Same proof as for bounded operators.

□

Direct Sums & Invariant Subspaces

Defn: (direct sum)

Let $A_i : \mathcal{D}(A_i) \to \mathcal{H}_i$, $i = 1, 2$. We define the direct sum $A := A_1 \oplus A_2 : \underbrace{\mathcal{D}(A_1) \oplus \mathcal{D}(A_2)}_{\text{subspace of } \mathcal{H}_1 \oplus \mathcal{H}_2} \to \mathcal{H}_1 \oplus \mathcal{H}_2$ via

$$A(\psi_1, \psi_2) = (A_1 \psi_1, A_2 \psi_2)$$

* If A_i is self-adjoint, then so is A.

* $(A - z\mathbb{1})^{-1} = (A_1 - z\mathbb{1})^{-1} \oplus (A_2 - z\mathbb{1})^{-1}$

Defn: (invariant subspace)

Let A be S.A. on $\mathcal{H}$. A closed vector subspace $I \subseteq \mathcal{H}$ is said to be invariant under A iff

$$(A - z\mathbb{1})^{-1} I \subseteq I \qquad (z \in \mathbb{C} \setminus \mathbb{R})$$

Prop: If $I \subseteq \mathcal{H}$ is invariant under a self-adjoint A, then so is $I^\perp$.

Now, for given invariant subspaces, we may restrict A to its invariant subspaces.

For $I \subseteq \mathcal{H}$ invariant under S.A. A, define $A_I : \mathcal{D}(A) \cap I \to I$ (a Hilbert space) via $A_I \psi = A\psi$ $\forall \psi \in \mathcal{D}(A) \cap I$.

Prop: A_I is also S.A.

Proof: Study $\Gamma(A_I) = \Gamma(A) \cap (I \times \mathcal{H})$ and use $V : \mathcal{H}^2 \to \mathcal{H}^2$ unitary from the characterization of closable ops.

□

So, for any invariant $I \subseteq \mathcal{H}$, writing $\mathcal{H} = I \oplus I^{\perp}$, we may decompose $A = A_I \oplus A_{I^\perp}$.

Prop:

Let $\{A_n : \mathcal{D}(A_n) \to \mathcal{H}_n\}_n$ (countable) be a sequence of S.A. ops. Define $A := \bigoplus_n A_n$ on $\mathcal{H} := \bigoplus_n \mathcal{H}_n$ with

$$\mathcal{D}(A) := \left\{ \psi \in \mathcal{H} : \ \psi_n \in \mathcal{D}(A_n) \text{ and } \sum_n \|A_n \psi_n\|^2_{\mathcal{H}_n} < \infty \right\}$$

Then,

① A is also S.A.

② $(A - z\mathbb{1})^{-1} = \bigoplus_n (A_n - z\mathbb{1})^{-1}$

③ $\sigma(A) = \overline{\bigcup_n \sigma(A_n)}$

Proof: ① Use $\left(\bigoplus_n A_n\right)^* = \bigoplus_n A_n^*$. Check R&S VIII for the rest. $\square$

Cyclic Subspaces and Decomposition of S.A. Operator

Def: (cyclic subspace)

Let A be S.A. on $\mathcal{H}$. Then $\{\varphi_n\}_{n=1}^{N}$ (finite or countably infinite) is called cyclic for A iff

$$\mathcal{H} = \overline{\operatorname{span}\left\{ (A - z\mathbb{1})^{-1} \varphi_n : \ z \in \mathbb{C} \setminus \mathbb{R},\ n \in \{1, \dots, N\} \right\}}$$

When $N=1$ we recover the cyclic vector. There always exists a cyclic collection by taking an ONB.

Theorem: (Decomposition)

Let A be S.A. on $\mathcal{H}$ (separable). Then, $\exists$ sequence of closed vector subspaces $\{\mathcal{H}_n\}_n \subseteq \mathcal{H}$ (will be countable since $\mathcal{H}$ is separable) which are mutually orthogonal and S.A. ops $A_n : \mathcal{D}(A_n) \to \mathcal{H}$ s.t.

① $\forall n$, $\exists \psi_n \in \mathcal{H}_n$ s.t. ψ_n is cyclic for A_n

② $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ and $A = \bigoplus_n A_n$

Proof: Let $\{\varphi_n\}_n$ be cyclic for A. Define

$$\psi_1 = \varphi_1 \quad \text{and} \quad \mathcal{H}_1 = \overline{\operatorname{span}\{(A-z\mathbb{1})^{-1}\psi_1 : z \in \mathbb{C}\setminus\mathbb{R}\}}$$

$\mathcal{H}_1$ is a closed invariant subspace of $\mathcal{H}$, and so define $A_1 := A_{\mathcal{H}_1}$.

For the inductive step, let $\tilde{\varphi}_{n+1}$ be the first element of $\{\varphi_k\}_k$ not in $\bigcup_{k\le n} \mathcal{H}_k$. Decompose $\mathcal{H} = \left(\bigcup_{k\le n} \mathcal{H}_k\right) \oplus \left(\bigcup_{k\le n} \mathcal{H}_k\right)^\perp$ and write $\tilde{\varphi}_{n+1} = \tilde{\varphi}^{(1)}_{n+1} + \underbrace{\tilde{\varphi}^{(2)}_{n+1}}_{=:\psi_{n+1}}$

Let $\mathcal{H}_{n+1}$ be cyclic subspace generated by ψ_{n+1}. Then, $\mathcal{H}_{n+1} \perp \left(\bigcup_{k\le n} \mathcal{H}_n\right)$ by the resolvent identity.

$\square$

Spectral Theorem

Theorem: (Diagonal operators)

Let $(X,\mathcal{F})$ be a measure space and μ a positive, finite measure. Let $f: X \to \mathbb{R}$ be $\mathcal{F}$-measurable. Define $M_f: \mathcal{D}(M_f) \longrightarrow L^2(X,\mu)$ via

$$\mathcal{D}(M_f) := \{\varphi \in L^2(X,\mu) : f\varphi \in L^2(X,\mu)\}$$

$$M_f\varphi := f\varphi \qquad \forall \varphi \in \mathcal{D}(M_f)$$

Then,

1. M_f is S.A.
2. $M_f \in \mathcal{B}(L^2(X,\mu))$ iff $f \in L^\infty(X,\mu)$, in which case $\|M_f\| = \|f\|_\infty$
3. $\sigma(M_f) = \underbrace{\{\lambda \in \mathbb{R} : \mu(f^{-1}((\lambda-\varepsilon,\lambda+\varepsilon))) > 0 \ \forall \varepsilon > 0\}}_{\text{essential range of } f}$

Proof: R & S VIII $\square$

Defn: (unitary equivalence)

Let $A_i : \mathcal{D}(A_i) \to \mathcal{H}_i$, $i=1,2$. A_1 is unitarily equivalent to A_2 iff $\exists$ unitary $U: \mathcal{H}_1 \to \mathcal{H}_2$ s.t.
1. $U\mathcal{D}(A_1) \subseteq \mathcal{D}(A_2)$
2. $UA_1U^{-1} = A_2$

Theorem: (Spectral thm. in cyclic case)

Let A be S.A. and $\psi \in \mathcal{H}$. **Then,** $\exists!$ finite positive measure $\mu_{A,\psi}$ on $\mathbb{R}$ s.t.

$$\circledast\ \mu_{A,\psi}(\mathbb{R}) = \|\psi\|^2 \qquad \circledast\ \langle \psi, (A - z\mathbb{1})^{-1}\psi \rangle = \int_{\mathbb{R}} \frac{1}{t-z}\, d\mu_{A,\psi}(t)$$

Proof: Write
$$V(z) := \operatorname{Im}\{\langle \psi, (A-z\mathbb{1})^{-1}\psi\rangle\} \qquad (z \in \mathbb{C}_+)$$
$$= \operatorname{Im}\{z\}\, \|(A - z\mathbb{1})^{-1}\psi\|^2 \qquad \text{(harmonic \& positive)}$$

By harmonic analysis, $\exists\, c \geq 0$ and pos. finite measure $\mu_{A,\psi}$ s.t.

$$V(z) = c \operatorname{Im}\{z\} + \operatorname{Im}\{z\} \int_{t \in \mathbb{R}} \underbrace{\frac{1}{(\operatorname{Re}\{z\} - t)^2 + \operatorname{Im}\{z\}^2}}_{\operatorname{Im}\left\{\frac{1}{t-z}\right\} \cdot \frac{1}{\operatorname{Im}\{z\}}} d\mu_{A,\psi}(t)$$

By $\operatorname{Im}\{z\} \to \infty$ estimate, $c = 0$. By dominated convergence, $\mu_{A,\psi}(\mathbb{R}) = \|\psi\|^2$.

$\square$

Theorem:

Let $\psi \in \mathcal{H}$ be <u>cyclic</u> for S.A. A. Then, A is unitarily equivalent to $M_{x \mapsto x}$ on $L^2(\mathbb{R}, \mu_{A,\psi})$. In particular, $\sigma(A) = \operatorname{spt}(\mu_{A,\psi})$.

From here, decompose $\mathcal{H}$ into cyclic subspaces and diagonalize the restriction of A to these subspaces.

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11.6 - Schrödinger Operators (Teschl)

Recall the basics $\mathcal{H} := L^2(\mathbb{R}^d \to \mathbb{C})$ are wavefunctions s.t. $\frac{|\psi(x)|^2}{\|\psi\|_{\mathcal{H}}}$ is a probability density on $\mathbb{R}^d$.

Time Translation

Let $\psi(t) : \mathbb{R} \to \mathcal{H}$ be the map from time to wavefunctions.
We know it follows the Schrödinger equation

$$i\,\partial_t \psi(t) = H\,\psi(t) \qquad \text{for some unbounded } H$$

Thus, $\psi(t) = e^{-itH}\psi(0)$ and $(\varphi, H\varphi)$ is expected energy in φ.

We may expect $H = \frac{p^2}{2m} + V(x)$ as in the classical case. But no, we quantize.

Quantization

Write $x \longmapsto X$ as the position op. on L^2 and
postulate $p \longmapsto P := -i\hbar\nabla$ as the momentum operator. Then,

$$H = P^2 + V(X) = \underset{\text{Laplacian}}{-\Delta} + V(X)$$

if we use the standard units $c = \hbar = 1$, $m = \frac{1}{2}$.

If there's a magnetic field, we write

$$H = (P - A(X))^2 + V(X).$$

First, let's investigate the case $A = V = 0$, the free particle.

The Laplacian

Consider $-\Delta$ on $L^2(\mathbb{R}^d)$ via $-\Delta = -\sum_j \partial_j^2$. We might expect to set

$$\mathfrak{D}(-\Delta) = \{\varphi \in L^2 : \varphi \text{ has } 2^{\text{nd}} \text{ derivatives in } L^2\}$$

This isn't big enough to ensure $-\Delta$ is ess. SA, so we add more.

Def: (weak derivative)

$f \in L^2(\mathbb{R}^d \to \mathbb{C})$ is weakly-differentiable with weak derivative $\psi : \mathbb{R}^d \to \mathbb{C}$ on j^{th} axis iff

$$\int_{\mathbb{R}^d} \overline{\partial_j \varphi} f = - \int_{\mathbb{R}^d} \overline{\varphi} \psi \qquad (\varphi \in C_c^\infty(\mathbb{R}^d \to \mathbb{C}))$$

$$\iff \langle \partial_j \varphi, f \rangle = \langle \varphi, \psi \rangle \qquad (\varphi \in C_c^\infty(\mathbb{R}^d \to \mathbb{C}))$$

Because φ vanishes at ∞, we have no boundary terms for integration by parts.

Weak derivatives are unique, and we say $\psi \overset{w}{=} \partial_j f$.

We then define

$$\mathcal{D}(-\Delta) := \{ \psi \in L^2 : \psi \text{ has weak second derivatives in } L^2 \}$$
$$=: \mathcal{H}^2(\mathbb{R}^d \to \mathbb{C}) \subseteq L^2$$

to be the 2nd Sobolev space (a Hilbert space).

The Fourier Transform

We'd like to define the Fourier Transform $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$(\mathcal{F}(\varphi))(p) := (2\pi)^{-d/2} \int_{x \in \mathbb{R}^d} e^{-i\langle p, x \rangle} \varphi(x)\, dx$$

However, it doesn't make sense to define this way on L^2, so we define on a dense subspace.

Def: (Schwartz Space)

$$\mathcal{S}(\mathbb{R}^d \to \mathbb{C}) := \left\{ \varphi \in C^\infty(\mathbb{R}^d \to \mathbb{C}) : \sup_x |x^\alpha (\partial^\beta \varphi)(x)| < \infty \quad \forall \alpha, \beta \in (\mathbb{N}_{\geq 0})^d \right\}$$

Then $C_c^\infty \subseteq \mathcal{S}$, and so $\mathcal{S}$ is dense in L^2.

Claim: $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is a well-defined bijection with

$$(\mathcal{F}^{-1} \hat{\varphi})(x) = (2\pi)^{-d/2} \int_{p \in \mathbb{R}^d} e^{i\langle p, x \rangle} \hat{\varphi}(p)\, dp$$

We know $\mathcal{F}^4 = \mathbb{1}$, $\mathcal{F}^2 = \text{reflection}$, $\|\mathcal{F}\varphi\|_{L^2} \overset{\text{Parseval}}{=} \|\varphi\|_{L^2}$ on $\mathcal{S}$

So, BLT theorem allows us to extend $\mathcal{F}$ to a unitary map $\mathcal{F} \in \mathcal{B}(L^2(\mathbb{R}^d \to \mathbb{C}))$. It has the properties

- $\sigma(\mathcal{F}) = \{\pm 1, \pm i\}$

- Explicitly, we find
$$(\mathcal{F}(\varphi))(p) := \underset{R\to\infty}{\overset{L^2 \text{ limit}}{\lim}} (2\pi)^{-d/2} \int_{B_R(0)} e^{-i\langle p, x\rangle} \varphi(x)\, dx$$

Def: (Sobolev space)

$\forall r \in \mathbb{N}$, $q \in [1, \infty)$, the r^{th} Sobolev space of order q is

$$H^{r,q}(\mathbb{R}^d) := \{\psi \in L^q(\mathbb{R}^d) : p \mapsto \|p\|^r (\mathcal{F}(\psi))(p) \in L^q(\mathbb{R}^d)\}$$

With $q=2$, we have the inner product

$$\langle \varphi, \psi\rangle_{H^r} := \sum_{\substack{\alpha,\beta \in (\mathbb{N}_{\geq 0})^d : \\ |\alpha| = |\beta| \leq r}} \langle \partial^\alpha \varphi, \partial^\beta \psi\rangle_{L^2}$$

Back to $-\Delta$

So, $\mathcal{D}(-\Delta) = H^2(\mathbb{R}^d)$ makes $-\Delta$ densely defined. Note that $\mathcal{F}$ makes $-\Delta$ unitarily equivalent to $M_{p\mapsto\|p\|^2}$ on $\underbrace{L^2(\mathbb{R}^d)}_{\text{momentum space}}$

via $\mathcal{F}(-\Delta)\mathcal{F}^* = M_{p\mapsto\|p\|^2}$

Prop:

$$\sigma(-\Delta) = \sigma_{a.c}(-\Delta) = [0, \infty)$$

Proof: Let $\psi \in \mathcal{H}$. We compute the spectral measure as follows:

$$\langle \psi, (-\Delta - z\mathbb{1})^{-1}\psi\rangle_{L^2} \overset{\mathcal{F}\text{ unitary}}{=} \langle \mathcal{F}\psi, \underbrace{\mathcal{F}(-\Delta - z\mathbb{1})^{-1}\mathcal{F}^*}_{M_{p\mapsto(\|p\|^2 - z)^{-1}}}\mathcal{F}\psi\rangle$$

$$= \int_{p\in\mathbb{R}^d} |\hat{\varphi}(p)|^2 \frac{1}{\|p^2\| - z}\, dp \overset{\text{polar coords}}{=} \int_{r=0}^\infty \underbrace{dr\, r^{d-1} C_d \int_{w\in S^{d-1}} d\Omega(w)\, |\hat{\varphi}(rw)|^2}_{=: d\tilde{\mu}(r)} \cdot \frac{1}{r^2 - z}$$

$$= \int_0^\infty d\tilde{\mu}(r) \frac{1}{r^2 - z}$$

$$\overset{\lambda = r^2}{=} \int_{\lambda=-\infty}^\infty \frac{1}{\lambda - z}\, d\mu_\psi(\lambda) \quad \text{with} \quad d\mu_\psi(\lambda) = \left(\tfrac{1}{2}\chi_{[0,\infty)}(\lambda)\lambda^{\frac{d}{2}-1} \cdot C_d \int_{w\in S^{d-1}} d\Omega(w)\, |\hat{\varphi}(\sqrt{\lambda}w)|^2\right) d\lambda$$

So, $\mu_{-\Delta,\psi}$ is a.c w.r.t. Lebesgue! $\square$

Claim: $C_c^\infty(\mathbb{R}^d)$ is a core for $-\Delta$.

Claim: $\exp(-it(-\Delta))$ is a unitary operator on L^2 $\forall t>0$ w/ integral kernel on $L^1 \cap L^2$ functions via

$$\exp(-it(-\Delta))(x,y) \underset{t>0}{=} (4\pi i t)^{-d/2} \exp\left(i \frac{\|x-y\|^2}{4t}\right) \qquad (x,y \in \mathbb{R}^d)$$

Claim: Let $\Omega \subseteq \mathbb{R}^d$ be compact. Let $\psi \in L^2$ be an initial state. Then, eventually ψ gets delocalized over time, i.e.

$$\lim_{t\to\infty} \| \chi_\Omega(X) e^{-it(-\Delta)} \psi \|^2 = 0$$

In fact, this holds $\forall$ ops w. only a.c. spectrum (RAGE Theorem)!

Claim: Heat kernel exists with

$$\exp(-t(-\Delta))(x,y) = (4\pi t)^{-d/2} \exp\left(-\frac{1}{4t}\|x-y\|^2\right) \qquad (x,y \in \mathbb{R}^d)$$

Claim: Note that

$$\frac{1}{\lambda - z} = \int_{t=0}^\infty e^{-t(\lambda - z)} dt \qquad \left(\begin{array}{l}\lambda \geq 0 \quad \text{and} \\ z \in \mathbb{C} \text{ w. } \mathrm{Re}\{z\} < 0\end{array}\right)$$

We may compute $-\Delta$'s resolvent through the functional calculus:

$$(-\Delta - z\mathbb{1})^{-1} = \int_{t=0}^\infty \exp(-t(-\Delta - z\mathbb{1}))\, dt$$

We may write this as an integral operator w. kernel

$$(-\Delta - z\mathbb{1})^{-1}(x,y) = \int_{t=0}^\infty (4\pi t)^{-d/2} \exp\left(-\frac{1}{4t}\|x-y\|^2 + zt\right) dt$$

$$= \frac{1}{2\pi}\left(\frac{\sqrt{-z}}{2\pi\|x-y\|}\right)^{\frac{d}{2}-1} \underbrace{K_{\frac{d}{2}-1}}_{\text{modified Bessel fn}}(\sqrt{-z}\,\|x-y\|)$$

Note the special cases:

$\underline{d=1}$ $$(-\Delta - z\mathbb{1})^{-1}(x,y) = \frac{1}{2\sqrt{-z}} e^{-\sqrt{-z}\|x-y\|}$$

$$\left(z \in \mathbb{C} \setminus [0,\infty)\right)$$

$\underline{d=3}$ $$(-\Delta - z\mathbb{1})^{-1}(x,y) = \frac{e^{-\sqrt{-z}\|x-y\|}}{4\pi\|x-y\|}$$

We have exponential decay of resolvent away from the spectrum: c.w. $\leq \frac{1}{|\mathrm{Im}\{z\}|}$ trivial bound

this is the Combes-Thomas Lemma.