MAT 520: Final

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I pledge my honor that I have not violated the Honor Code during this examination.

Problem 1

Define the operator K on $\mathcal{H} := L^2([0,1])$ via

$$(K\psi)(x) := \int_{y=x}^{1} \left(\int_0^y \psi(z)dz \right) dy \quad (x \in [0,1], \psi \in \mathcal{H})$$

Show that:

(a) K is self-adjoint.

(b) K is compact.

(c) Find the spectrum of K.

Solution

Proof. Let V be the operator on $\mathcal{H}$ given by

$$(V\psi)(x) := \int_0^x \psi(y)dy \quad (x \in [0,1], \psi \in \mathcal{H})$$

V is exactly the operator we studied in Problem 7 on Problem Set 8, where we proved the following properties:

1. $V \in \mathcal{B}(\mathcal{H})$ is a well-defined operator.
2. V^* is the operator given by

$$(V^*\psi)(x) := \int_x^1 \psi(y)dy \quad (x \in [0,1], \psi \in \mathcal{H})$$

Now, we note that $K = |V|^2 = V^*V$, since for all $x \in [0,1]$ and all $\psi \in \mathcal{H}$, we have

$$\begin{aligned}(V^*V\psi)(x) &= \int_x^1 (V\psi)(y)dy = \int_x^1 \left(\int_0^y \psi(z)dz \right) dy \\ &= (K\psi)(x)\end{aligned}$$

From here, we may prove everything we need to show about K.

(a) Since $K = |V|^2$, we immediately find that K is self-adjoint.

(b) We will show that V is compact, from which compactness of K follows immediately. Note that for each ψ, we see that for all $x \in [0,1]$, letting χ_S denote the indicator function of a set S,

$$(V\psi)(x) = \int_0^1 \chi_{[0,x]}(y)\psi(y)dy$$

If we write $K : [0,1]^2 \to \mathbb{C}$ to be given by $K(x,y) = \chi_{[0,x]}(y)$, it is apparent that $K \in L^2([0,1]^2)$ (it is bounded on a compact domain) and that

$$(V\psi)(x) = \int_0^1 K(x,y)\psi(y)dy$$

In this form, we recognize V to be a Hilbert-Schmidt operator, which is immediately compact. So, since the compact operators form a $*$-closed two-sided idea, V^*V is also compact. Therefore, K is compact.

(c) Since K is compact, the Riesz-Schauder theorem (Theorem 9.42 in the lecture notes) tells us that $\sigma(K) = \{0\} \cup \sigma_p(K)$, and so we must compute the nonzero eigenvalues of K. To this end, suppose that ψ is an eigenvector of K with eigenvalue $\lambda \neq 0$ (since K is positive, then $\lambda > 0$). Then, for a.e. $x \in [0,1]$ we have that

$$\lambda\psi(x) = (K\psi)(x) = \int_x^1 \left(\int_0^t \psi(s)ds \right) dt$$

Note that

$$|\psi(y) - \psi(x)| \leq \frac{1}{|\lambda|} \int_x^y \left| \int_0^t \psi(s)ds \right| dt \leq \frac{1}{|\lambda|} \int_x^y \int_0^t |\psi(s)| ds dt \leq \frac{1}{|\lambda|} \int_x^y \left(\int_0^1 |\psi| \right) dt$$

Using our favorite Holder estimate $\|\psi\|_{L^1} = \int_0^1 |\psi| \leq \|\psi\|$ (where $\|\cdot\|$ always denotes the $\mathcal{H}$-norm), we see that

$$|\psi(y) - \psi(x)| \leq \frac{\|\psi\|}{|\lambda|} |y - x|$$

In particular, ψ is Lipschitz and so differentiable a.e.. Taking a derivative of our initial expression and applying the fundamental theorem of calculus, we see that

$$\lambda\psi'(x) = -\int_0^x \psi(s)ds$$

From this we see that $\psi'(0) = 0$. Applying very similar logic as above, we have that

$$|\psi'(y) - \psi'(x)| \leq \frac{1}{|\lambda|} \int_x^y |\psi(s)| ds$$

ψ is Lipschitz, and so continuous, which means it is bounded on $[0,1]$, i.e. $|\psi(s)| \leq M < \infty$ for $s \in [0,1]$. Therefore ψ' is $\frac{M}{|\lambda|}$-Lipschitz, which means that ψ' is a.e. differentiable. So, we may take another derivative and see that for a.e. $x \in [0,1]$,

$$\lambda\psi''(x) = -\psi(x) \implies \psi(x) = C_1 \cos(x/\sqrt{\lambda}) + C_2 \sin(x/\sqrt{\lambda})$$

for some constants C_1, C_2. We know that $\psi'(0) = 0$, and so $C_2 = 0$. Also, since $(K\psi)(1) = 0$ we have $\psi(1) = 0$. Therefore,

$$\cos(1/\sqrt{\lambda}) = 0 \implies \frac{1}{\sqrt{\lambda}} = \left(k + \frac{1}{2}\right)\pi \text{ for some } k \in \mathbb{N} \cup \{0\}$$

The above holds for any $k \in \mathbb{N} \cup \{0\}$ (note that we cannot have $k < -\frac{1}{2}$ since the LHS is positive, and so we are restricted to nonnegative integers for k). So, we may enumerate the nonzero eigenvalues as

$$\lambda_k := \left(\frac{2}{(2k+1)\pi} \right)^2 \quad (k \in \mathbb{N} \cup \{0\})$$

with corresponding eigenfunctions

$$\psi_k \propto \cos\left(\left(k + \frac{1}{2}\right)\pi \right) \quad (k \in \mathbb{N} \cup \{0\})$$

Therefore, the spectrum of K equals

$$\sigma(K) = \left\{ \left(\frac{2}{(2k+1)\pi} \right)^2 : \ k \in \mathbb{N} \cup \{0\} \right\} \cup \{0\}$$

■

Problem 2

Prove Kuiper's theorem: Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, and let $A \in \mathcal{B}(\mathcal{H})$ be invertible. Show that there is an operator-norm-continuous map $\gamma : [0,1] \to \mathcal{B}(\mathcal{H})$ such that:

1. $\gamma(0) = A$,
2. $\gamma(1) = \mathbb{1}$,
3. $\gamma(t)$ is invertible for all $t \in [0,1]$.

Solution

Proof. Since A is invertible, the polar decomposition (Theorem 9.25 in the lecture notes) says that we may express

$$A = U|A|$$

where $|A|$ is invertible and $U := A|A|^{-1}$ is unitary. Lemma 8.5 in the lecture notes tells us that since U is unitary, $\sigma(U) \subseteq \mathbb{S}^1$. The function

$$\log : \mathbb{S}^1 \to i[0, 2\pi]$$

is bounded and Borel-measurable on $\sigma(U)$; as U is normal we may apply the Borel functional calculus to find $V := \log(U)$ such that $U = e^V$. By the spectral mapping theorem, $\sigma(V) \subseteq i[0, 2\pi]$ and so $W := -iV$ is self-adjoint. We simply define $\gamma : [0,1] \to \mathcal{B}(\mathcal{H})$ via

$$\gamma(t) := e^{i(1-t)W}(t\mathbb{1} + (1-t)|A|)$$

To see norm-continuity, let $\varepsilon > 0$. Then, for all $s, t \in [0,1]$ with $|s-t| < \delta$, we have

$$\|\gamma(s) - \gamma(t)\| = \left\| e^{i(1-s)W}(s\mathbb{1} + (1-s)|A|) - e^{i(1-t)W}(t\mathbb{1} + (1-t)|A|) \right\|$$

Let $M := \max\{\||A|\|, 1\}$. Adding and subtracting $e^{i(1-s)W}(t\mathbb{1} + (1-t)|A|)$ and applying the triangle rule and submultiplicativity of the operator norm,

$$\begin{aligned} \|\gamma(s) - \gamma(t)\| &\leq \left\| e^{i(1-s)W} \right\| \cdot \|(s-t)\mathbb{1} + (t-s)|A|\| + \|t\mathbb{1} + (1-t)|A|\| \cdot \left\| e^{i(1-t)W} - e^{i(1-s)W} \right\| \\ &\leq \left\| e^{i(1-s)W} \right\| \cdot M|s-t| + M \left\| e^{i(1-t)W} - e^{i(1-s)W} \right\| \end{aligned}$$

We know that since W is self-adjoint, so is $(1-s)W$, which means that $e^{i(1-s)W}$ is unitary. By Lemma 8.5 in the lecture notes, $\left\| e^{i(1-s)W} \right\| = 1$. So,

$$\|\gamma(s) - \gamma(t)\| \leq M \left(|s-t| + \left\| e^{i(1-t)W} - e^{i(1-s)W} \right\| \right)$$

Since the operator exponential $e^{\cdot} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is operator-norm-continuous and $\|i(1-t)W - i(1-s)W\| = |s-t|\|W\|$, there is a $\delta > 0$ small enough that this expression is at most ε. So, γ is continuous in the operator norm. We verify the 3 desired properties next.

1. We have $\gamma(0) = e^{iW}|A| = e^V|A| = U|A| = A$.
2. We have $\gamma(1) = e^0\mathbb{1} = \mathbb{1}\mathbb{1} = \mathbb{1}$.
3. Note that since W is self-adjoint, so is $(1-t)W$, which means that $e^{i(1-t)W}$ is unitary and therefore invertible. Also, since $|A|$ is positive and invertible, $\sigma(|A|) \subseteq (0, \infty)$. Therefore, by the spectral mapping theorem applied to the map $z \mapsto t + (1-t)z$, we have that $\sigma(t\mathbb{1} + (1-t)|A|) = (t, \infty)$ for all $t \in (0,1)$. So, $t\mathbb{1} + (1-t)|A|$ is invertible for all $t \in (0,1)$, which means that $\gamma(t)$ is invertible for all $t \in (0,1)$. Since $\gamma(0) = A$ and $\gamma(1) = \mathbb{1}$ are also invertible, the result follows.

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Problem 3

This question is divided into three independent parts.

(a) Prove (with the spectral theorem) that if $A \in \mathcal{B}(\mathcal{H})$ then the following are equivalent:

  i. $\langle \psi, A\psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$
  ii. $A = A^*$ and $\sigma(A) \subseteq [0, \infty)$
  iii. There exists some $B \in \mathcal{B}(\mathcal{H})$ s.t. $A = |B|^2$

(b) Prove Stone's formula: if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint and

$$\widetilde{\chi}_{[a,b]}(\lambda) := \begin{cases} 1 & \lambda \in (a,b) \\ 0 & \lambda \notin [a,b] \\ \frac{1}{2} & \lambda \in \{a,b\} \end{cases} \qquad (\lambda \in \mathbb{R})$$

then

$$\widetilde{\chi}_{[a,b]}(A) = \underset{\varepsilon \to 0^+}{\text{s-lim}} \frac{1}{2\pi i} \int_{\lambda=a}^{b} \left[(A - (\lambda + i\varepsilon)\mathbb{1})^{-1} - (A - (\lambda - i\varepsilon)\mathbb{1})^{-1} \right] d\lambda$$

(c) Show that if $A \in \mathcal{B}(\mathcal{H})$ is normal and $\psi \in \mathcal{H}$ is a cyclic vector for A then it is also cyclic for A^*.

Solution

Proof. (a) (i $\implies$ ii) Suppose first that $\langle \psi, A\psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$. Then, we have that $\langle \psi, A\psi \rangle \in \mathbb{R}$, and so

$$\langle \psi, A^*\psi \rangle = \overline{\langle A^*\psi, \psi \rangle} = \overline{\langle \psi, A\psi \rangle} = \langle \psi, A\psi \rangle \implies \langle \psi, (A - A^*)\psi \rangle = 0 \quad (\psi \in \mathcal{H})$$

By Theorem 7.11 in the lecture notes, $A = A^*$, and so $\sigma(A) \subseteq \mathbb{R}$. We may apply the spectral theorem in its multiplicative form (Corollary to Theorem VII.3 in R&S) to find a finite measure space (M, μ), a bounded function F on M, and a unitary map $U : \mathcal{H} \to L^2(M, \mu)$ such that

$$(UAU^{-1})(f)(m) = F(m)f(m) \quad (f \in L^2(M, \mu), m \in M)$$

We know that $\sigma(A)$ is the essential range of F (this is because $\sigma(UAU^{-1})$ is the essential range of F by Theorem 11.35 in the lecture notes, and the spectrum is a unitary invariant). This immediately tells us that F is real-valued μ-a.e. since $\sigma(A) \subseteq \mathbb{R}$. Suppose by way of contradiction that $-\lambda \in \sigma(A)$ for some $\lambda > 0$. Then, $-\lambda$ is in the essential range of F, and so

$$\mu(\{m \in M : \ F(m) \in (-3\lambda/2, -\lambda/2)\}) > 0 \implies \mu(F^{-1}((-\infty, 0))) > 0$$

where $F^{-1}(\cdot)$ denotes the preimage. Let $f := \chi_{F^{-1}((-\infty,0))}$ be the indicator function on the set $F^{-1}((-\infty, 0))$; we know that $f \in L^2(M, \mu)$ since (M, μ) is a finite measure space and so all indicator functions are integrable. Thus,

$$\begin{aligned} 0 &> \int_{F^{-1}(-\infty,0))} F(m) d\mu(m) = \int_M F(m)f(m) d\mu(m) = \int_M \overline{f(m)} F(m) f(m) d\mu \\ &= \int_M \overline{f(m)} (UAU^{-1}f)(m) d\mu(m) = \left\langle f, (UAU^{-1})f \right\rangle_{L^2(M,\mu)} \end{aligned}$$

where the first inequality is strict precisely since $\mu(F^{-1}((-\infty, 0))) > 0$ and F is strictly negative on this set, and the other equalities follow since $f = \overline{f} = f^2$ for indicator functions f. Since U is unitary and so $U^{-1} = U^*$, we have

$$\left\langle U^{-1}f, AU^{-1}f \right\rangle_{\mathcal{H}} = \left\langle U^*f, AU^{-1}f \right\rangle_{\mathcal{H}} = \left\langle f, (UAU^{-1})f \right\rangle_{L^2(M,\mu)} < 0$$

Letting $\psi = U^{-1}f$, this tells us that $\left\langle U^{-1}f, AU^{-1}f\right\rangle_{\mathcal{H}} = \langle \psi, A\psi\rangle_{\mathcal{H}} \geq 0$ by hypothesis. So,

$$0 \leq \left\langle U^{-1}f, AU^{-1}f\right\rangle_{\mathcal{H}} < 0,$$

a contradiction. So, $-\lambda \notin \sigma(A)$ for all $\lambda > 0$; equivalently, $\sigma(A) \subseteq [0, \infty)$.

(ii $\implies$ iii) Suppose now that $A = A^*$ and $\sigma(A) \subseteq [0, \infty)$. Since the map sending $z \mapsto \sqrt{z}$ is continuous on $\sigma(A)$, we may apply the continuous functional calculus (Theorem VII.1 in R&S) to find $B := \sqrt{A} \in \mathcal{B}(\mathcal{H})$; since $z \mapsto \sqrt{z}$ takes only real values over $\sigma(A)$, we also see that $B^* = \sqrt{A}$. So,

$$|B|^2 = B^*B = \sqrt{A}\sqrt{A} = A$$

where the last equality is by the homomorphism property of the functional calculus.

(iii $\implies$ i) Suppose now that $A = |B|^2$ for some $B \in \mathcal{B}(\mathcal{H})$. Then, for all $\psi \in \mathcal{H}$ we have

$$\langle \psi, A\psi\rangle = \left\langle \psi, |B|^2\psi\right\rangle = \langle \psi, B^*B\psi\rangle = \langle B\psi, B\psi\rangle = \|B\psi\|^2 \geq 0$$

In particular, (i) holds.

(b) We start with the following lemma.

Lemma 1. *For all $E \in \mathbb{R}$,*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\lambda=a}^{b} \mathbb{Im}\left\{\frac{1}{E - \lambda - i\varepsilon}\right\} d\lambda = \widetilde{\chi}_{[a,b]}(E)$$

Proof of Lemma. When $E \notin [a, b]$, then for small enough ε we know that $\lambda \mapsto \mathbb{Im}\left\{\frac{1}{E-\lambda-i\varepsilon}\right\}$ is holomorphic over $[a, b]$. By Cauchy's integral theorem, this means that the integral evaluates to 0. When $E \in (a, b)$, we may apply the residue theorem for small enough ε. I have no idea how to prove this lemma, I saw it in the Lecture notes. I should have taken complex analysis haha ■

We also note that for $E, \lambda, \varepsilon \in \mathbb{R}$,

$$\mathbb{Im}\left\{\frac{1}{E - \lambda - i\varepsilon}\right\} = 2i\left[\frac{1}{E - \lambda - i\varepsilon} - \frac{1}{E - \lambda + i\varepsilon}\right]$$

So, letting $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f_\varepsilon(z) := \frac{1}{2\pi i} \int_{\lambda=a}^{b} \left[(z - \lambda - i\varepsilon)^{-1} - (z - \lambda + i\varepsilon)^{-1}\right] d\lambda,$$

the above lemma tells us that $f_\varepsilon \to \widetilde{\chi}_{[a,b]}$ pointwise. Since each $\{f_\varepsilon\}_\varepsilon$ is bounded and each f_ε is measurable, we see by the Borel functional calculus (Theorem VII.2(d) in R&S) that $f_\varepsilon(A) \to \widetilde{\chi}_{[a,b]}(A)$ strongly. This is exactly what Stone's formula states.

(c) Let $A \in \mathcal{B}(\mathcal{H})$ be normal and $\psi \in \mathcal{H}$ be cyclic for A. Then,

$$\{p(A)\psi : \ p \text{ is a polynomial}\}$$

is dense in $\mathcal{H}$. We wish to show that

$$\overline{\{p(A)\psi : \ p \text{ is a polynomial}\}} \subseteq \overline{\{p(A^*)\psi : \ p \text{ is a polynomial}\}}$$

as this will show the required density. Since complex conjugation is continuous, there is a sequence of polynomials $\{p_n\}_n$ that approximates $z \mapsto \overline{z}$ pointwise on $\sigma(A^*)$. Therefore, by the functional calculus (which we may apply by normality of A), we see that $p_n(A^*) \to (A^*)^* = A$ strongly. Therefore, since multiplication is jointly continuous in the strong operator topology (Problem 25 on Problem Set 7), we see that $p_n(A^*)^k \to A^k$ strongly for all $k \in \mathbb{N} \cup \{0\}$. By the $*$-homomorphism property of the functional calculus, $(p_n)^k(A^*) \to A^k$ strongly as well, and so $(p_n)^k(A^*)\psi \to A^k\psi$. Since each $(p_n)^k$ is itself a polynomial, we see that

$$A^k\psi \in \overline{\{p(A^*)\psi : \ p \text{ is a polynomial}\}} \quad (k \in \mathbb{N} \cup \{0\})$$

So, since the set $\overline{\{p(A^*)\psi : \ p \text{ is a polynomial}\}}$ is a linear space, we find that

$$\{p(A)\psi : \ p \text{ is a polynomial}\} \subseteq \overline{\{p(A^*)\psi : \ p \text{ is a polynomial}\}}$$

from which the result follows by taking closures. ■

Problem 4

This problem is divided into multiple parts which are independent of each other.

(a) Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $z \in \mathbb{C}$ with $\mathbb{Im}\{z\} \neq 0$. Show that

$$U := (A + \overline{z}\mathbb{1})(A + z\mathbb{1})^{-1}$$

is unitary.

(b) Show that for any $A \in \mathcal{B}(\mathcal{H})$, $\ker\left(|A|^2\right) = \ker(A)$.

(c) Show that for any $A \in \mathcal{B}(\mathcal{H})$, if $\dim \operatorname{im}(A) = 1$ then there are $\varphi, \psi \in \mathcal{H} \setminus \{0\}$ such that

$$A\xi = \langle \varphi, \xi \rangle \psi \quad (\xi \in \mathcal{H}).$$

Proceed to calculate: (i) $\|A\|$, (ii) A^*, (iii) $\sigma(A)$.

(d) Show that if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint and unitary, then there are two orthogonal projections P, Q such that

$$A = P - Q$$

and that this yields a $\mathbb{Z}_2$ grading of the Hilbert space as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

(e) Let $\mathcal{H} := \ell^2(\mathbb{N})$ and $R \in \mathcal{B}(\mathcal{H})$ be the unilateral right shift operator

$$(R\psi)(n) := \begin{cases} \psi(n-1) & n \geq 2 \\ 0 & n = 1 \end{cases} \quad (\psi \in \mathcal{H})$$

Compute $|R|^2$ and $|R^*|^2$. If $\{\delta_n\}_{n\in\mathbb{N}}$ is the standard basis of $\mathcal{H}$, calculate the following expressions:

$$\sum_{n=1}^{\infty} \langle \delta_n, R\delta_n \rangle, \sum_{n=1}^{\infty} \langle \delta_n, R^*\delta_n \rangle, \sum_{n=1}^{\infty} \left\langle \delta_n, |R|^2\delta_n \right\rangle, \sum_{n=1}^{\infty} \left\langle \delta_n, |R^*|^2\delta_n \right\rangle, \sum_{n=1}^{\infty} \left\langle \delta_n, \left(|R|^2 - |R^*|^2\right)\delta_n \right\rangle$$

Interpreting these expressions naively as traces, what can you conclude about cyclicity in this infinite setting?

Solution

Proof. (a) Note that $A + z\mathbb{1}$ and $A + \overline{z}\mathbb{1}$ are both invertible since $z, \overline{z} \notin \sigma(A)$. So, U is invertible and therefore $\operatorname{im}(U) = \mathcal{H}$. Now, we may compute that

$$|U|^2 = U^*U = \left((A + \overline{z}\mathbb{1})(A + z\mathbb{1})^{-1}\right)^* (A+\overline{z}\mathbb{1})(A+z\mathbb{1})^{-1} = ((A+z\mathbb{1})^{-1})^*(A+z\mathbb{1})(A+\overline{z}\mathbb{1})(A+z\mathbb{1})^{-1}$$

Clearly, $A + z\mathbb{1}$ and $A + \overline{z}\mathbb{1}$ commute. Also, since $(T^*)^{-1} = (T^{-1})^*$ for all invertible T, we may say

$$|U|^2 = ((A + z\mathbb{1})^*)^{-1}(A + \overline{z}\mathbb{1})(A + z\mathbb{1})(A + z\mathbb{1})^{-1} = (A + \overline{z}\mathbb{1})^{-1}(A + \overline{z}\mathbb{1}) = \mathbb{1}$$

So, $\langle U\psi, U\varphi \rangle = \left\langle \psi, |U|^2\varphi \right\rangle = \langle \psi, \varphi \rangle$ for all $\psi, \varphi \in \mathcal{H}$. By Claim 9.4 in the lecture notes, U is unitary.

(b) Suppose first that $\psi \in \ker\left(|A|^2\right)$, and so $|A|^2\psi = 0$. Then,

$$0 = \left\langle \psi, |A|^2\psi \right\rangle = \langle \psi, A^*A\psi \rangle = \langle A\psi, A\psi \rangle = \|A\psi\|^2 \implies A\psi = 0,$$

and so $\psi \in \ker(A)$. Conversely, suppose that $\psi \in \ker(A)$. Then, $A\psi = 0$, and so $A^*(A\psi) = 0$. Since $A^*A = |A|^2$, we see that $|A|^2\psi = 0$ and $\psi \in \ker\left(|A|^2\right)$.

(c) Since $\dim \operatorname{im}(A) = 1$, then there is some unit vector $\psi \in \mathcal{H} \setminus \{0\}$ such that $\operatorname{im}(A) = \operatorname{span}\{\psi\}$ (in particular, $\{\psi\}$ is a normalized basis for $\operatorname{im}(A)$). So, for each $\xi \in \mathcal{H}$ we know that $A\xi = \lambda(\xi)\psi$ for some map $\lambda : \mathcal{H} \to \mathbb{C}$. We note that λ must be linear since for all $\xi, \eta \in \mathcal{H}$ and all $\alpha \in \mathbb{C}$,

$$A(\xi + \alpha\eta) = A\xi + \alpha A\eta \implies \lambda(\xi + \alpha\eta)\psi = (\lambda(\xi) + \alpha\lambda(\eta))\psi \implies \lambda(\xi + \alpha\eta) = \lambda(\xi) + \alpha\lambda(\eta),$$

where the last implication follows since $\psi \neq 0$. Furthermore, λ must be bounded since A is, and therefore $\lambda \in \mathcal{H}^*$ since it is a continuous linear functional on $\mathcal{H}$. By Riesz representation (Theorem 7.10 in the lecture notes), there is some $\varphi \in \mathcal{H}$ such that

$$\lambda(\xi) = \langle \varphi, \xi \rangle \quad (\xi \in \mathcal{H})$$

Since A is not identically 0 (its range is nonzero), neither is λ, which means $\varphi \neq 0$ as well. So, we see that there are two $\varphi, \psi \in \mathcal{H} \setminus \{0\}$ for which $A\xi = \langle \varphi, \xi \rangle \psi$ for all $\xi \in \mathcal{H}$ (i.e. $A = \psi \otimes \varphi^*$). To compute the desired quantities, note that for each unit vector ξ,

$$\|A\xi\| = \| \langle \varphi, \xi \rangle \psi\| = | \langle \varphi, \xi \rangle | \leq \|\varphi\|\|\xi\| = \|\varphi\|$$

by Cauchy-Schwartz. However, for the unit vector $\xi := \frac{\varphi}{\|\varphi\|}$, we see that

$$\|A\xi\| = \frac{1}{\|\varphi\|} | \langle \varphi, \varphi \rangle | = \frac{1}{\|\varphi\|} \|\varphi\|^2 = \|\varphi\|$$

Therefore, $\|A\| = \sup\{\|A\xi\| : \|\xi\| = 1\} = \|\varphi\|$. Next, we claim that A^* is the map sending $\xi \mapsto \langle \psi, \xi \rangle \varphi$ (i.e. $A^* = \varphi \otimes \psi^*$). To see this, note that for all $\xi, \eta \in \mathcal{H}$,

$$\langle \xi, A\eta \rangle = \langle \varphi, \eta \rangle \langle \xi, \psi \rangle = \left\langle \overline{\langle \xi, \psi \rangle} \varphi, \eta \right\rangle = \langle \langle \psi, \xi \rangle \varphi, \eta \rangle = \langle A^*\xi, \eta \rangle$$

Lastly, to compute $\sigma(A)$, we note that A is finite-rank and therefore compact, and so $\sigma(A) = \{0\} \cup \sigma_p(A)$ by the Riesz-Schauder theorem (Theorem 9.42 in the lecture notes, though I will remark that A is invertible and 0 is not in the spectrum if $\dim(\mathcal{H}) = 1$). To compute the point spectrum, we see that

$$\lambda \in \sigma_p(A) \iff \exists \xi \in \mathcal{H} \text{ s.t. } A\xi = \lambda\xi \iff \exists \xi \in \mathcal{H} \text{ s.t. } \langle \varphi, \xi \rangle \psi = \lambda\xi$$

Clearly, ξ must be a multiple of ψ for this to happen. So, letting $\xi = \alpha\psi$ for $\alpha \neq 0$,

$$\lambda \in \sigma_p(A) \iff \exists \alpha \in \mathbb{C}\setminus\{0\} \text{ s.t. } \langle \varphi, \alpha\psi \rangle \psi = \lambda\alpha\psi \iff \exists \alpha \in \mathbb{C}\setminus\{0\} \text{ s.t. } \alpha \langle \varphi, \psi \rangle = \alpha\lambda \iff \langle \varphi, \psi \rangle = \lambda$$

Therefore, we see that

$$\sigma(A) = \{0, \langle \varphi, \psi \rangle\}$$

(d) Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint and unitary. Then, $\sigma(A) \subseteq \{-1, 1\}$. Letting $\chi_\cdot(A)$ be the projection-valued measure for A, we can apply the Borel functional calculus via the map $z \mapsto z$ to see that

$$A = \int_{\mathbb{R}} z d\chi_{\{z\}}(A)$$

Since the only values of the spectrum may be -1 or 1, these are the only two values of z where $\chi_{\{z\}}(A)$ may be nonzero (in other words, $\operatorname{spt}(\chi_\cdot(A)) = \sigma(A) \subseteq \{-1, 1\}$). So, we see that

$$A = (1)\chi_{\{1\}}(A) + (-1)\chi_{\{-1\}}(A) = \chi_{\{1\}}(A) - \chi_{\{-1\}}(A) =: P - Q$$

Clearly, both P and Q are orthogonal projections since indicator functions are idempotent and real-valued. Now, note that by the properties of a projection-valued measure (Definition 10.17 in the lecture notes),

$$\mathbb{1} = \chi_{\{1\}}(A) + \chi_{\{-1\}}(A) + \chi_{\mathbb{R}\setminus\{\pm 1\}}(A) = P + Q$$

since $\chi_{\mathbb{R}\setminus\{\pm 1\}}(A) = 0$. Since $P+Q = \mathbb{1}$, we see that any $\psi \in \mathcal{H}$ can be expressed as $\psi = \mathbb{1}\psi = P\psi + Q\psi$ with $P\psi \in \operatorname{im}(P)$ and $Q\psi \in \operatorname{im}(Q)$. By Problem 6 on Problem Set 7, we see that $P + Q = \mathbb{1} \implies P \perp Q \implies \operatorname{im}(P) \perp \operatorname{im}(Q)$, and so this expression is unique. Thus,

$$\mathcal{H} = \operatorname{im}(P) \oplus \operatorname{im}(Q)$$

Letting $\mathcal{H}_+ := \operatorname{im}(P)$ and $\mathcal{H}_- := \operatorname{im}(Q)$, we get the desired $\mathbb{Z}_2$ grading of $\mathcal{H}$ via P and Q.

(e) It is a bit simpler for me to express R in the position basis as

$$R\delta_n := \delta_{n+1} \quad (n \in \mathbb{N})$$

and extended linearly. We claim that $R^* = L$, the unilateral left shift operator defined on the basis by

$$L\delta_n := \begin{cases} \delta_{n-1} & n > 1 \\ 0 & n = 1 \end{cases}$$

and extended linearly. To see that they are adjoints, let $\varphi, \psi \in \ell^2(\mathbb{N})$ be arbitrary. We may therefore express

$$\varphi \equiv \sum_{n\in\mathbb{N}} \varphi_n \delta_n \quad \text{and} \quad \psi \equiv \sum_{n\in\mathbb{N}} \psi_n \delta_n$$

for $\varphi_n, \psi_n \in \mathbb{C}$. As such, we see that

$$\langle L\varphi, \psi\rangle = \left\langle \sum_{n>1} \varphi_n \delta_{n-1}, \psi \right\rangle = \left\langle \sum_{n\in\mathbb{N}} \varphi_{n+1}\delta_n, \psi \right\rangle = \sum_{n\in\mathbb{N}} \overline{\varphi_{n+1}}\psi_n$$

and

$$\langle \varphi, R\psi\rangle = \left\langle \varphi, \sum_{n\in\mathbb{N}} \psi_n \delta_{n+1} \right\rangle = \left\langle \varphi, \sum_{n>1} \psi_{n-1}\delta_n \right\rangle = \sum_{n>1} \overline{\varphi_n}\psi_{n-1} = \sum_{n\in\mathbb{N}} \overline{\varphi_{n+1}}\psi_n,$$

where the last equality simply relabeled indices. So, $\langle L\varphi, \psi\rangle = \langle \varphi, R\psi\rangle$; since this holds for all $\varphi, \psi \in \ell^2(\mathbb{N})$, they are indeed adjoints. We may now compute $|R|^2 = LR$ and $|R^*|^2 = RL$. For any $n > 1$, we have that

$$LR\delta_n = L\delta_{n+1} = \delta_n \quad \text{and} \quad RL\delta_n = R\delta_{n-1} = \delta_n$$

However, we note that

$$LRe_1 = Le_2 = e_1 \quad \text{yet} \quad RLe_1 = R0 = 0$$

since $Le_1 = 0$. As such, we find that $|R|^2 = LR = \mathbb{1}$, whereas $|R^*|^2$ is defined on the basis as

$$|R^*|^2\delta_n = RL\delta_n = \begin{cases} \delta_n & n > 1 \\ 0 & n = 1 \end{cases}$$

and extended linearly.

We may now compute the desired traces. We see that

$$\langle \delta_n, A\delta_n\rangle = \begin{cases} 0 & A = R, R^* \\ 1 & A = |R|^2 \\ 0 \text{ if } n = 1,\ 1 \text{ otherwise} & A = |R^*|^2 \\ 1 \text{ if } n = 1,\ 0 \text{ otherwise} & A = |R|^2 - |R^*|^2 \end{cases}$$

So,

$$\sum_{n=1}^{\infty} \langle \delta_n, R\delta_n \rangle = \sum_{n=1}^{\infty} \langle \delta_n, R^* \delta_n \rangle = 0, \qquad \sum_{n=1}^{\infty} \left\langle \delta_n, |R|^2 \delta_n \right\rangle = \sum_{n=1}^{\infty} \left\langle \delta_n, |R^*|^2 \delta_n \right\rangle = \infty$$

and

$$\sum_{n=1}^{\infty} \left\langle \delta_n, \left(|R|^2 - |R^*|^2\right) \delta_n \right\rangle = 1$$

I am not sure what is meant by cyclicity. We see that R, R^*, and $|R|^2 - |R^*|^2$ are trace-class, and we also know that δ_0 is a cyclic vector for R I suppose. No idea what else we can say that relates to cyclicity. $\blacksquare$

Problem 5

The following question has two independent parts:

(a) Let X, Y be normed spaces and $A : X \to Y$ linear. Suppose that whenever $\{\psi_n\}_n \subseteq X$ converges weakly to zero, $\{A\psi_n\} \subseteq Y$ converges weakly to zero. Show that A is bounded.

(b) Let X, Y, Z be Banach spaces. Let $A : X \to Y$ and $J : Y \to Z$ be linear. Suppose that J is bounded and injective, and JA is bounded. Show that A is bounded.

Solution

Proof. (a) Suppose by way of contradiction that A were not bounded. Then, there is some sequence $\{\psi_n\}_n \subseteq X$ of unit vectors such that $\|A\psi_n\| \to \infty$. Since every bounded sequence contains a weakly-convergent subsequence, there is some $\{\psi_{n_k}\}_k$ that converges weakly. Therefore, by hypothesis we have that $\{\psi_{An_k}\}_k$. By Proposition 5.12 in the lecture notes, this means that $\{A\psi_{n_k}\}_k$ is norm-bounded. This contradicts that $\|A\psi_n\| \to \infty$, and so it must be that A is bounded.

(b) We will show that $\Gamma(A) \subseteq X \times Y$ is closed, since that will imply A is bounded by the closed graph theorem (Theorem 3.37 in the lecture notes). So, let $\{(\psi_n, A\psi_n)\}_n \subseteq \Gamma(A)$ be a sequence that converges to some $(\psi, \varphi) \in X \times Y$, and so $\psi_n \to \psi$ and $A\psi_n \to \varphi$; we must show that $A\psi = \varphi$ and the result will follow. Since JA is bounded and therefore continuous we know that $\psi_n \to \psi \implies (JA)(\psi_n) \to (JA)\psi$. Also, since J is bounded and therefore continuous we know that $A\psi_n \to \varphi \implies J(A\psi_n) \to J\varphi$. By the uniqueness of limits, this means that $(JA)\psi = J\varphi$. However, since J is injective, the only way this is possible is if $A\psi = \varphi$. Thus, $\Gamma(A) \in \text{Closed}(X \times Y)$ and A is bounded. ■