MAT 520: Midterm

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I pledge my honor that I have not violated the Honor Code during this examination.

Problem 1

Let X, Y be two Banach spaces. Define on the Cartesian product $X \times Y$ coordinate-wise addition and scalar multiplication. For $p \in [1, \infty]$, define

$$\|(x,y)\|_p := \begin{cases} \max\{\|x\|_X, \|y\|_Y\} & \text{if } p = \infty \\ (\|x\|_X^p + \|y\|_Y^p)^{1/p} & \text{if } p \in [1, \infty) \end{cases}$$

(a) Show that with these definitions, $X \times Y$ is a Banach space (i.e., show it is a complete normed vector space).

(b) Show that all p-norms are equivalent on $X \times Y$.

Solution

Proof.

Lemma 1. *Let $p \in [1, \infty]$. For all $(x, y) \in X \times Y$,*

$$\|x\|_X \leq \|(x,y)\|_p \leq \|x\|_X + \|y\|_Y$$

and similarly

$$\|y\|_Y \leq \|(x,y)\|_p \leq \|x\|_X + \|y\|_Y$$

Proof of Lemma 1. The result clearly holds when $p = \infty$ since a maximum is $\geq$ both of its arguments but will be equal to one of them, and so $\leq$ to the sum. So, suppose that $p \in [1, \infty)$. We have

$$\|(x,y)\|_p = (\|x\|_X^p + \|y\|_Y^p)^{1/p} \geq (\|x\|_X^p)^{1/p} = \|x\|_X$$

and similarly $\|(x, y)\|_p \geq \|y\|_Y$, where for the above we used that norms are nonnegative and $(\cdot)^{1/p}$ is monotonic. We now want to show the upper bound. Write $a := \frac{\|x\|_X}{\|x\|_X + \|y\|_Y}$ and $b := \frac{\|y\|_Y}{\|x\|_X + \|y\|_Y}$. Then, $a + b = 1$. Furthermore, since $a \leq 1$ and $b \leq 1$, we know that $a^p \leq a$ and $b^p \leq b$. So,

$$a^p + b^p \leq a + b = 1 \implies \|x\|_X^p + \|y\|_Y^p \leq (\|x\|_X + \|y\|_Y)^p \implies (\|x\|_X^p + \|y\|_Y^p)^{1/p} \leq \|x\|_X + \|y\|_Y$$

So, the result of the lemma holds. ∎

(a) Clearly, $X \times Y$ is a vector space. We verify that $\|\cdot\|_p$ defines a norm.

1. Note that by homgeneity of the norms on X and Y,

$$\|\alpha(x,y)\|_p = \begin{cases} \max\{\|\alpha x\|_X, \|\alpha y\|_Y\} = \max\{|\alpha|\|x\|_X, |\alpha|\|y\|_Y\} = |\alpha| \max\{\|x\|_X, \|y\|_Y\} & \text{if } p = \infty \\ (\|\alpha x\|_X^p + \|\alpha y\|_Y^p)^{1/p} = (|\alpha|^p\|x\|_X^p + |\alpha|^p\|y\|_Y^p)^{1/p} = |\alpha|^{p(1/p)} (\|x\|_X^p + \|y\|_Y^p)^{1/p} & \text{if } p \in [1, \infty) \end{cases}$$

   Since $|\alpha|^{p(1/p)} = |\alpha|$, in either case, we see that

$$\|\alpha(x,y)\|_p = |\alpha|\|(x,y)\|_p$$

2. Suppose now that $\|(x, y)\|_p = 0$. Lemma 1 tells us that $\|x\|_X = \|y\|_Y = 0$, and so both x and y are 0 by positivity of the X and Y norms. Therefore, $(x, y) = 0$ in $X \times Y$, and so this norm is positive.

3. Let $(x, y), (a, b) \in X \times Y$. If $p = \infty$, then

$$\|(x, y) + (a, b)\|_p = \max\{\|x + a\|_X, \|y + b\|_Y\} \leq \max\{\|x\|_X + \|a\|_X, \|y\|_Y + \|b\|_Y\}$$

We know that $\|x\|_X + \|a\|_X \leq \|(x, y)\|_p + \|(a, b)\|_p$ by Lemma 1, and similarly $\|y\|_Y + \|b\|_Y \leq \|(x, y)\|_p + \|(a, b)\|_p$. So, the maximum is certainly also $\leq \|(x, y)\|_p + \|(a, b)\|_p$, revealing that

$$\|(x, y) + (a, b)\|_p \leq \|(x, y)\|_p + \|(a, b)\|_p$$

Suppose now that $p \in [1, \infty)$. We know that

$$\|(x, y) + (a, b)\|_p = (\|x + a\|_X^p + \|y + b\|_Y^p)^{1/p}$$

By Minkowski's inequality and the triangle inequality, this is bounded by

$$\|(x, y) + (a, b)\|_p \leq (\|x\|_X^p + \|y\|_Y^p)^{1/p} + (\|a\|_X^p + \|b\|_Y^p)^{1/p} = \|(x, y)\|_p + \|(a, b)\|_p$$

So, we find that $\| \cdot \|_p$ satisfies the triangle inequality for all $p \in [1, \infty]$.

So, $X \times Y$ is indeed a normed vector space. To see that it is complete, let $\{(x_n, y_n)\}_n \subseteq X \times Y$ be Cauchy. Let $\epsilon > 0$ be arbitrary. Then, there is some $N \in \mathbb{N}$ such that for all $n, m > N$,

$$\|(x_n, y_n) - (x_m, y_m)\|_p = \|(x_n - x_m, y_n - y_m)\|_p < \epsilon$$

By the Lemma, we know that

$$\epsilon > \|(x_n - x_m, y_n - y_m)\|_p \geq \|x_n - x_m\|_X$$

and similarly for Y. Since such an N exists for all $\epsilon > 0$, we find that $\{x_n\}_n \subseteq X$ is Cauchy in X and $\{y_n\}_n \subseteq Y$ is Cauchy in Y. Since these are Banach spaces and therefore complete, they converge in their norms to elements $x \in X$ and $y \in Y$, respectively. We must show that $(x_n, y_n) \to (x, y)$ in the $\| \cdot \|_p$ norm. To do this, let $\epsilon > 0$ be fresh. Then, there is some $N_X, N_Y \in \mathbb{N}$ such that for all $n > N_X$ and $m > N_Y$,

$$\|x - x_n\|_X < \frac{\epsilon}{2} \quad \text{and} \quad \|y - y_m\|_Y < \frac{\epsilon}{2}$$

by definition of convergence. So, for all $n > N := \max\{N_X, N_Y\}$, we have by Lemma 1 that

$$\|(x, y) - (x_n, y_n)\|_p = \|(x - x_n, y - y_n)\|_p \leq \|x - x_n\|_X + \|y - y_n\|_Y < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since such an N exists for all $\epsilon > 0$, we see that $\|(x, y) - (x_n, y_n)\|_p \to 0$ as desired.

(b) Now, let $p, q \in [1, \infty]$. To see that they are equivalent, we may just apply Lemma 1 to see that for all $(x, y) \in X \times Y$,

$$\|(x, y)\|_p \leq \|x\|_X + \|y\|_Y \leq \|(x, y)\|_q + \|(x, y)\|_q = 2\|(x, y)\|_q$$

Identical logic shows that

$$\|(x, y)\|_q \leq \|x\|_X + \|y\|_Y \leq \|(x, y)\|_p + \|(x, y)\|_p = 2\|(x, y)\|_p,$$

and so

$$\frac{1}{2}\|(x, y)\|_p \leq \|(x, y)\|_q \leq 2\|(x, y)\|_p$$

Since this holds for all $(x, y) \in X \times Y$, the p and q norms are equivalent. ■

Problem 2

Let $M : X \to Y$ be a continuous map between normed spaces X, Y such that $M(0) = 0$ and

$$M\left(\frac{1}{2}(x+\tilde{x})\right) = \frac{1}{2}M(x) + \frac{1}{2}M(\tilde{x}) \quad (\forall x, \tilde{x} \in X).$$

Show that M is linear.

Solution

Proof. Firstly, note that with $\tilde{x} = 0$ we see that for all $x \in X$,

$$M(x) = M\left(\frac{1}{2}(2x+0)\right) = \frac{1}{2}M(2x) + \frac{1}{2}M(0) = \frac{1}{2}M(2x) \implies M(2x) = 2M(x)$$

Now, let $x_1, x_2 \in X$ be arbitrary. Then, defining $x := 2x_1$ and $\tilde{x} = 2x_2$, we may apply the hypothesis and our above conclusion to see

$$M(x_1 + x_2) = M\left(\frac{1}{2}(2x_1 + 2x_2)\right) = \frac{1}{2}M(2x_1) + \frac{1}{2}M(2x_2) = M(x_1) + M(x_2),$$

where for the last equality we used that $M(2a) = 2M(a)$. So, M is additive.

We know by additivity of M that $M(nx) = M(x) + \ldots + M(x) = nM(x)$ for all $n \in \mathbb{N}$, and so $M(x) = \frac{1}{n}M(nx)$. Replacing x with $\frac{x}{n}$, we also see that $nM\left(\frac{1}{n}x\right) = M(x)$. Furthermore, we know that

$$M(x) + M(-x) = M(0) = 0 \implies M(x) = -M(-x)$$

Together, the above facts tell us that for all nonzero $n \in \mathbb{Z}$,

$$M(x) = nM\left(\frac{1}{n}x\right) = \frac{1}{n}M(nx)$$

To conclude the proof, let $\alpha \in \mathbb{R}$ and $x \in X$ be arbitrary and nonzero. Let $\epsilon > 0$ be arbitrary. Since M is continuous at αx, there is some $\delta > 0$ such that for all $\tilde{x} \in X$ with $\|\alpha x - \tilde{x}\|_X \leq \delta$, we have $\|M(\alpha x) - M(\tilde{x})\|_Y < \epsilon$. Now, may find a rational number $\beta \in \mathbb{Q}$ such that $|\alpha - \beta| < \max\left\{\frac{\delta}{\|x\|_X}, \frac{\epsilon}{\|Mx\|_Y}\right\}$ by the density of the rationals. If we express $\beta = \frac{n}{m}$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, we see by additivity and our previous conclusions about M that

$$M(\beta x) = M\left(\frac{n}{m}x\right) = \frac{1}{m}(nx) = \frac{n}{m}M(x) = \beta M(x)$$

So, since

$$\|\alpha x - \beta x\|_X = |\alpha - \beta| \cdot \|x\|_X \leq \frac{\delta}{\|x\|_X} \cdot \|x\|_X = \delta,$$

we see that

$$\|M(\alpha x) - M(\beta x)\|_Y < \epsilon \implies \|M(\alpha x) - \beta M(x)\|_Y < \epsilon$$

So, by the triangle inequality,

$$\begin{aligned}\|M(\alpha x) - \alpha M(x)\|_Y &\leq \|M(\alpha x) - \beta M(x)\|_Y + \|\beta M(x) - \alpha M(x)\|_Y \\ &< \epsilon + |\beta - \alpha| \cdot \|M(x)\|_Y \leq \epsilon + \frac{\epsilon}{\|Mx\|_Y} \cdot \|Mx\|_Y = 2\epsilon\end{aligned}$$

So, $\|M(\alpha x) - \alpha M(x)\|_Y < \epsilon$ for all $\epsilon > 0$, which means that $M(\alpha x) = \alpha M(x)$. Therefore, M is linear. ■

Problem 3

Provide an example (no further explanation or proof is necessary) for each of the following:

(a) A normed vector space which is not a Banach space.

(b) A linear functional that is not continuous.

(c) A topological vector space which is not locally convex.

(d) A Banach space whose closed unit ball is compact.

(e) A Banach space which is not reflexive.

Solution

Proof. (a) Let $X = \mathbb{C}^\infty$ be the space of infinite complex sequences such that only finitely many elements are nonzero. Equip X with any norm (such as the norm $\|x\|_\infty = \max_{n\in\mathbb{N}}\{|x(n)|\}$, where we know the max exists because only finitely many elements $x(n)$ are nonzero). Let $\{e_j\}_{j\in\mathbb{N}}$ be the elements of X with a 1 in the j^{th} element and 0's everywhere else. Then, we may note that $\{e_j\}_{j\in\mathbb{N}}$ forms a Hamel basis for X since any element of X has a unique representation as a *finite* linear combination of e_j's (namely, $x = \sum_{\substack{n\in\mathbb{N}\\ x(n)\neq 0}} x(n)e_n$). We know that X is infinite-dimensional and has a Hamel basis, which by Problem 15 on Problem set 2 means that X can't be Banach.

(b) Let $X = \mathbb{C}^\infty$ be the space of infinite complex sequences such that only finitely many elements are nonzero, and endow it with the ℓ^∞ norm as above. Define $f : X \to \mathbb{C}$ via

$$f(x) := \sum_{n\in\mathbb{N}} nx(n),$$

where we know this sum to converge since $x(n)$ is nonzero for finitely many n. Since

$$f(\alpha x + y) = \sum_{n\in\mathbb{N}} n \cdot (\alpha x + y)(n) = \sum_{n\in\mathbb{N}} \alpha nx(n) + ny(n) = \alpha f(x) + f(y),$$

this functional is linear. However, it is not bounded, since for any n we may always find an element of unit norm e_n such that $f(e_n) = n$. Since linear maps between normed spaces are bounded iff continuous, f cannot be continuous.

(c) Let $p \in (0, 1)$. Consider $\ell^p(\mathbb{N} \to \mathbb{C})$, which is the space of all complex sequences such that

$$\sum_{n\in\mathbb{N}} |x(n)|^p < \infty$$

With elementwise addition and scalar multiplication, this is a vector space. Endow $\ell^p(\mathbb{N} \to \mathbb{C})$ with the metric

$$d(x, y) := \sum_{n\in\mathbb{N}} |x(n) - y(n)|^p$$

This is a TVS (in fact it is an F-space) but is not locally convex. See Rudin Problem 5 from Chapter 3.

(d) Consider the Banach space $X = \mathbb{C}^n$ with the Euclidean norm. Then, the closed unit ball is a closed and bounded subset of $\mathbb{C}^n$, which by Heine-Borel means that it is compact.

(e) Consider the Banach space $\ell^1(\mathbb{N} \to \mathbb{C})$, which is the space of all complex sequences such that

$$\sum_{n\in\mathbb{N}} |x(n)| < \infty$$

From Problem 2(c) on Problem Set 4, we know $(\ell^1)^* = \ell^\infty$. However, that problem produces an element of $(\ell^\infty)^* = (\ell^1)^{**}$ that is **not** an element of $J(\ell^1)$, and so

$$J(\ell^1) \subsetneq (\ell^1)^{**}$$

In particular, ℓ^1 is not reflexive. ■

Problem 4

Show that if X, Y are Banach spaces and $A \in \mathcal{B}(X \to Y)$, then if $x_n \to x$ weakly in X, then $Ax_n \to Ax$ weakly in Y.

Solution

Proof.
Suppose that $x_n \to x$ weakly. We wish to show that $Ax_n \to Ax$ weakly in Y. So, let $\lambda \in Y^*$ be arbitrary. By construction, both λ and A are linear and continuous. So, $\lambda \circ A$ is also linear and continuous since it is a composition of linear and continuous maps. Thus, $\lambda \circ A \in X^*$. By Lemma 5.11 in the lecture notes, since $x_n \to x$ weakly we know that $\Lambda x_n \to \Lambda x$ in Open($\mathbb{C}$) for all $\Lambda \in X^*$. Letting $\Lambda = \lambda \circ A$, we have that

$$\lambda(Ax_n) \to \lambda(Ax) \quad \text{in Open}(\mathbb{C})$$

Since this holds for all $\lambda \in Y^*$, we may apply the converse of Lemma 5.11 to find that $Ax_n \to Ax$ weakly in Y, as desired. ■

Problem 5

In a Banach algebra $\mathcal{A}$, let $a, b \in \mathcal{A}$. Show that if $ab = ba$, then

$$\sigma(a+b) \subseteq \sigma(a) + \sigma(b) \quad \text{and} \quad \sigma(ab) \subseteq \sigma(a)\sigma(b).$$

Find examples where these containments are strict, and find examples when these containments are false if $ab \neq ba$.

Solution

Proof. Suppose that $z \in \sigma(ab)$. If $z = 0$, then $ab \notin \mathcal{G}_\mathcal{A}$, which means that either $a \notin \mathcal{G}_\mathcal{A}$ or $b \notin \mathcal{G}_\mathcal{A}$; in either case, we get that $0 \in \sigma(a)\sigma(b)$. So, suppose without loss of generality that $z \neq 0$. Then, $z\mathbb{1} - ab \notin \mathcal{G}_\mathcal{A} \implies \mathbb{1} - \frac{1}{z}ab \notin \mathcal{G}_\mathcal{A}$. By the contrapositive to Lemma 6.5 in the lecture notes, we see that $\left\|\frac{1}{z}ab\right\| \geq 1 \implies \|ab\| \geq |z|$ i got no idea how to do this :)

To see examples where the containments are strict, consider $\mathcal{A} = \mathbb{C}^{2\times 2}$ to be the Banach algebra of square 2×2 matrices. Note that for diagonal matrices $M := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, we always have that $\sigma(M) = \{\lambda_1, \lambda_2\}$. To see this, note that the matrix $M - \lambda\mathbb{1} = \begin{bmatrix} \lambda_1 - \lambda & 0 \\ 0 & \lambda_2 - \lambda \end{bmatrix}$ is noninvertible if and only if $(\lambda_1 - \lambda)(\lambda_2 - \lambda) = 0 \iff \lambda \in \{\lambda_1, \lambda_2\}$. Let

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then, we see that $\sigma(A) = \{0, 1\}$ and $\sigma(B) = \{0, -1\}$. However, $A + B = 0$, and so $\sigma(A + B) = \sigma(0) = \{0\}$ (the spectral radius is 0 by Gelfand's formula, and so the nonempty spectrum therefore must contain only 0). So, $1 \in \sigma(A) + \sigma(B)$, yet $1 \notin \sigma(A + B)$. To see a counterexample in the multiplicative case, let

$$A = B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then, $\sigma(A) = \sigma(B) = \{-1, 1\}$. However, $AB = \mathbb{1}$, and so $\sigma(AB) = \{1\}$. This tells us that $-1 \in \sigma(A)\sigma(B)$ yet $-1 \notin \sigma(AB)$. Thus, the containments may be strict.

To see examples where the containments may not hold for noncommuting elements, we must use nondiagonal matrices. Let

$$A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

We may confirm that $\sigma(A) = \sigma(B) = \{0\}$; indeed, if $\lambda \in \sigma(A)$ then $\det(A - \lambda\mathbb{1}) = 0 \implies \lambda^2 - 1(0) = 0 \implies \lambda^2 = 0 \implies \lambda = 0$ and similarly for B. However, we see that 1 is in both $\sigma(A + B)$ and $\sigma(AB)$. To see this, note that

$$A + B - \mathbb{1} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \implies \det(A + B - \mathbb{1}) = 1^2 - (-1)^2 = 0,$$

and so $A + B - \mathbb{1} \notin \mathcal{G}_\mathcal{A} \implies 1 \in \sigma(A + B)$. Similarly,

$$AB - \mathbb{1} = \begin{bmatrix} 1 - 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \implies \det(AB - \mathbb{1}) = 0 - 0 = 0$$

and so $AB - \mathbb{1} \notin \mathcal{G}_\mathcal{A} \implies 1 \in \sigma(AB)$. We see therefore that $\sigma(A + B) \not\subseteq \sigma(A) + \sigma(B)$ and also that $\sigma(AB) \not\subseteq \sigma(A)\sigma(B)$ in this case. ■